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## A fully parametric approach for solving quantile regressions with time-varying coefficients

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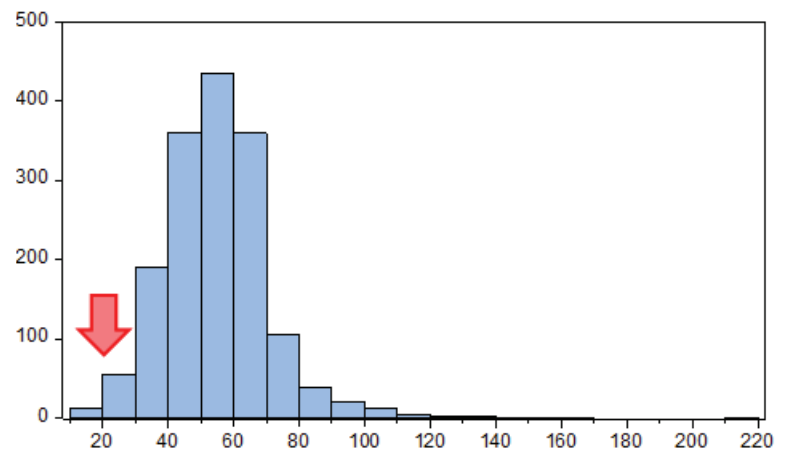
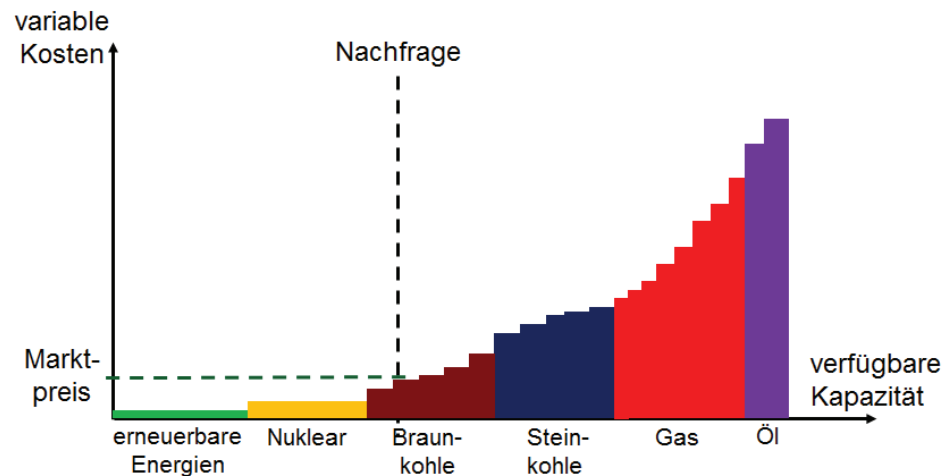
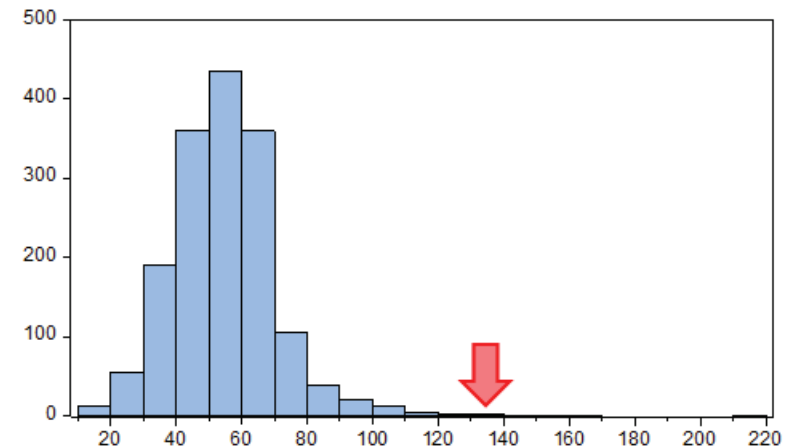
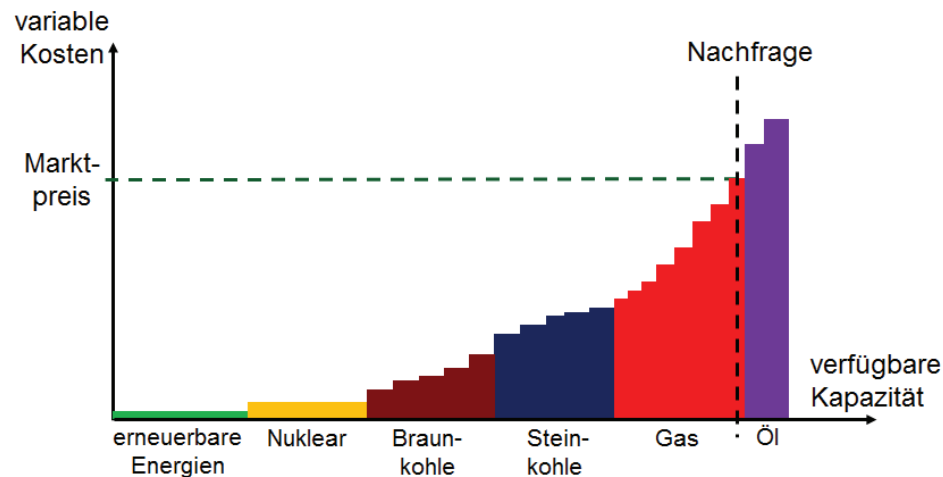
Derek Bunn, London Business School

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Hannover, June 2016

# Motivation

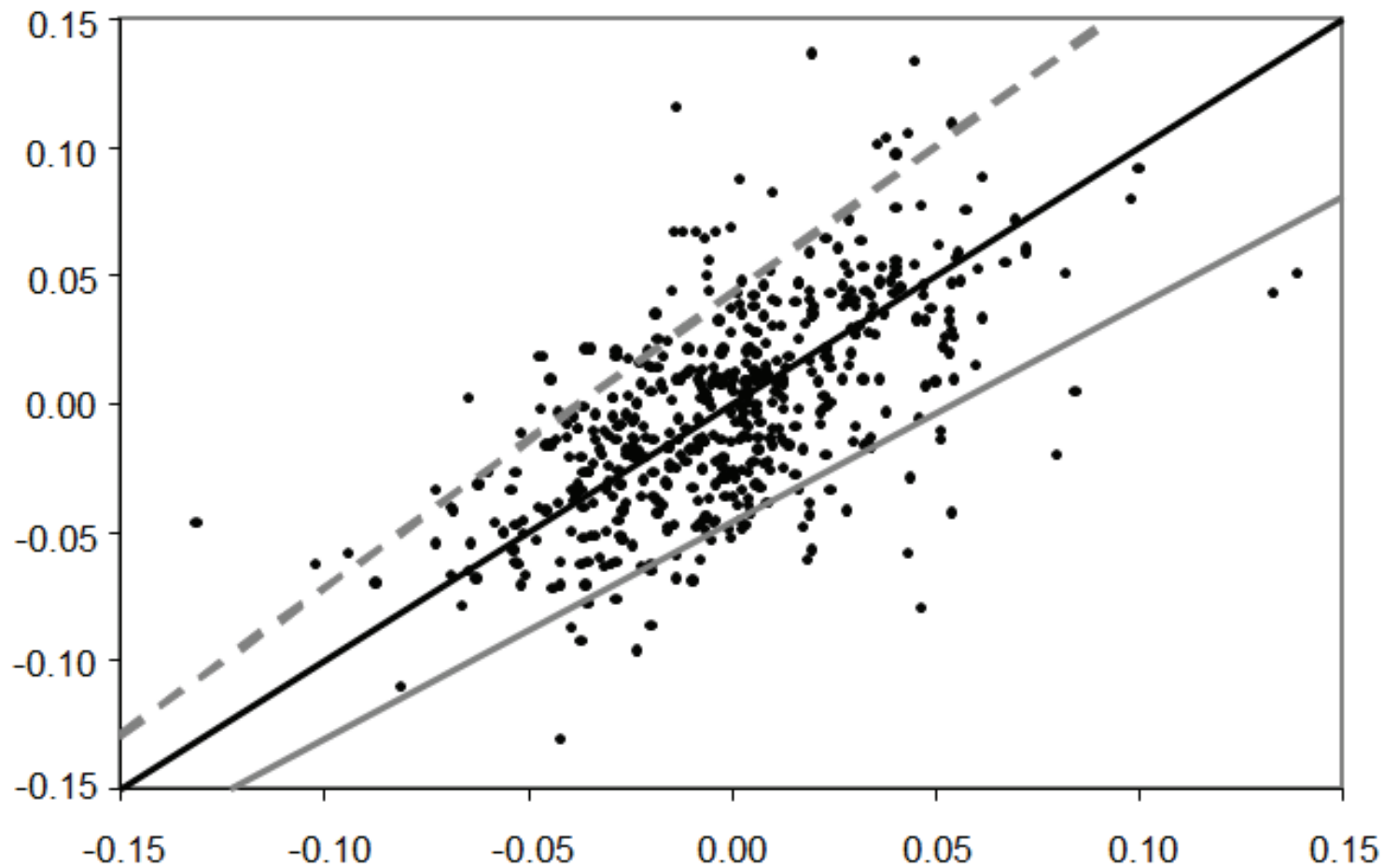
Different electricity price “regions” are influenced by different fundamental variables, due to different input mix:



## General formulation of the quantile regression

$$y_t = f(\beta, x_t) + u_t \quad (1)$$

$$q_\alpha(y_t | \beta^\alpha, x_t) = f_t(\beta^\alpha, x_t) + u_t^\alpha \quad (2)$$



## Why do we need a parametric approach?

- Estimation of quantile regressions in the literature is non-parametric and often a 2-stage approach (Bunn et al. (2012), Honda (2004), Kim (2007), De Rossi & Harvey (2006), Gerlach et al. (2011)):
  1. I: Estimate  $\beta^\alpha$  by a non-parametric approach, so no assumption about the distribution of residuals  $u_t$  is made.
  2. II: After step one is completed, separately estimate the variance of residuals  $u_t$  (GARCH(1,1)CAViaR models).
- The two-step approach is criticized in the literature for inconsistency
- The model specification in Equation 2 does not explain typical features of financial data like: volatility clustering, extreme tails. These must be separately modeled in a second step.

## Why do we need dynamic coefficients?

- In Karakatsani & Bunn (2010) and Paraschiv et al. (2014) it is shown that: the time-varying feature of coefficients allow adaption of dependent variable to exogenous variables
- **Via adaption, the typical volatility clustering pattern of financial data is removed**
- Extension of the classical quantile regression model: allowance for time-varying coefficients

$$q_{\alpha}(y_t|\beta_{\mathbf{t}}^{\alpha}, x_t) = f_t(\beta_{\mathbf{t}}^{\alpha}, x_t) + u_t^{\alpha} \quad (3)$$

- Time-variability of coefficients:
  - Short-term: demand/supply shocks
  - long-term: political changes, structural change of the markets

## Theoretical background

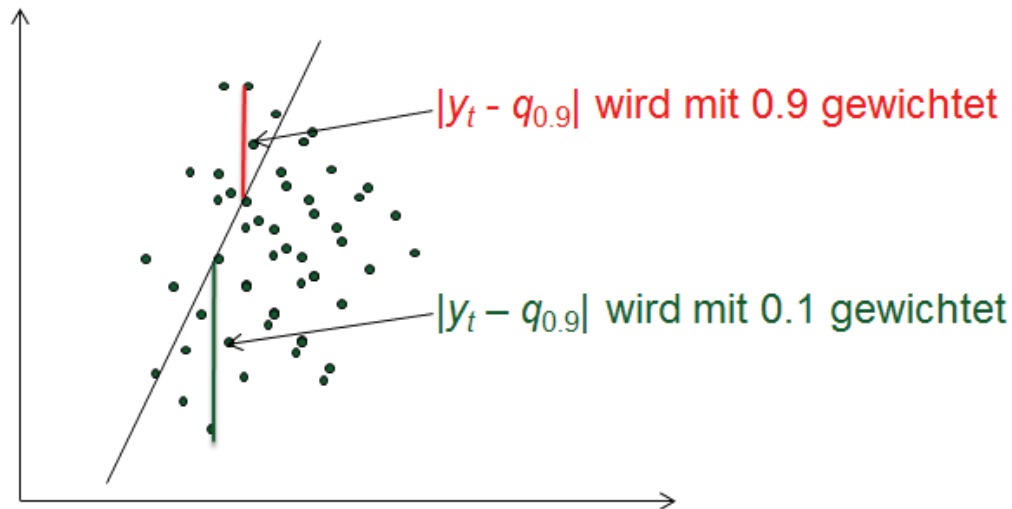
- The conditional  $\alpha \in (0, 1)$ - quantile of observations  $y_t$  is:

$$q_\alpha(y_t|\beta^\alpha, x_t) = f_t(\beta^\alpha, x_t) \quad (4)$$

- For each  $\alpha \in (0, 1)$  we compute the absolute distances between  $y_t$  and the quantile  $q_\alpha$  (vgl. Hao/Naiman (2007)):

$$d_\alpha(y_t, q_\alpha) = \left\{ \begin{array}{ll} (\alpha - 1)|y_t - q_\alpha|, & y_t < q_\alpha \\ \alpha|y_t - q_\alpha|, & y_t \geq q_\alpha \end{array} \right\} \quad (5)$$

- Example for  $\alpha = 0.9$ :



## The Skewed-Laplace connection

- We look for the value  $q_\alpha$  that minimises the mean distance from  $y_t$ :  $\mathbb{E}[d_\alpha(y_t, q_\alpha)]$ . **The minimum occurs when  $q_\alpha$  is the  $\alpha$ 's quantile of  $y_t$ .**  $q_\alpha$  depends on  $\beta^\alpha$  which is the solution to:

$$\min_{\beta} \sum_t \rho_\alpha(y_t - f_t(\beta, x_t)), \quad (6)$$

where  $\rho(\cdot)$  is a loss function specified as:

$$\rho_\alpha(u_t) = u_t \cdot (\alpha - \mathbf{1}(u < 0)) \quad (7)$$

with  $u_t = y_t - f_t(\beta, x_t)$ .

- **We assume that residuals  $u_t$  are SL-distributed.** The  $SL(\mu, \tau, \alpha)$  has a density function:

$$\mathbb{L}_\alpha(\beta, \tau; y, x) \propto \tau^{-n} \exp \left\{ -\tau^{-1} \sum_{t=1}^n (y_t - f_t(\beta, x_t)) \times [\alpha - \mathbf{1}_{(-\infty, 0)}(y_t - f_t(\beta, x_t))] \right\}. \quad (8)$$

- We observe that the sum which must be minimised in Equation (6) is contained in the exponent of the likelihood. Thus, **the maximum likelihood estimation for  $\beta$  in (6) is equivalent to the quantile estimator in (6).**

## The time-varying coefficients feature

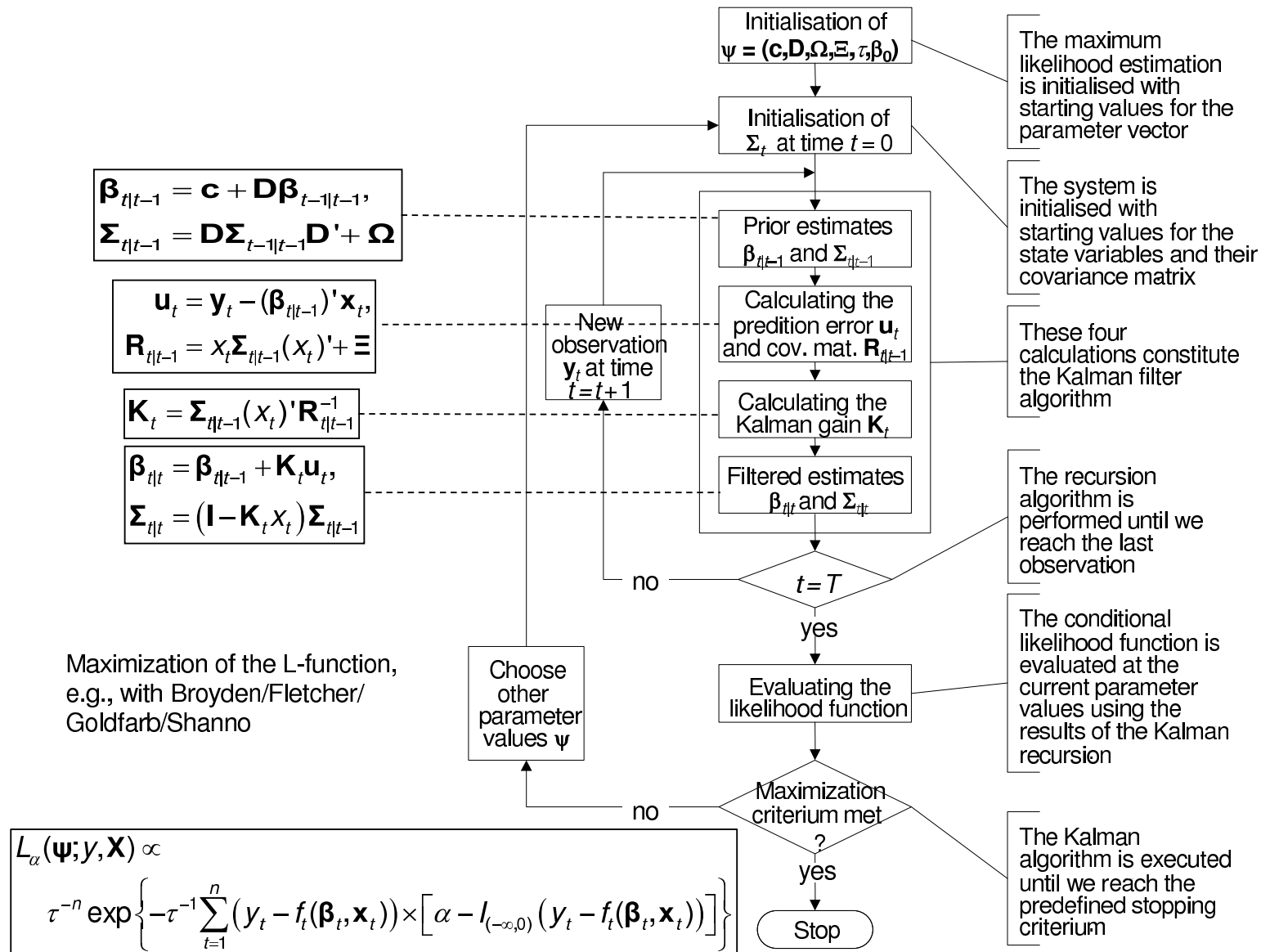
- We extend the model by introducing time-varying coefficients.
- We formulate a **state space model with time-dependent coefficients recursively filtered with Kalman Filter** (Kalman (1960)) and model parameters are estimated with maximum likelihood.
- The assumption that  $u_t$  follow a  $SL(\mu, \tau, \alpha)$  distribution is used here to estimate parametrically the quantiles of the dependent variables  $y_t$ .
- The likelihood function is similar to the specification in (8) but we account in addition for time-varying  $\beta_t$ .
- The state space formulation reads:

$$y_t^\alpha = (\beta_t^\alpha)' x_t + u_t \tag{9}$$

$$\beta_t^\alpha = c + D\beta_{t-1}^\alpha + w_t \tag{10}$$

where  $u_t \sim SL(0, \tau, \alpha)$ ,  $\Xi$  is the variance of residuals  $u_t$  and  $w_t \sim N(0, \Omega)$ .

# Algorithm estimation procedure



## Estimation procedure

- We reformulate the problem:

$$\begin{aligned}\ln L_\alpha(\Psi; y, x) &\propto -n \ln \tau - \frac{1}{\tau} \sum_{t=1}^n u_t \times [\alpha - \mathbf{1}_{(-\infty, 0)}(u_t)] \\ &= -n \ln \tau - \frac{1}{\tau} \left[ \alpha \sum_{t=1}^n \{u_t | u_t \geq 0\} + (\alpha - 1) \sum_{t=1}^n \{u_t | u_t < 0\} \right] \\ &= -n \ln \tau - \frac{1}{\tau} \left[ \alpha \sum_{t=1}^n \{u_t | u_t \geq 0\} + (1 - \alpha) \sum_{t=1}^n \{-u_t | u_t < 0\} \right] \\ &= -n \ln \tau - \frac{\alpha}{\tau} \sum_{t=1}^n v_t^1 - \frac{(1 - \alpha)}{\tau} \sum_{t=1}^n v_t^2\end{aligned}\tag{11}$$

as non-linear optimization problem with linear constraints

- We further represent  $v_t^1 := \max\{u_t, 0\}$ ,  $v_t^2 := \max\{-u_t, 0\}$  by the following linear restrictions:

$$\begin{aligned}\min \quad & n \ln \tau + \frac{\alpha}{\tau} \sum_{t=1}^n v_t^1 + \frac{(1 - \alpha)}{\tau} \sum_{t=1}^n v_t^2 \\ \text{s.t.} \quad & v_t^1 \geq y_t - (\beta_{t|t-1})' x_t, & t = 1, \dots, n \\ & v_t^2 \geq (\beta_{t|t-1})' x_t - y_t, & t = 1, \dots, n \\ & v_t^1, v_t^2 \geq 0, & t = 1, \dots, n\end{aligned}\tag{12}$$

## The problem of “embedded betas”

- Based on the new reformulation:

$$\begin{aligned} \min \quad & n \ln \tau + \frac{\alpha}{\tau} \sum_{t=1}^n v_t^1 + \frac{(1-\alpha)}{\tau} \sum_{t=1}^n v_t^2 \\ \text{s.t.} \quad & -v_t^1 + v_t^2 + y_t - (\beta_{t|t-1})' x_t = 0, \quad t = 1, \dots, n \\ & v_t^1, v_t^2 \geq 0, \quad t = 1, \dots, n \end{aligned} \tag{13}$$

we define

$$\begin{aligned} f(\tilde{\Psi}) &:= n \ln \tau + \frac{\alpha}{\tau} \sum_{t=1}^n v_t^1 + \frac{(1-\alpha)}{\tau} \sum_{t=1}^n v_t^2 \\ c_t(\tilde{\Psi}) &:= -v_t^1 + v_t^2 + y_t - \left( \beta_{t|t}(\tilde{\Psi}) \right)' x_t \end{aligned}$$

which gives

$$\min_{\tilde{\Psi}} f(\tilde{\Psi}) \quad \text{s.t.} \quad c_t(\tilde{\Psi}) = 0; \quad v_t^1, v_t^2 \geq 0, \quad t = 1, \dots, n \tag{14}$$

- By integrating the constraints in the objective we can solve the problem by *Augmented Lagrangian Method* (we follow Nocedal/Wright, 2006, Chapter. 17). The relaxed problem is:

$$\Phi(\tilde{\Psi}, \Lambda; \mu) := f(\tilde{\Psi}) + \frac{\mu_k}{2} \sum_{t=1}^n \left( c_t(\tilde{\Psi}) \right)^2 - \sum_{t=1}^n \lambda_t c_t(\tilde{\Psi}) \tag{15}$$

## Case study: Fundamental model for quantiles of electricity prices

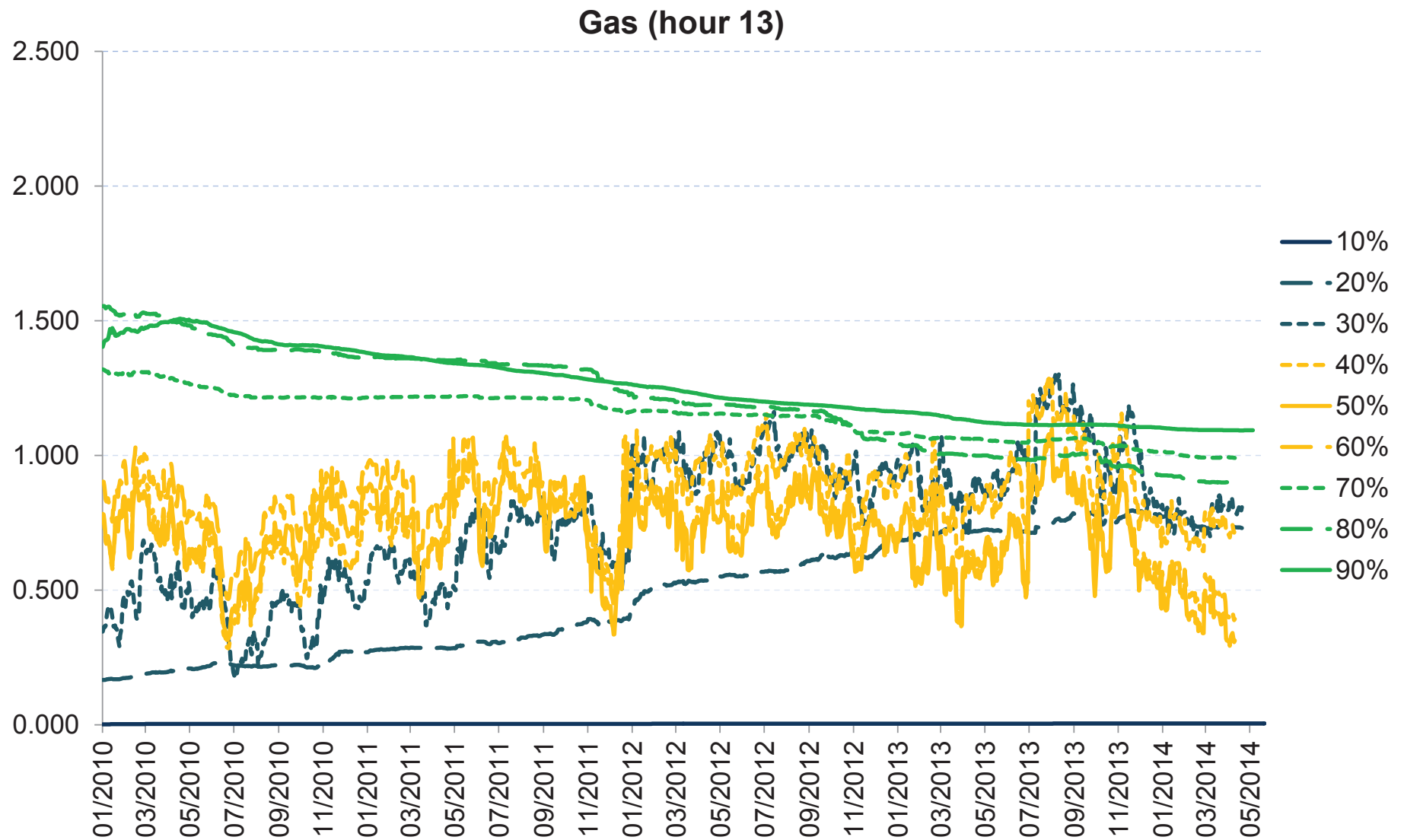
- Electricity day-ahead prices in Germany are explained by the following fundamental variables: demand, reserve margin, coal, gas, oil, CO2 certificate prices, wind, PV.
- 24 data sets have been constructed for each hour of a day between 01/01/2010–31/05/2014 to take into account the typical intra-day seasonality of electricity prices.
- We assume that renewable energies have a direct and indirect effect on electricity prices:

$$P_t^\alpha = \beta_{0t}^\alpha + \beta_{1t}^\alpha P_{t-1} + \beta_{2t}^\alpha Wind_t + \beta_{3t}^\alpha PV_t + \beta_{4t}^\alpha Demand_t + \beta_{5t}^\alpha Coal_t + \beta_{6t}^\alpha Gas_t + \beta_{7t}^\alpha Oil_t + \beta_{8t}^\alpha CO2_t + \beta_{9t}^\alpha Reserve_t \quad (16)$$

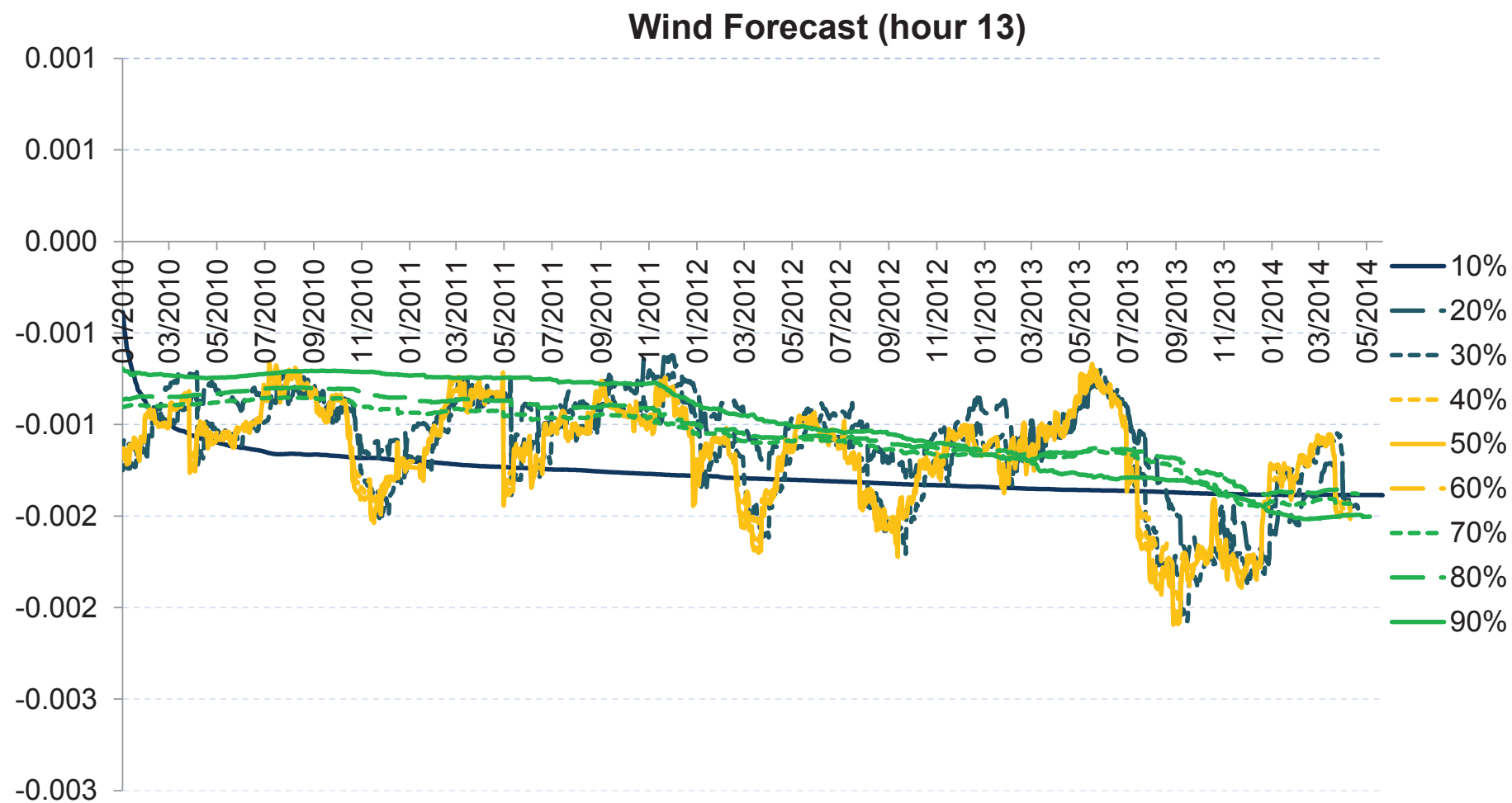
$$\beta_{jt}^\alpha = c + D\beta_{j(t-1)}^\alpha + \varepsilon_j^\alpha Wind_t + \gamma_j^\alpha PV_t + \phi_j^\alpha Reserve_t + w_t \quad \text{for } j = \{1, \dots, 9\}; \quad (17)$$

where for  $j = \{1, \dots, 4; 7, 8, 9\}$   $\varepsilon_j$ ,  $\gamma_j$  and  $\phi_j$  take the value 0.

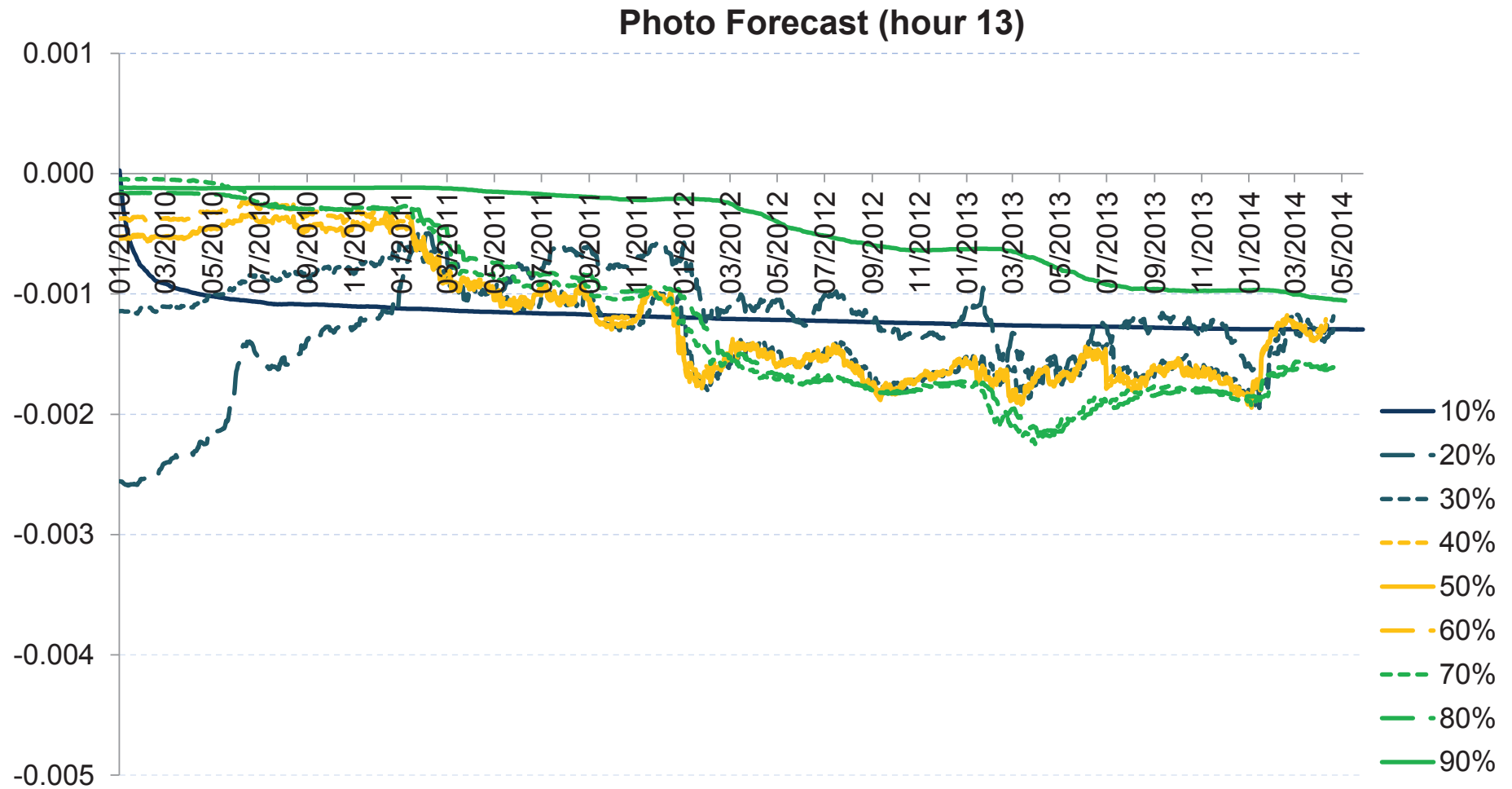
## Results hour 13. Time varying betas for gas



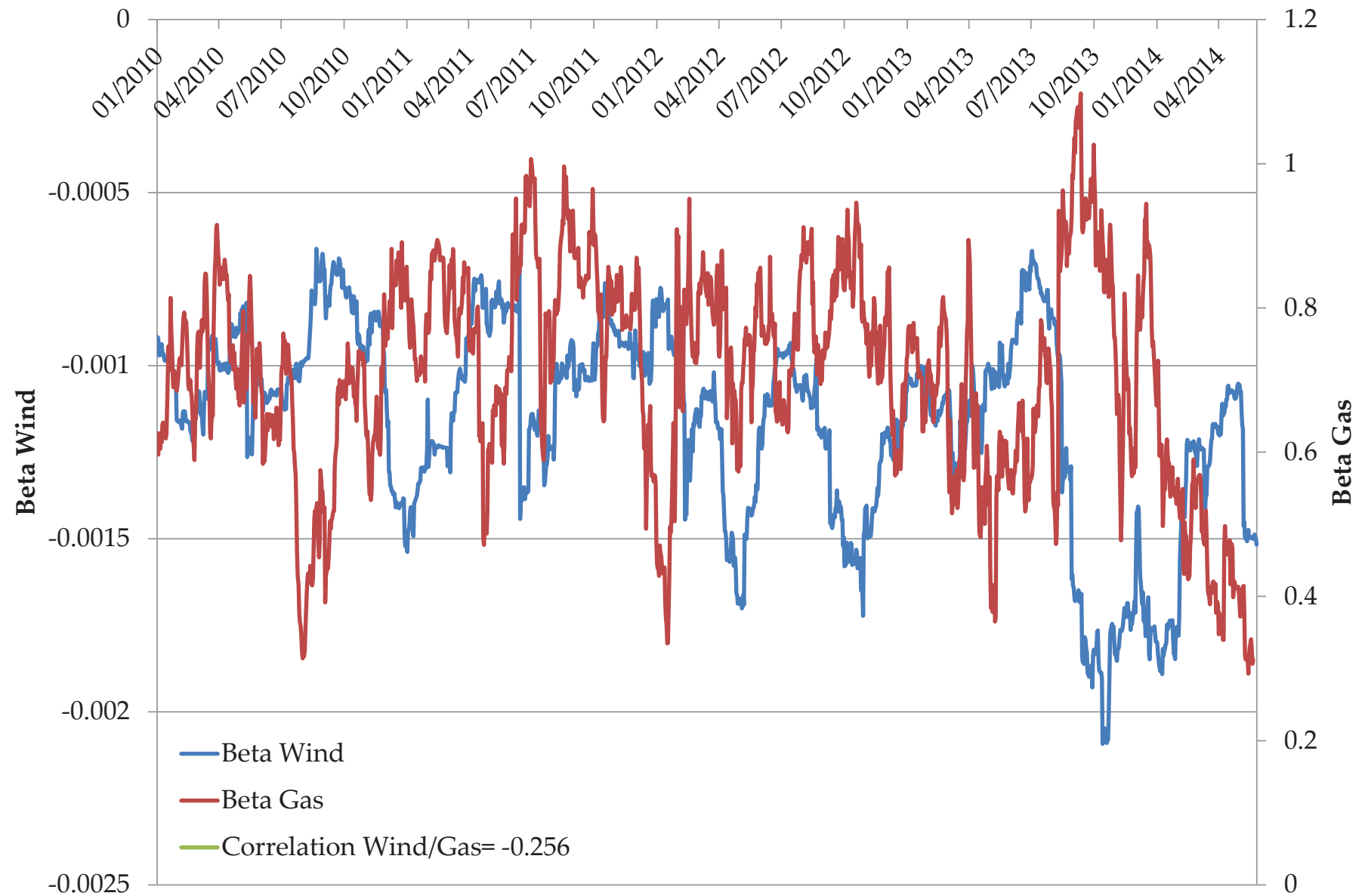
# Results hour 13. Time varying betas for wind



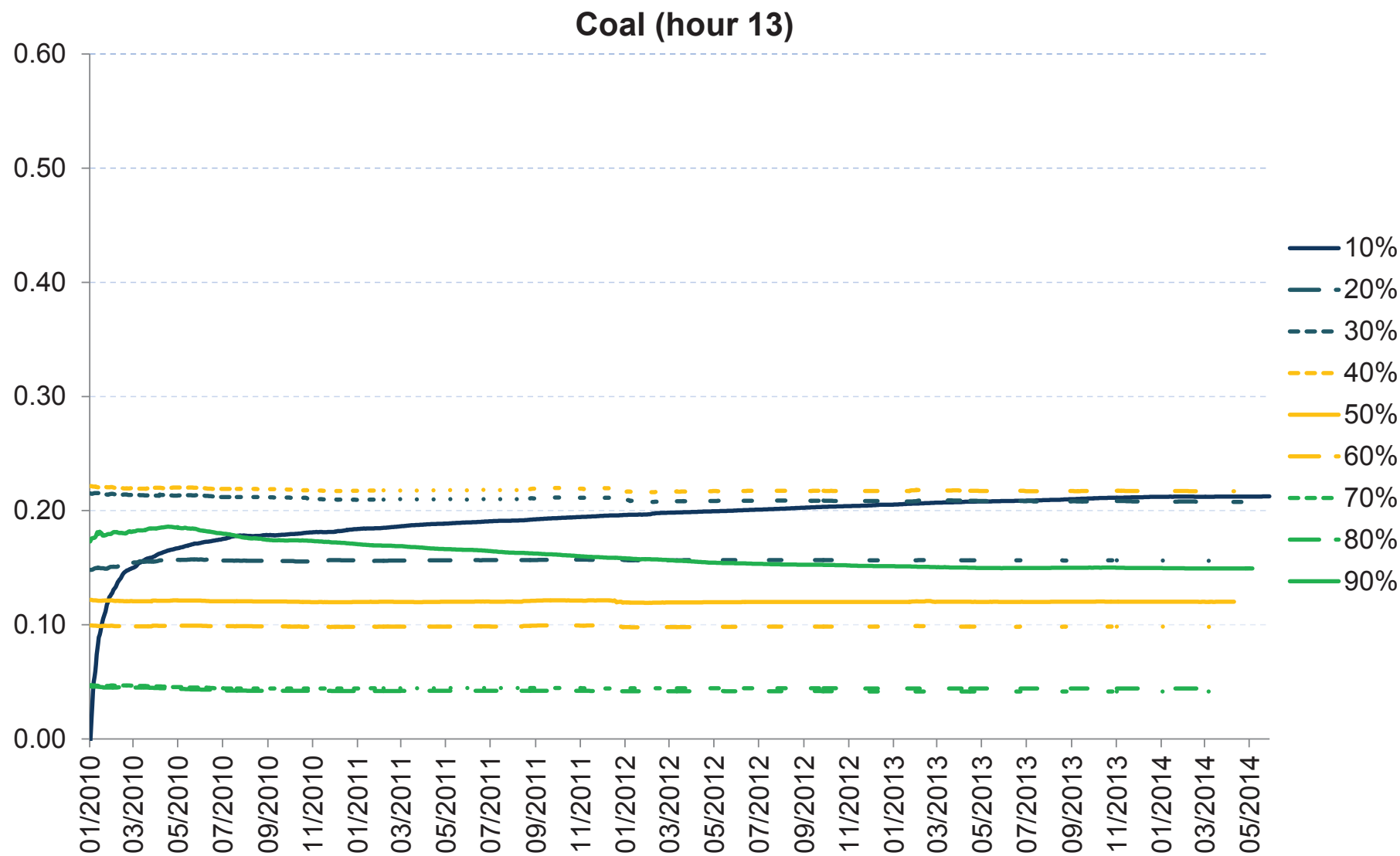
## Results hour 13. Time varying betas for PV



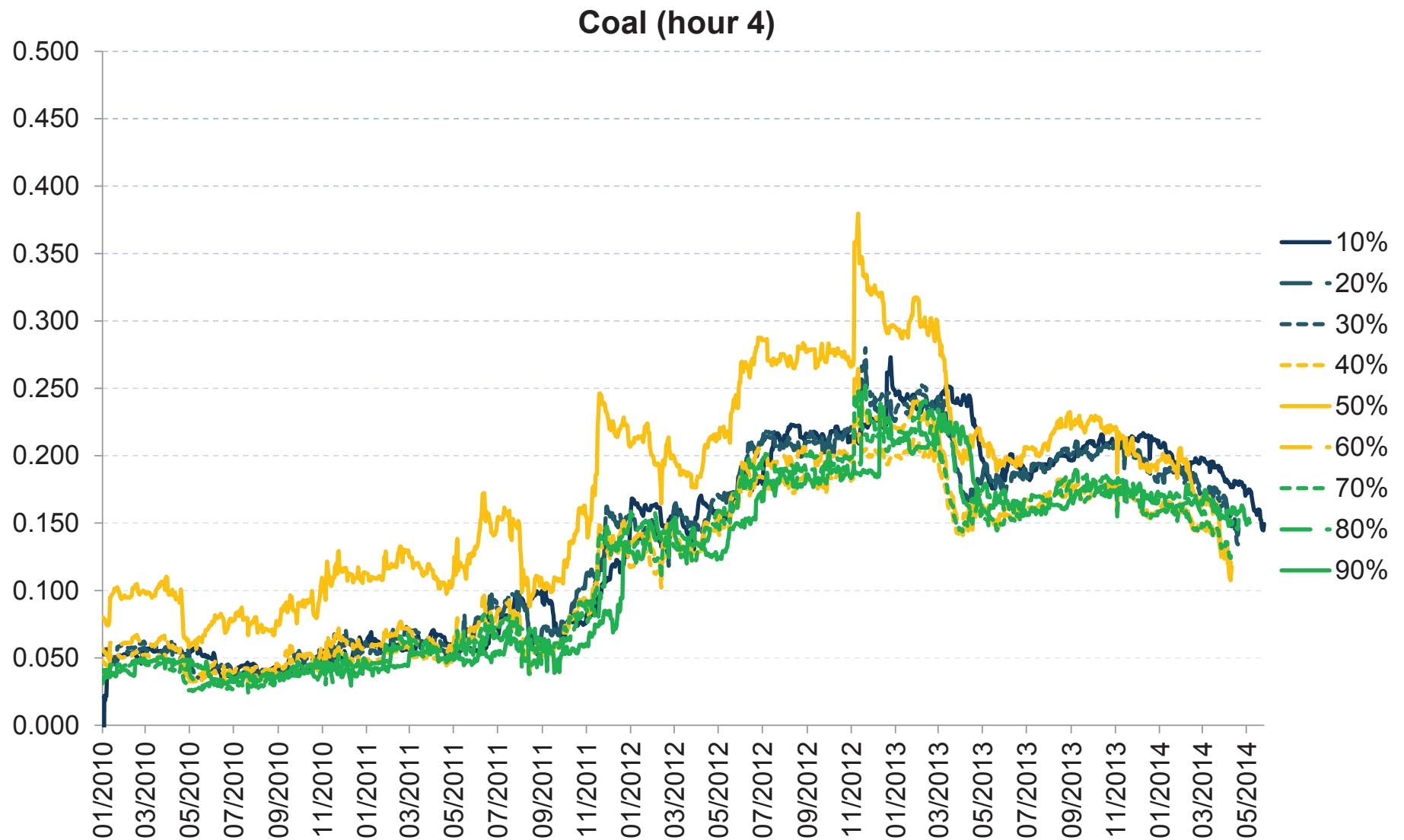
## Correlations betas wind and gas



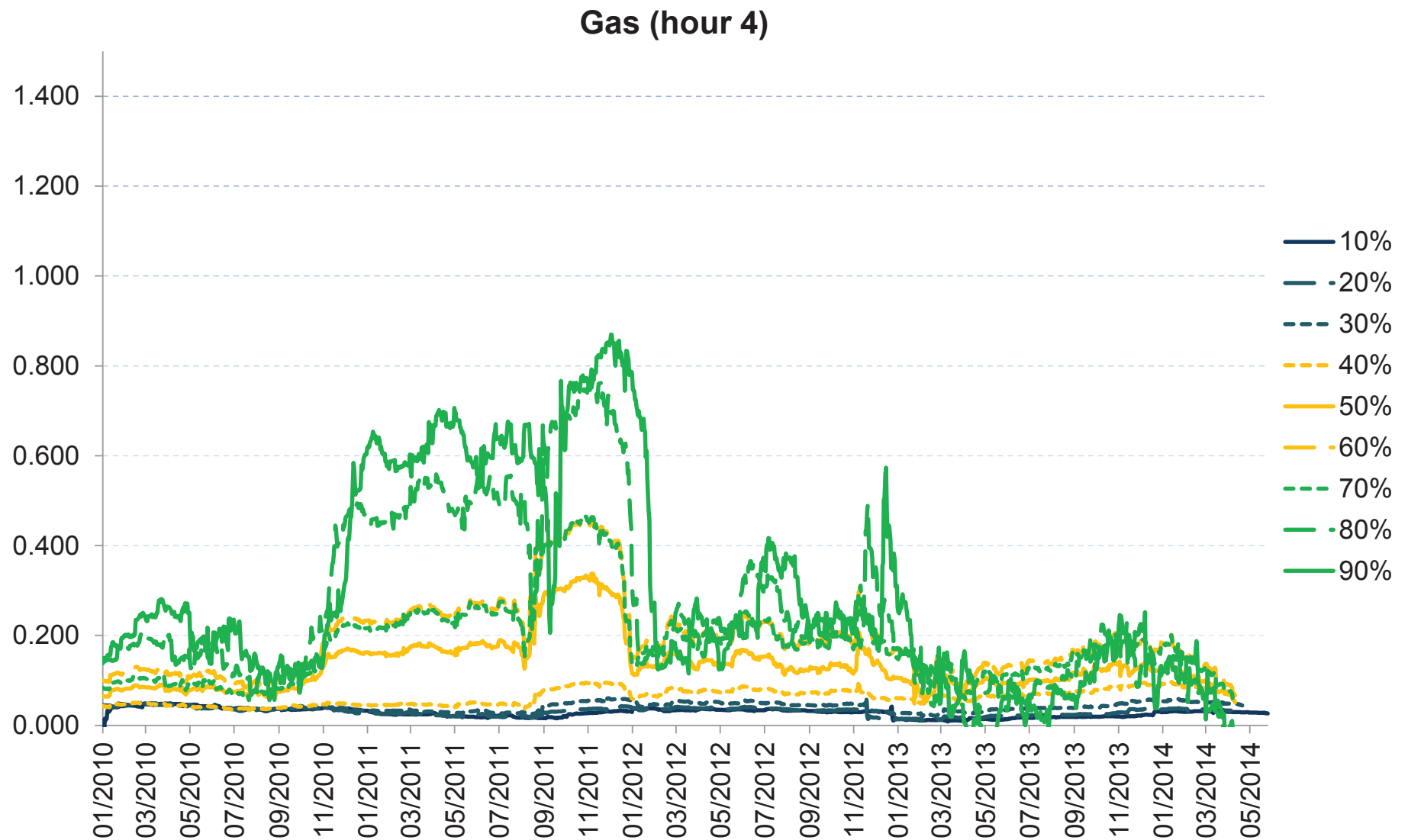
# Results hour 13. Time varying betas for coal



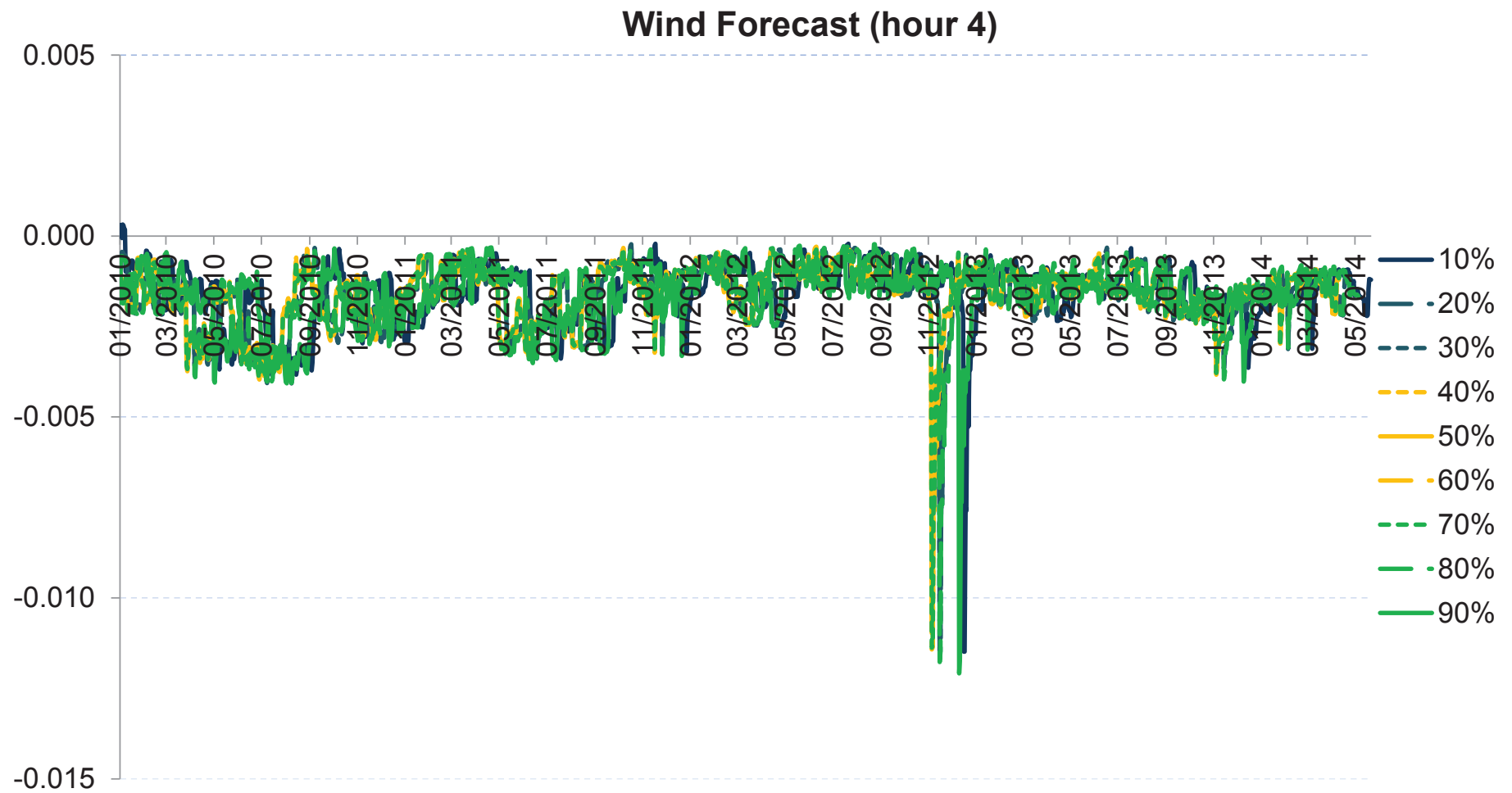
## Results hour 4. Time-varying betas for coal



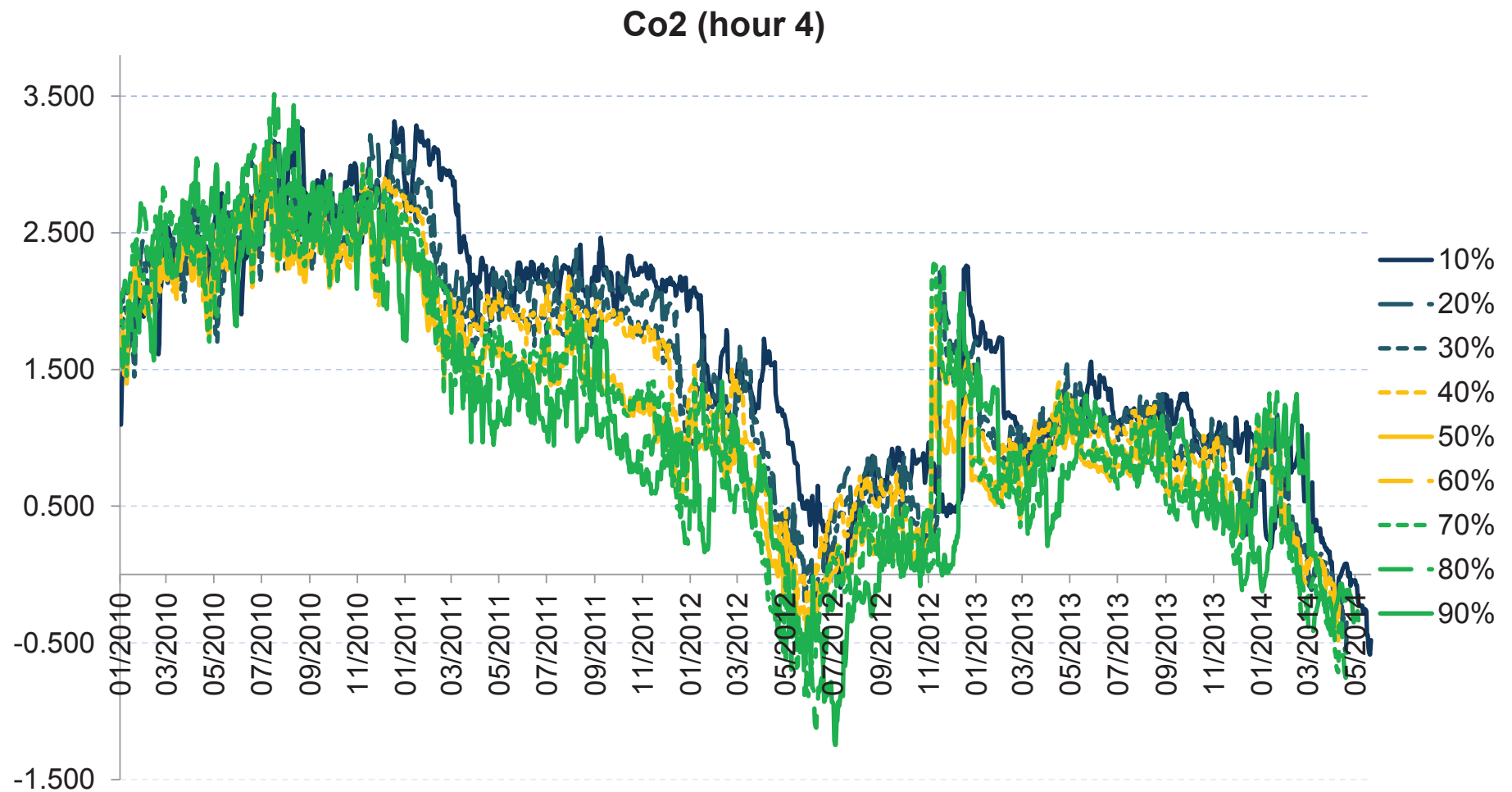
## Results hour 4. Time-varying betas for gas



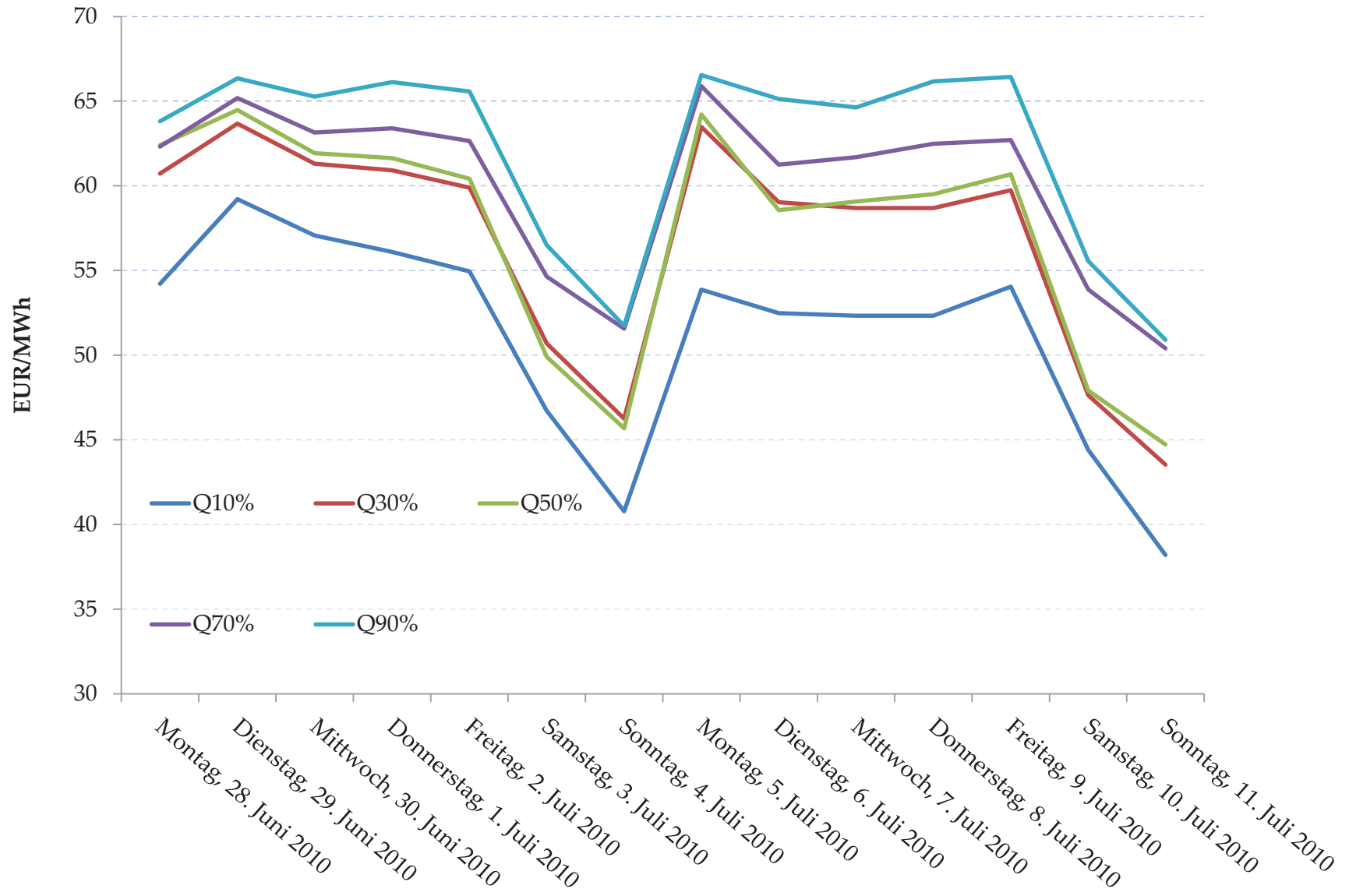
## Results hour 4. Time-varying betas for wind



## Results hour 4. Time-varying betas for carbon prices



## Fitted price quantiles



## Indirect effect of renewables on electricity prices

| Dependent variable: $\beta_{t,Gas}$                             |                       |                      |                      |                        |                        |
|-----------------------------------------------------------------|-----------------------|----------------------|----------------------|------------------------|------------------------|
| Method: Least Squares                                           |                       |                      |                      |                        |                        |
|                                                                 | Q 10%                 | Q 30%                | Q 50%                | Q 70%                  | Q 90%                  |
| C                                                               | 0.0000<br>(23.0451)   | 0.0103<br>(4.1064)   | 0.0370<br>(7.1436)   | 0.2507<br>(21.0069)    | 0.2131<br>(17.5405)    |
| $D_{Gas}$                                                       | 0.9888<br>(1795.4470) | 0.9872<br>(277.3021) | 0.9561<br>(149.5016) | 0.8012<br>(84.3599)    | 0.8510<br>(99.6584)    |
| $\varepsilon_{Wind}^{\alpha} * 10^8$                            | 0.0018<br>(2.8295)    | -1.4500<br>(-0.9128) | -3.8600<br>(-1.8863) | -10.0000<br>(-6.1723)  | -11.8000<br>(-5.9069)  |
| $\gamma_{PV}^{\alpha} * 10^8$                                   | 0.0037<br>(5.6391)    | 2.3300<br>(1.4540)   | -2.3200<br>(-1.3332) | -21.7000<br>(-13.3793) | -25.4000<br>(-12.2649) |
| $\phi_{Reserve}^{\alpha} * 10^8$                                | 0.0011<br>(2.8067)    | 1.7500<br>(1.7686)   | 2.4600<br>(1.9042)   | -3.6200<br>(-3.5875)   | -3.1600<br>(-2.5796)   |
| $R^2$                                                           | 0.9510                | 0.9863               | 0.9338               | 0.8889                 | 0.9229                 |
| Test for omitted variables (Wind, PV, Reserve Margin), p-values |                       |                      |                      |                        |                        |
| F-Statistik                                                     | 0.0000                | 0.1225               | 0.0279               | 0.0000                 | 0.0000                 |
| Likelihood Ratio                                                | 0.0000                | 0.1216               | 0.0276               | 0.0000                 | 0.0000                 |

## Conclusion

- We propose a **novel approach for quantile regressions with time-varying coefficients based on a fully parametric approach**.
  - Time-varying coefficients are recursively filtered with Kalman Filter and model parameters are estimated by maximum likelihood.
  - To deal with the non-differentiability of the likelihood function, we formulated the problem as a non-linear optimisation with constraints.
  - We further moved the constraints into the objective and solved a relaxed problem with the Augmented Lagrangian Method
- We showed the importance of modelling the time-varying feature in the context of a **fundamental model for quantiles of electricity prices**
  - We found that marginal effects of market fundamentals on electricity prices differ accross price quantiles and between different hours within one day.
  - A good understanding of the risk drivers of electricity prices is important for an optimal production planning, for the fine tuning of bids in electricity trading and generally for a lower exposure to risk.

## References

- Gerlach, R., Chen, C. W. S. & Chan, N. C. Y. (2011), “Bayesian time-varying quantile forecasting for value-at-risk in financial markets”, *Journal of Business & Economic Statistics* 29, 481–492.
- Honda, T. (2004), “Quantile regression in varying coefficient models”, *Journal of Statistical Planning and Inference* 121, 113–125.
- Kim, M. O. (2007), “Quantile regression with varying coefficients”, *The Annals of Statistics* 35(1), 92–108.
- Rossi, G. D. & Harvey, A. (2006), “Time-varying quantiles”, Working paper University of Cambridge CWPE 0649.