A Generalized Framework for Folding Codes and Applications to Proofs of Proximity

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Institute of Computer Science

$P(x, w)$ $V(x)$

$\mathtt{Com}$ →

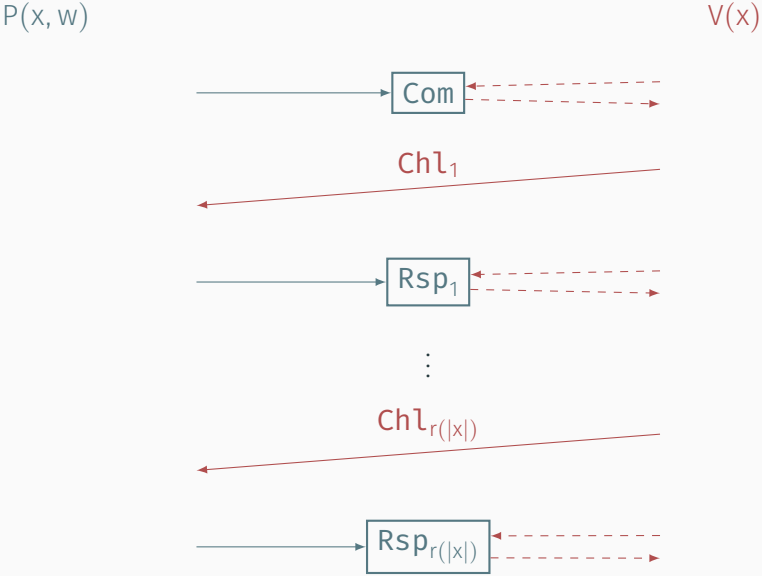
← $\mathtt{Chl}_1$

$\mathtt{Rsp}_1$ →

⋮

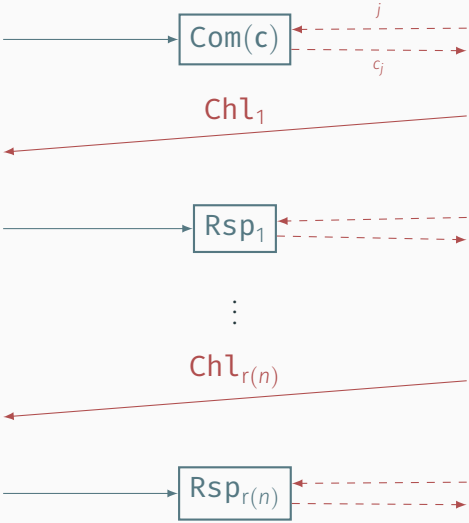
← $\mathtt{Chl}_{r(|x|)}$

$\mathtt{Rsp}_{r(|x|)}$ →


P(x, w)
V(x)
Com
Chl1
Rsp1
Chlr(|x|)
Rspr(|x|)

$\mathsf{P}((C \subseteq \mathbb{F}_q^n, \mathtt{Com}(\mathbf{c})), \mathbf{c} \in \mathbb{F}_q^n)$

$\mathsf{V}((C, \mathtt{Com}(\mathbf{c})))$



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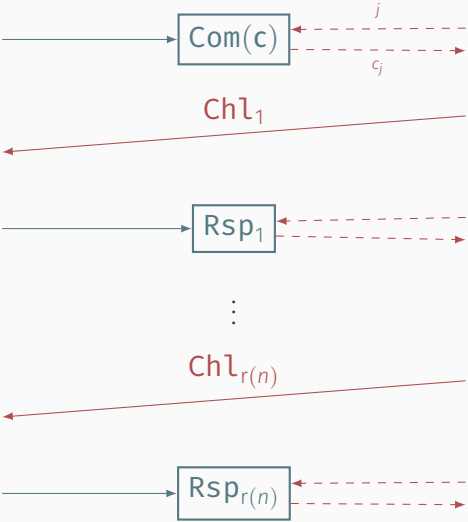

Accept (1) if $\mathbf{c} \in C$, otherwise Reject (0) with pb. $\geq 1 - \varepsilon(\frac{d(\mathbf{c},C)}{n})$.

Interactive Oracle Proofs (IOP) [Ben-Sasson, Chiesa, and Spooner 2016]

IP

An *interactive proof* system for a relation $\mathcal{R}$ with round complexity $r : \{0,1\}^* \to \mathbb{N}$ and soundness $\varepsilon : \{0,1\}^* \to [0,1]$ is a tuple (P, V), where P, V are probabilistic algorithms that satisfies the following properties.

- **Completeness**: For every $(x, w) \in \mathcal{R}$, $\langle P(x, w), V(w)\rangle$ is a transcript of $r(|x|)$ rounds and
$$\Pr\left[\langle P(x, w), V(x)\rangle \Rightarrow \mathbf{1}\right] = 1$$
- **Soundness**: For every $x \notin \mathcal{L}(\mathcal{R})$ and every $\tilde{P}$, $\langle \tilde{P}(x), V(w)\rangle$ is a transcript of $r(|x|)$ rounds and
$$\Pr\left[\langle \tilde{P}(x), V(x)\rangle \Rightarrow \mathbf{1}\right] \leq \varepsilon(x).$$

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$$\Pr\left[\langle \tilde{P}(x), V(x)\rangle \Rightarrow \mathbf{1}\right] \leq \varepsilon(x).$$
- For every $x \in \{0,1\}^*$ and $0 \leq i \leq r(x)$, the i-th response of P is of the form $\mathtt{Rsp}_i = P(w, \{(j, \mathtt{Chl}_j)\}_{j<i}, \mathtt{state}_i)$ and the i-th challenge of V is of the form $\mathtt{Chl}_i = V^{\mathcal{O}_i(\mathtt{Rsp}_0,\ldots,\mathtt{Rsp}_i)}(x, \{(j, \mathtt{Chl}_j)\}_{j<i}, \mathtt{state}'_i)$.

What do we want? Why?

rounds complexity:	$\mathrm{r}(n)$
proof length:	$\mathrm{l}(n)$
query complexity:	$\mathrm{q}(n)$
prover complexity:	$\mathrm{t_P}(n)$
verifier complexity:	$\mathrm{t_V}(n)$

What do we want? Why?

rounds complexity:	$\mathsf{r}(n)$	$\leq \log(n)$
proof length:	$\mathsf{l}(n)$	$\leq O(n)$
query complexity:	$\mathsf{q}(n)$	$< O(n)$
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Goal

Construct IOPP protocols with *sublinear* complexity (for the verifier) for *broad* families of codes.

Applications to...

- Succint Arguments: (zk-)SNARGs and (zk-)SNARKs,
- Non-Interactive Zero-Knowledge: compilers from SNARGs to (zk-)SNARGs and NIZKs,
- Integrity Verification: delegating computation to a non-trusted server over a network...

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How do folding protocols work?

From the geometric intuition...

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...to (linear) algebra and representation theory.

Preliminaries

How do folding protocols work?

The typical example: FRI [Ben-Sasson, Bentov, et al. 2017]

Let $\nu > \mathrm{r} \geq 0$, q an odd prime power such that $2^\nu | q-1$, $\alpha \in \mathbb{F}_q^\times$ with $\mathrm{ord}(\alpha) = 2^\nu$.
Set $n = 2^\nu$, $k = 2^{\mathrm{r}}$, $\mathcal{P} = \{\alpha^j \mid j \in \{0, \ldots, n-1\}\}$, and $C = \mathrm{RS}[\mathcal{P}, k]$.

For $f(X) = \sum_{i=0}^{m} f_i X^i \in \mathbb{F}_q[X]$, do we have $\deg f = m < k$?

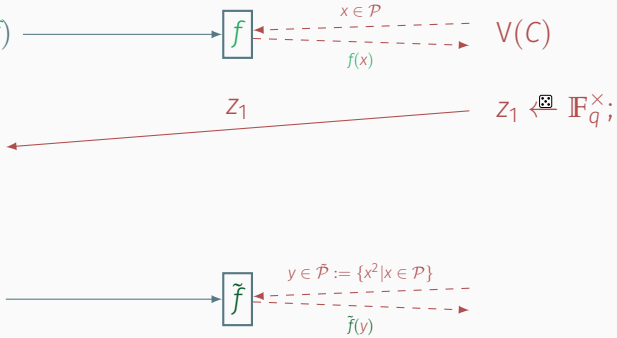
The typical example: FRI [Ben-Sasson, Bentov, et al. 2017]

$$f(X) = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} f_{2i} X^{2i} + \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} f_{2i+1} X^{2i+1}$$

$$= \underbrace{\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} f_{2i} (X^2)^i}_{g(X^2)} + X \underbrace{\sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} f_{2i+1} (X^2)^i}_{h(X^2)};$$

$$\tilde{f}(Y) = \mathsf{Fold}[f, z_1](Y)$$

$$= g(Y) + z_1 h(Y);$$



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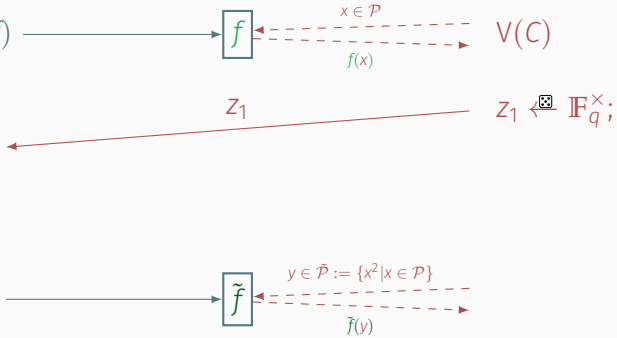
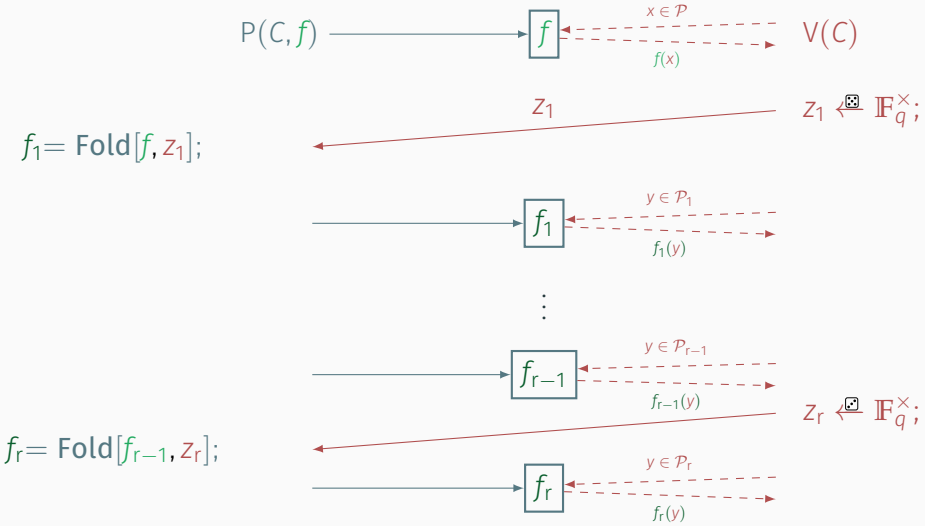


If $f \in C = \mathrm{RS}[\mathcal{P}, k]$, then $\tilde{f} = \mathbf{Fold}[f, z_1] \in \tilde{C} = \mathrm{RS}[\tilde{\mathcal{P}}, k/2]$.

The typical example: FRI [Ben-Sasson, Bentov, et al. 2017]



The typical example: FRI [Ben-Sasson, Bentov, et al. 2017]

Round consistency check for verification: For $y \neq 0$, verify whether

$$2 \cdot f_{i+1}(y^2) = (1 + z_1 y^{-1}) f_i(y) + (1 + z_i y^{-1}) f_i(-y),$$

which costs 1 query to f_{i+1} plus 2 queries to f_i.

Folding AG-Codes (intuition) [Bordage et al. 2020]

Suppose $N = \prod_{j=1}^{r-1} p_j$, let $\Gamma_0 = \Gamma \cong \mathbb{Z}/N\mathbb{Z}$, Γ_{i+1} such that $\Gamma_{i+1}/\Gamma_i \cong \mathbb{Z}/p_i\mathbb{Z}$, denote $\mathcal{X}_i : y^{\prod_{j=i}^{r-1} p_j} = \prod_{l=1}^{m}(x - a_l)$, $P_\infty^{(i)}$ the place at infinity on $\mathcal{X}_i$, $\pi_{i+1} : \mathcal{X}_i \to \mathcal{X}_{i+1}$.

[Proposition 29, Bordage et al. 2020]

Let $i \geq 0$. Set $D_i = d_i P_\infty^{(i)}$ for some integer d_i. Then $L_{\mathcal{X}_i}(D_i) = \bigoplus_{j=0}^{p_i-1} \nu_i^j \pi_i^*(L_{\mathcal{X}_{i-1}}(E_{i,j}))$ with $E_{i,j} = \left\lfloor \frac{1}{p_i} \pi_{i*}(D_i - j \cdot \mathrm{Div}_{\mathcal{X}_i}((\nu_i))) \right\rfloor$, for $0 \leq j \leq p_i - 1$, and some well chosen balancing function $\nu_i \in \mathbb{F}_q(\mathcal{X}_i)$ "compatible" with D_i.

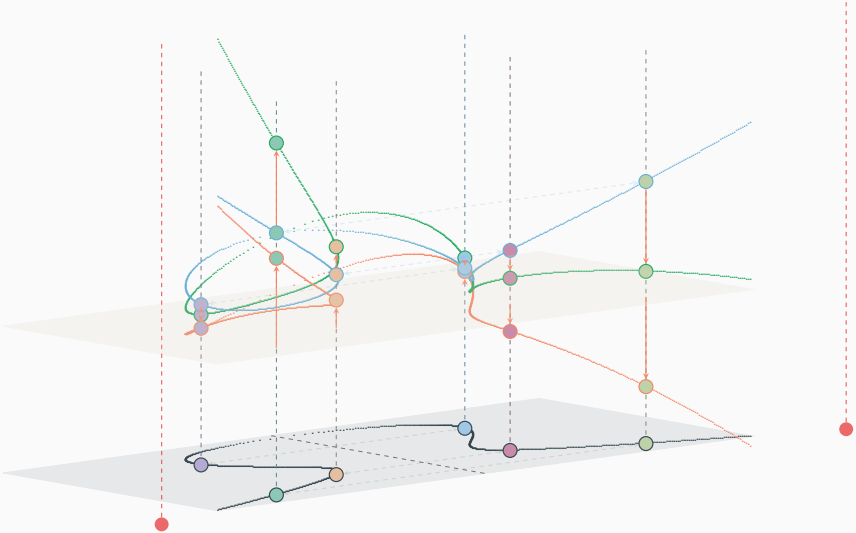
Corollary for real-world purposes

If $C = C(\mathcal{X}, \mathcal{P}, D)$ with $\mathcal{P} \subset \mathcal{X}$ a disjoint union of Γ-orbits of maximum size and D a Γ-invariant divisor, then for any $\mathbf{z} = (z_1, z_2) \in \mathbb{F}_q$, the operator $\mathbf{Fold}[f, \mathbf{z}] = \sum_{j=0}^{p_i-1} z_1^j f_j + \sum_{j=0}^{p_i-1} z_2^{j+1} \nu_{i+1,j} f_j$ induces a projection from $C(\mathcal{X}_i, \mathcal{P}_i, D_i)$ to $C(\mathcal{X}_i/\Gamma_i, \pi_{i+1}(\mathcal{P}_i), E_i)$, resulting in an *efficient* IOPP protocol.

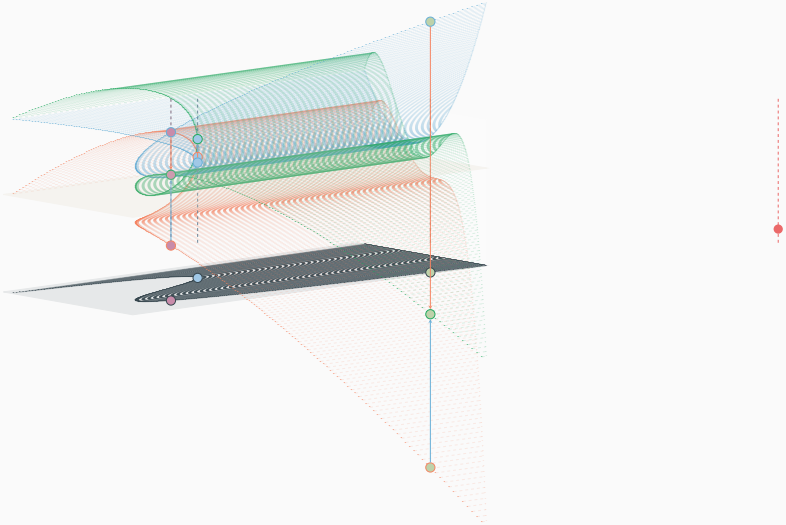
Preliminaries

From the geometric intuition...

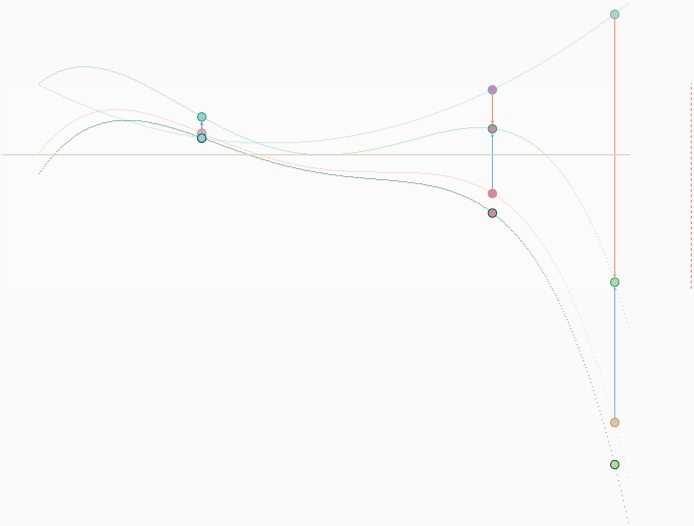
Geometric Intuition: Starting Curve



Geometric Intuition: Quotient Curve



Geometric Intuition: New Codeword



Generalized Framework

...to (linear) algebra and representation theory.

Staring at this generator matrix of an $[36, 14]_{37}$-code

$$\begin{pmatrix}
9 & 9 & 9 & 9 & 26 & 26 & 26 & 26 & 7 & 7 & 7 & 7 & 1 & 1 & 1 & 1 & 26 & 26 & 26 & 26 & 16 & 16 & 16 & 16 & 34 & 34 & 34 & 34 & 9 & 9 & 9 & 9 & 16 & 16 & 16 & 16 \\
7 & 7 & 7 & 7 & 17 & 17 & 17 & 17 & 13 & 13 & 13 & 13 & 4 & 4 & 4 & 4 & 35 & 35 & 35 & 35 & 32 & 32 & 32 & 32 & 6 & 6 & 6 & 6 & 8 & 8 & 8 & 8 & 34 & 34 & 34 & 34 \\
26 & 26 & 26 & 26 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 3 & 16 & 16 & 16 & 16 & 3 & 3 & 3 & 3 & 27 & 27 & 27 & 27 & 25 & 25 & 25 & 25 & 3 & 3 & 3 & 3 & 26 & 26 & 26 & 26 \\
12 & 12 & 12 & 12 & 14 & 14 & 14 & 14 & 32 & 32 & 32 & 32 & 27 & 27 & 27 & 27 & 14 & 14 & 14 & 14 & 17 & 17 & 17 & 17 & 24 & 24 & 24 & 24 & 15 & 15 & 15 & 15 & 9 & 9 & 9 & 9 \\
34 & 34 & 34 & 34 & 12 & 12 & 12 & 12 & 33 & 33 & 33 & 33 & 34 & 34 & 34 & 34 & 16 & 16 & 16 & 16 & 34 & 34 & 34 & 34 & 26 & 26 & 26 & 26 & 1 & 1 & 1 & 1 & 33 & 33 & 33 & 33 \\
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35 & 25 & 2 & 12 & 22 & 21 & 15 & 16 & 8 & 11 & 29 & 26 & 4 & 24 & 33 & 13 & 17 & 28 & 20 & 9 & 16 & 22 & 21 & 15 & 17 & 28 & 20 & 9 & 3 & 18 & 34 & 19 & 17 & 28 & 20 & 9 \\
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15 & 21 & 22 & 16 & 26 & 29 & 11 & 8 & 34 & 18 & 3 & 19 & 1 & 31 & 36 & 6 & 26 & 29 & 11 & 8 & 2 & 25 & 35 & 12 & 33 & 24 & 4 & 13 & 15 & 21 & 22 & 16 & 2 & 25 & 35 & 12 \\
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\end{pmatrix}$$

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$$
\begin{pmatrix}
9 & 9 & 9 & 9 & 26 & 26 & 26 & 26 & 7 & 7 & 7 & 7 & 1 & 1 & 1 & 1 & 26 & 26 & 26 & 26 & 16 & 16 & 16 & 16 & 34 & 34 & 34 & 34 & 9 & 9 & 9 & 9 & 16 & 16 & 16 & 16 \\
7 & 7 & 7 & 7 & 17 & 17 & 17 & 17 & 13 & 13 & 13 & 13 & 4 & 4 & 4 & 4 & 35 & 35 & 35 & 35 & 32 & 32 & 32 & 32 & 6 & 6 & 6 & 6 & 8 & 8 & 8 & 8 & 34 & 34 & 34 & 34 \\
26 & 26 & 26 & 26 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 3 & 16 & 16 & 16 & 16 & 3 & 3 & 3 & 3 & 27 & 27 & 27 & 27 & 25 & 25 & 25 & 25 & 3 & 3 & 3 & 3 & 26 & 26 & 26 & 26 \\
12 & 12 & 12 & 12 & 14 & 14 & 14 & 14 & 32 & 32 & 32 & 32 & 27 & 27 & 27 & 27 & 14 & 14 & 14 & 14 & 17 & 17 & 17 & 17 & 24 & 24 & 24 & 24 & 15 & 15 & 15 & 15 & 9 & 9 & 9 & 9 \\
34 & 34 & 34 & 34 & 12 & 12 & 12 & 12 & 33 & 33 & 33 & 33 & 34 & 34 & 34 & 34 & 16 & 16 & 16 & 16 & 34 & 34 & 34 & 34 & 26 & 26 & 26 & 26 & 1 & 1 & 1 & 1 & 33 & 33 & 33 & 33 \\
8 & 11 & 29 & 26 & 1 & 6 & 36 & 31 & 10 & 23 & 27 & 14 & 1 & 6 & 36 & 31 & 1 & 6 & 36 & 31 & 8 & 11 & 29 & 26 & 10 & 23 & 27 & 14 & 8 & 11 & 29 & 26 & 8 & 11 & 29 & 26 \\
35 & 25 & 2 & 12 & 22 & 21 & 15 & 16 & 8 & 11 & 29 & 26 & 4 & 24 & 33 & 13 & 17 & 28 & 20 & 9 & 16 & 22 & 21 & 15 & 17 & 28 & 20 & 9 & 3 & 18 & 34 & 19 & 17 & 28 & 20 & 9 \\
19 & 3 & 18 & 34 & 3 & 18 & 34 & 19 & 36 & 31 & 1 & 6 & 16 & 22 & 21 & 15 & 30 & 32 & 7 & 5 & 32 & 7 & 5 & 30 & 3 & 18 & 34 & 19 & 15 & 16 & 22 & 21 & 13 & 4 & 24 & 33 \\
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27 & 10 & 27 & 10 & 35 & 2 & 35 & 2 & 22 & 15 & 22 & 15 & 4 & 33 & 4 & 33 & 22 & 15 & 22 & 15 & 8 & 29 & 8 & 29 & 5 & 32 & 5 & 32 & 15 & 22 & 15 & 22 & 27 & 10 & 27 & 10 \\
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24 & 4 & 13 & 33 & 17 & 9 & 20 & 28 & 5 & 7 & 32 & 30 & 4 & 13 & 33 & 24 & 35 & 12 & 2 & 25 & 4 & 13 & 33 & 24 & 8 & 26 & 29 & 11 & 1 & 31 & 36 & 6 & 32 & 30 & 5 & 7
\end{pmatrix}
$$

Staring at this generator matrix of an $[36, 14]_{37}$-code

9	9	9	9	26	26	26	26	7	7	7	7	1	1	1	1	26	26	26	26	16	16	16	16	34	34	34	34	9	9	9	9	16	16	16	16
7	7	7	7	17	17	17	17	13	13	13	13	4	4	4	4	35	35	35	35	32	32	32	32	6	6	6	6	8	8	8	8	34	34	34	34
26	26	26	26	4	4	4	4	3	3	3	3	16	16	16	16	3	3	3	3	27	27	27	27	25	25	25	25	3	3	3	3	26	26	26	26
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Staring at this generator matrix of an $[36, 14]_{37}$-code

9	9	9	9	26	26	26	26	7	7	7	7	1	1	1	1	26	26	26	26	16	16	16	16	34	34	34	34	9	9	9	9	16	16	16	16
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35	25	2	12	22	21	15	16	8	11	29	26	4	24	33	13	17	28	20	9	16	22	21	15	17	28	20	9	3	18	34	19	17	28	20	9
19	3	18	34	3	18	34	19	36	31	1	6	16	22	21	15	30	32	7	5	32	7	5	30	3	18	34	19	15	16	22	21	13	4	24	33
23	27	14	10	29	26	8	11	14	10	23	27	27	14	10	23	29	26	8	11	27	14	10	23	31	1	6	36	1	6	36	31	23	27	14	10
3	34	3	34	10	27	10	27	9	28	9	28	1	36	1	36	10	27	10	27	4	33	4	33	16	21	16	21	3	34	3	34	4	33	4	33
27	10	27	10	35	2	35	2	22	15	22	15	4	33	4	33	22	15	22	15	8	29	8	29	5	32	5	32	15	22	15	22	27	10	27	10
21	16	21	16	30	7	30	7	25	12	25	12	16	21	16	21	4	33	4	33	16	21	16	21	27	10	27	10	1	36	1	36	25	12	25	12
15	21	22	16	26	29	11	8	34	18	3	19	1	31	36	6	26	29	11	8	2	25	35	12	33	24	4	13	15	21	22	16	2	25	35	12
24	4	13	33	17	9	20	28	5	7	32	30	4	13	33	24	35	12	2	25	4	13	33	24	8	26	29	11	1	31	36	6	32	30	5	7

Staring at this generator matrix of an $[36, 14]_{37}$-code

9	9	9	9	26	26	26	26	7	7	7	7	1	1	1	1	26	26	26	26	16	16	16	16	34	34	34	34	9	9	9	9	16	16	16	16
7	7	7	7	17	17	17	17	13	13	13	13	4	4	4	4	35	35	35	35	32	32	32	32	6	6	6	6	8	8	8	8	34	34	34	34
26	26	26	26	4	4	4	4	3	3	3	3	16	16	16	16	3	3	3	3	27	27	27	27	25	25	25	25	3	3	3	3	26	26	26	26
12	12	12	12	14	14	14	14	32	32	32	32	27	27	27	27	14	14	14	14	17	17	17	17	24	24	24	24	15	15	15	15	9	9	9	9
34	34	34	34	12	12	12	12	33	33	33	33	34	34	34	34	16	16	16	16	34	34	34	34	26	26	26	26	1	1	1	1	33	33	33	33
8	11	29	26	1	6	36	31	10	23	27	14	1	6	36	31	1	6	36	31	8	11	29	26	10	23	27	14	8	11	29	26	8	11	29	26
35	25	2	12	22	21	15	16	8	11	29	26	4	24	33	13	17	28	20	9	16	22	21	15	17	28	20	9	3	18	34	19	17	28	20	9
19	3	18	34	3	18	34	19	36	31	1	6	16	22	21	15	30	32	7	5	32	7	5	30	3	18	34	19	15	16	22	21	13	4	24	33
23	27	14	10	29	26	8	11	14	10	23	27	27	14	10	23	29	26	8	11	27	14	10	23	31	1	6	36	1	6	36	31	23	27	14	10
3	34	3	34	10	27	10	27	9	28	9	28	1	36	1	36	10	27	10	27	4	33	4	33	16	21	16	21	3	34	3	34	4	33	4	33
27	10	27	10	35	2	35	2	22	15	22	15	4	33	4	33	22	15	22	15	8	29	8	29	5	32	5	32	15	22	15	22	27	10	27	10
21	16	21	16	30	7	30	7	25	12	25	12	16	21	16	21	4	33	4	33	16	21	16	21	27	10	27	10	1	36	1	36	25	12	25	12
15	21	22	16	26	29	11	8	34	18	3	19	1	31	36	6	26	29	11	8	2	25	35	12	33	24	4	13	15	21	22	16	2	25	35	12
24	4	13	33	17	9	20	28	5	7	32	30	4	13	33	24	35	12	2	25	4	13	33	24	8	26	29	11	1	31	36	6	32	30	5	7

Staring at this generator matrix of an $[18, 8]_{37}$-code

9	9	9	9	26	26	26	26	7	7	7	7	1	1	1	1	26	26	26	26	16	16	16	16	34	34	34	34	9	9	9	9	16	16	16	16
7	7	7	7	17	17	17	17	13	13	13	13	4	4	4	4	35	35	35	35	32	32	32	32	6	6	6	6	8	8	8	8	34	34	34	34
26	26	26	26	4	4	4	4	3	3	3	3	16	16	16	16	3	3	3	3	27	27	27	27	25	25	25	25	3	3	3	3	26	26	26	26
12	12	12	12	14	14	14	14	32	32	32	32	27	27	27	27	14	14	14	14	17	17	17	17	24	24	24	24	15	15	15	15	9	9	9	9
34	34	34	34	12	12	12	12	33	33	33	33	34	34	34	34	16	16	16	16	34	34	34	34	26	26	26	26	1	1	1	1	33	33	33	33
8	11	29	26	1	6	36	31	10	23	27	14	1	6	36	31	1	6	36	31	8	11	29	26	10	23	27	14	8	11	29	26	8	11	29	26
35	25	2	12	22	21	15	16	8	11	29	26	4	24	33	13	17	28	20	9	16	22	21	15	17	28	20	9	3	18	34	19	17	28	20	9
19	3	18	34	3	18	34	19	36	31	1	6	16	22	21	15	30	32	7	5	32	7	5	30	3	18	34	19	15	16	22	21	13	4	24	33
23	27	14	10	29	26	8	11	14	10	23	27	27	14	10	23	29	26	8	11	27	14	10	23	31	1	6	36	1	6	36	31	23	27	14	10
3	34	3	34	10	27	10	27	9	28	9	28	1	36	1	36	10	27	10	27	4	33	4	33	16	21	16	21	3	34	3	34	4	33	4	33
27	10	27	10	35	2	35	2	22	15	22	15	4	33	4	33	22	15	22	15	8	29	8	29	5	32	5	32	15	22	15	22	27	10	27	10
21	16	21	16	30	7	30	7	25	12	25	12	16	21	16	21	4	33	4	33	16	21	16	21	27	10	27	10	1	36	1	36	25	12	25	12
15	21	22	16	26	29	11	8	34	18	3	19	1	31	36	6	26	29	11	8	2	25	35	12	33	24	4	13	15	21	22	16	2	25	35	12
24	4	13	33	17	9	20	28	5	7	32	30	4	13	33	24	35	12	2	25	4	13	33	24	8	26	29	11	1	31	36	6	32	30	5	7

Commutator Relation

Let $G \in \mathbb{F}_{37}^{14\times 36}$ be the generator matrix from before.

What information can we get from relations of the form $LG = GR$ for $(L, R) \in \mathrm{GL}_k(\mathbb{F}_q) \times \mathrm{GL}_n(\mathbb{F}_q)$?

Commutator Relation

$$L_2 G = G R_2$$

for

$$L_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 36 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 36 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 36 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 36 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 36 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 36
\end{pmatrix},$$

$$R_2 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

As left-representation?...

$$L_2 = P_2 D_2 P_2^{-1}$$

$$= \begin{pmatrix}
0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0
\end{pmatrix}
\begin{pmatrix}
36&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&36&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&36&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&36&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&36&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&36&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}
\begin{pmatrix}
0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
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0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0
\end{pmatrix}.$$

… as right-representation?…

$$R_2 = Q_2 E_2 Q_2^{-1}$$

$$= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
36 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 36 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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19 & 0 & 18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 19 & 0 & 19
\end{pmatrix}.$$

...and as a pair of representations!

$$P_2{}^{-1}GQ_2 = 2 \cdot \begin{pmatrix}
8 & 11 & 1 & 6 & 10 & 23 & 1 & 6 & 1 & 6 & 8 & 11 & 10 & 23 & 8 & 11 & 8 & 11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
35 & 25 & 22 & 21 & 8 & 11 & 4 & 24 & 17 & 28 & 16 & 22 & 17 & 28 & 3 & 18 & 17 & 28 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
19 & 3 & 3 & 18 & 36 & 31 & 16 & 22 & 30 & 32 & 32 & 7 & 3 & 18 & 15 & 16 & 13 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
23 & 27 & 29 & 26 & 14 & 10 & 27 & 14 & 29 & 26 & 27 & 14 & 31 & 1 & 1 & 6 & 23 & 27 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
15 & 21 & 26 & 29 & 34 & 18 & 1 & 31 & 26 & 29 & 2 & 25 & 33 & 24 & 15 & 21 & 2 & 25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
24 & 4 & 17 & 9 & 5 & 7 & 4 & 13 & 35 & 12 & 4 & 13 & 8 & 26 & 1 & 31 & 32 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 9 & 26 & 26 & 7 & 7 & 1 & 1 & 26 & 26 & 16 & 16 & 34 & 34 & 9 & 9 & 16 & 16 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 7 & 17 & 17 & 13 & 13 & 4 & 4 & 35 & 35 & 32 & 32 & 6 & 6 & 8 & 8 & 34 & 34 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 26 & 26 & 4 & 4 & 3 & 3 & 16 & 16 & 3 & 3 & 27 & 27 & 25 & 25 & 3 & 3 & 26 & 26 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 14 & 14 & 32 & 32 & 27 & 27 & 14 & 14 & 17 & 17 & 24 & 24 & 15 & 15 & 9 & 9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 34 & 34 & 12 & 12 & 33 & 33 & 34 & 34 & 16 & 16 & 34 & 34 & 26 & 26 & 1 & 1 & 33 & 33 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 34 & 10 & 27 & 9 & 28 & 1 & 36 & 10 & 27 & 4 & 33 & 16 & 21 & 3 & 34 & 4 & 33 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 27 & 10 & 35 & 2 & 22 & 15 & 4 & 33 & 22 & 15 & 8 & 29 & 5 & 32 & 15 & 22 & 27 & 10 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21 & 16 & 30 & 7 & 25 & 12 & 16 & 21 & 4 & 33 & 16 & 21 & 27 & 10 & 1 & 36 & 25 & 12
\end{pmatrix}.$$

...and as a pair of representations!

$$P_2^{-1} G Q_2 = 2 \cdot \begin{pmatrix}
8 & 11 & 1 & 6 & 10 & 23 & 1 & 6 & 1 & 6 & 8 & 11 & 10 & 23 & 8 & 11 & 8 & 11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
35 & 25 & 22 & 21 & 8 & 11 & 4 & 24 & 17 & 28 & 16 & 22 & 17 & 28 & 3 & 18 & 17 & 28 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
19 & 3 & 3 & 18 & 36 & 31 & 16 & 22 & 30 & 32 & 32 & 7 & 3 & 18 & 15 & 16 & 13 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
23 & 27 & 29 & 26 & 14 & 10 & 27 & 14 & 29 & 26 & 27 & 14 & 31 & 1 & 1 & 6 & 23 & 27 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
15 & 21 & 26 & 29 & 34 & 18 & 1 & 31 & 26 & 29 & 2 & 25 & 33 & 24 & 15 & 21 & 2 & 25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
24 & 4 & 17 & 9 & 5 & 7 & 4 & 13 & 35 & 12 & 4 & 13 & 8 & 26 & 1 & 31 & 32 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 9 & 26 & 26 & 7 & 7 & 1 & 1 & 26 & 26 & 16 & 16 & 34 & 34 & 9 & 9 & 16 & 16 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 7 & 17 & 17 & 13 & 13 & 4 & 4 & 35 & 35 & 32 & 32 & 6 & 6 & 8 & 8 & 34 & 34 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 26 & 26 & 4 & 4 & 3 & 3 & 16 & 16 & 3 & 3 & 27 & 27 & 25 & 25 & 3 & 3 & 26 & 26 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 14 & 14 & 32 & 32 & 27 & 27 & 14 & 14 & 17 & 17 & 24 & 24 & 15 & 15 & 9 & 9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 34 & 34 & 12 & 12 & 33 & 33 & 34 & 34 & 16 & 16 & 34 & 34 & 26 & 26 & 1 & 1 & 33 & 33 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 34 & 10 & 27 & 9 & 28 & 1 & 36 & 10 & 27 & 4 & 33 & 16 & 21 & 3 & 34 & 4 & 33 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21 & 16 & 30 & 7 & 25 & 12 & 16 & 21 & 4 & 33 & 16 & 21 & 27 & 10 & 1 & 36 & 25 & 12
\end{pmatrix}.$$

A Decomposition Lemma to Compute Folded Codes

Lemma [Pasquereau]

Let $G \in \mathbb{F}_q^{k\times n}$ be the generator matrix of a linear code. Suppose $LG = GR$ for some $L \in \mathrm{GL}_k(\mathbb{F}_q)$ and $R \in \mathrm{GL}_n(\mathbb{F}_q)$. If L and R are diagonalizable matrices, then there exist a pair of invertible matrices $(P, Q) \in \mathrm{GL}_k(\mathbb{F}_q) \times \mathrm{GL}_n(\mathbb{F}_q)$, such that

$$P^{-1}GQ = \begin{pmatrix} G_{i_1,j_1} & 0 & \dots & 0 \\ 0 & G_{i_2,j_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & G_{i_t,j_t} \end{pmatrix},$$

where $t = |\{\lambda_i : 1 \le i \le r\} \cap \{\mu_j : 1 \le j \le s\}|$ is the size of the intersection between the spectra of L and R, $G_{i_u,j_u} \in \mathbb{F}_q^{k_{i_u}\times n_{j_u}}$. What's more, k_{i_u} and n_{i_u} are the respective geometric dimensions of the left and right eigenspaces associated to the same eigenvalue λ_u in the intersection.
In particular there exists a family $\{Q_u\}_{1\le u\le t}$ of projections $Q_u : C \to C'_u$, where C is the code generated by G, and C'_u is an $[n_{j_u}, k_{i_u}, d_u]$-code, such that

$$\forall \mathbf{x} \in \mathbb{F}_q^n, (\mathbf{x} \in C) \iff (\forall u \in \{1, \dots, t\}, \mathbf{x}Q_u \in C'_u).$$

Recovering the FRI's folding map

$$\Phi : \begin{cases} \mathbb{F}_q^\times & \hookrightarrow & \mathrm{Aut}(\mathbb{P}^1) \\ \alpha & \mapsto & \varphi_\alpha : (x:y) \mapsto (\alpha x : y) \end{cases}$$

$$\mathsf{L} : \begin{cases} \mathrm{Aut}(\mathbb{P}^1) & \to & \mathrm{GL}(L_{\mathbb{P}^1}(D)) \cong \mathrm{GL}_k(\mathbb{F}_q) \\ \varphi & \mapsto & \varphi^* : f \mapsto f \circ \varphi \end{cases}$$

$$\mathsf{R} : \begin{cases} \mathrm{Aut}(\mathbb{P}^1) & \to & \mathrm{GL}(\mathbb{F}_q[\Gamma]) \cong \mathrm{GL}_n(\mathbb{F}_q) \\ \varphi & \mapsto & \sum_x a_{(x:1)} \mathsf{e}_{(x:1)} \mapsto \sum_x a_{(x:1)} \mathsf{e}_{\varphi(x:1)} \end{cases}$$

with D a divisor of degree $0 \leq k-1 \leq q-1$ supported by $\mathbb{F}_q^\times$-inv. points.

For a primitive element $\alpha \in \mathbb{F}_q^\times$, $\mathsf{L}(\Phi(\alpha))G = G\mathsf{R}(\Phi(\alpha))$. If k and $n = q-1$ are divisible by 2, applying the lemma to $L = \mathsf{L}(\Phi(\alpha^{\frac{q-1}{2}}))$, $R = \mathsf{R}(\Phi(\alpha^{\frac{q-1}{2}}))$, induces a folding map $\mathsf{Fold}[\mathsf{c}, z] = \mathsf{c}(Q_1 + zQ_2)$.

Conclusion: What does this framework bring?

- *Generality*: focus on generator matrices and code automorphism groups.
- *Coding Theory Techniques:* study efficiency and soundness using coding theory tools.
- *Step-by-Step Approach*: a class of code does not need to be 'closed' under folding.
- *Folding Codes* OR *Folding Structure*: free to either fix G or fix (L, R).
- *Other Metrics*: consider different classes of isometries.

- *Code Automorphism*: computing a "good" (L, R)-pair is still difficult.
- *Control of Sufficient Conditions:* less guarantees that the folded codes are balanced or have good minimal distance.
- *Code Equivalence:* can be computationally difficult to merge the projectors.

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Thank you!

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