

A Generalized Framework for Folding Codes and Applications to Proofs of Proximity

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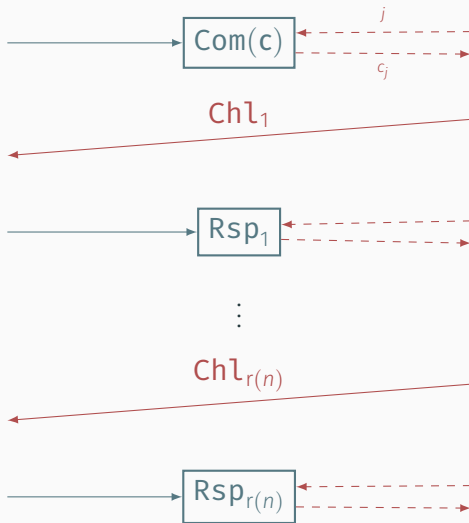


University of St.Gallen

Institute of Computer Science

$$P(C \subseteq \mathbb{F}_q^n, \mathbf{c} \in \mathbb{F}_q^n)$$

$$V(C)$$



$$P(C \subseteq \mathbb{F}_q^n, \mathbf{c} \in \mathbb{F}_q^n)$$

$$V(C)$$



Chl_1



$\vdots$

$\text{Chl}_{r(n)}$



Accept (1) if

$\mathbf{c} \in C$, otherwise

Reject (0) with
pb. $\geq 1 - \varepsilon(\frac{d(\mathbf{c}, C)}{n})$.

What do we want? Why?

rounds complexity: $r(n) \leq \log(n)$

proof length: $l(n) \leq O(n)$

query complexity: $q(n) < O(n)$

prover complexity: $t_P(n) \leq O(n)$

verifier complexity: $t_V(n) < O(n)$

Goal

Construct IOPP protocols with
sublinear complexity (for the verifier)
for *broad* families of codes.

Applications to...

- Succinct Arguments: (zk-)SNARGs and (zk-)SNARKs,
- Non-Interactive Zero-Knowledge: compilers from SNARGs to (zk-)SNARGs and NIZKs,
- Integrity Verification: delegating computation to a non-trusted server over a network...

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Motivation

Preliminaries

How do folding protocols work?

From the geometric intuition...

Generalized Framework

...to (linear) algebra and representation theory.

Preliminaries

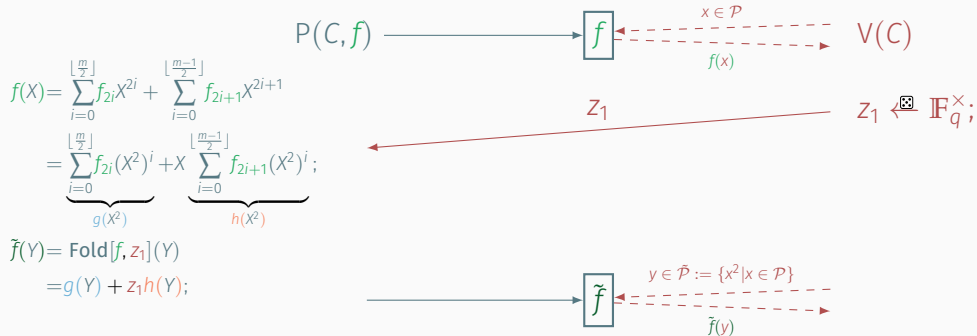
How do folding
protocols work?

The typical example: FRI [Ben-Sasson, Bentov, et al. 2017]

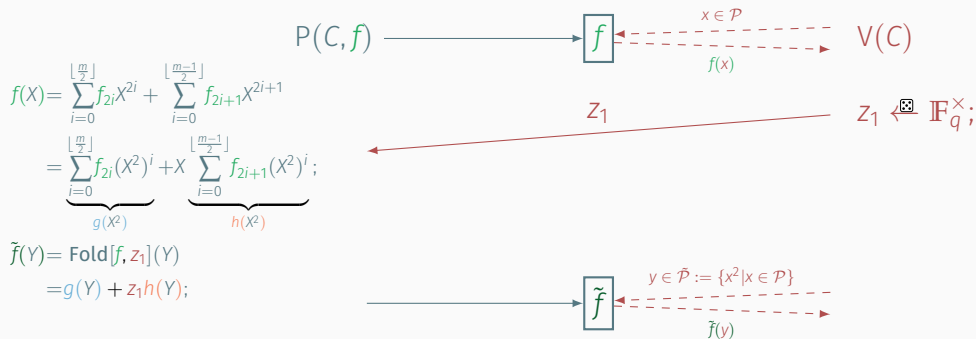
Let $\nu > r \geq 0$, q an odd prime power such that $2^\nu | q - 1$, $\alpha \in \mathbb{F}_q^\times$ with $\text{ord}(\alpha) = 2^\nu$.
Set $n = 2^\nu$, $k = 2^r$, $\mathcal{P} = \{\alpha^j \mid j \in \{0, \dots, n-1\}\}$, and $C = \text{RS}[\mathcal{P}, k]$.

For $f(X) = \sum_{i=0}^m f_i X^i \in \mathbb{F}_q[X]$, do we have $\deg f = m < k$?

The typical example: FRI [Ben-Sasson, Bentov, et al. 2017]

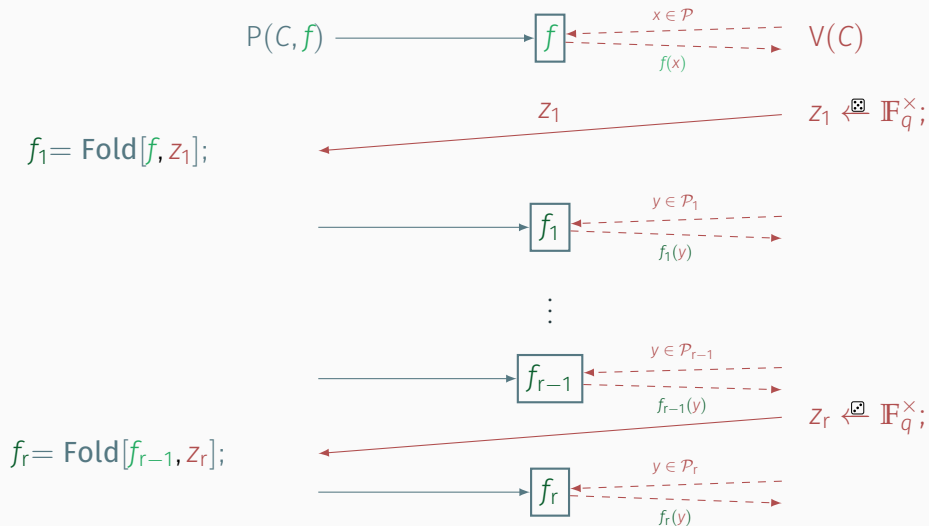


The typical example: FRI [Ben-Sasson, Bentov, et al. 2017]



If $f \in \mathcal{C} = \text{RS}[\mathcal{P}, k]$, then $\tilde{f} = \mathbf{Fold}[f, z_1] \in \tilde{\mathcal{C}} = \text{RS}[\tilde{\mathcal{P}}, k/2]$.

The typical example: FRI [Ben-Sasson, Bentov, et al. 2017]



The typical example: FRI [Ben-Sasson, Bentov, et al. 2017]

Round consistency check for verification: For $y \neq 0$, verify whether

$$2 \cdot f_{i+1}(y^2) = (1 + z_i y^{-1}) f_i(y) + (1 + z_i y^{-1}) f_i(-y),$$

which costs 1 query to f_{i+1} plus 2 queries to f_i .

Folding AG-Codes (intuition) [Bordage et al. 2020]

Suppose $N = \prod_{j=1}^{r-1} p_j$, let $\Gamma_0 = \Gamma \cong \mathbb{Z}/N\mathbb{Z}$, Γ_{i+1} such that $\Gamma_{i+1}/\Gamma_i \cong \mathbb{Z}/p_i\mathbb{Z}$, denote $\mathcal{X}_i : y^{\prod_{j=i}^{r-1} p_j} = \prod_{l=1}^m (x - a_l)$, $P_\infty^{(i)}$ the place at infinity on $\mathcal{X}_i$, $\pi_{i+1} : \mathcal{X}_i \rightarrow \mathcal{X}_{i+1}$.

[Proposition 29, Bordage et al. 2020]

Let $i \geq 0$. Set $D_i = d_i P_\infty^{(i)}$ for some integer d_i . Then $L_{\mathcal{X}_i}(D_i) = \bigoplus_{j=0}^{p_i-1} \nu_i^j \pi_i^*(L_{\mathcal{X}_{i-1}}(E_{i,j}))$ with $E_{i,j} = \left\lfloor \frac{1}{p_i} \pi_{i*}(D_i - j \cdot \text{Div}_{\mathcal{X}_i}((\nu_i))) \right\rfloor$, for $0 \leq j \leq p_i - 1$, and some well chosen balancing function $\nu_i \in \mathbb{F}_q(\mathcal{X}_i)$ “compatible” with D_i .

Corollary for real-world purposes

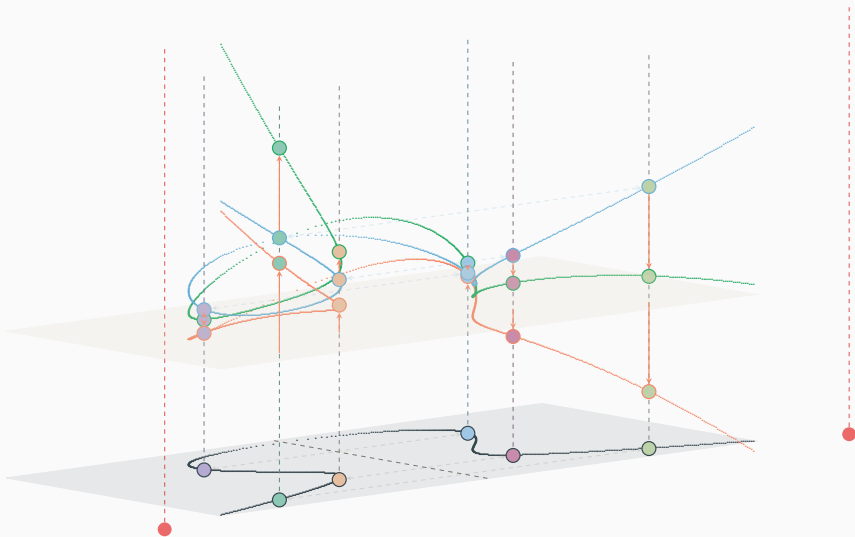
If $C = C(\mathcal{X}, \mathcal{P}, D)$ with $\mathcal{P} \subset \mathcal{X}$ a disjoint union of Γ -orbits of maximum size and D a Γ -invariant divisor, then for any $\mathbf{z} = (z_1, z_2) \in \mathbb{F}_q$, the operator

Fold $[f, \mathbf{z}] = \sum_{j=0}^{p_i-1} z_1^j f_j + \sum_{j=0}^{p_i-1} z_2^{j+1} \nu_{i+1,j} f_j$ induces a projection from $C(\mathcal{X}_i, \mathcal{P}_i, D_i)$ to $C(\mathcal{X}_i/\Gamma_i, \pi_{i+1}(\mathcal{P}_i), E_i)$, resulting in an *efficient* IOPP protocol.

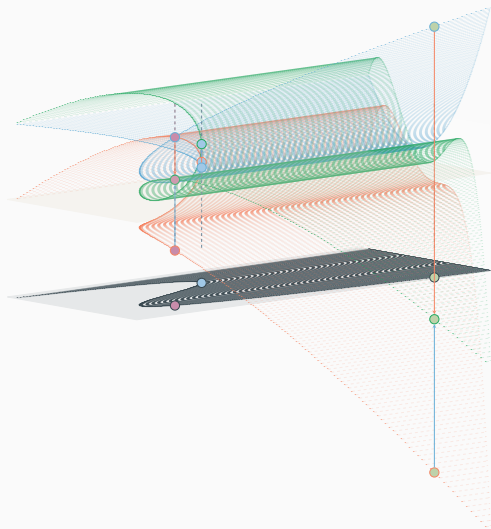
Preliminaries

From the geometric
intuition...

Geometric Intuition: Starting Curve



Geometric Intuition: Quotient Curve



Generalized Framework

...to (linear) algebra and
representation theory.

Staring at this generator matrix of an $[36, 14]_{37}$ -code

$$\begin{pmatrix} 9 & 9 & 9 & 9 & 26 & 26 & 26 & 26 & 7 & 7 & 7 & 7 & 1 & 1 & 1 & 1 & 26 & 26 & 26 & 26 & 16 & 16 & 16 & 16 & 34 & 34 & 34 & 34 & 9 & 9 & 9 & 9 & 16 & 16 & 16 & 16 \\ 7 & 7 & 7 & 7 & 17 & 17 & 17 & 17 & 13 & 13 & 13 & 13 & 4 & 4 & 4 & 4 & 35 & 35 & 35 & 35 & 32 & 32 & 32 & 32 & 6 & 6 & 6 & 6 & 8 & 8 & 8 & 8 & 34 & 34 & 34 & 34 \\ 26 & 26 & 26 & 26 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 3 & 16 & 16 & 16 & 16 & 3 & 3 & 3 & 3 & 27 & 27 & 27 & 27 & 25 & 25 & 25 & 25 & 3 & 3 & 3 & 3 & 26 & 26 & 26 & 26 \\ 12 & 12 & 12 & 12 & 14 & 14 & 14 & 14 & 32 & 32 & 32 & 32 & 27 & 27 & 27 & 27 & 14 & 14 & 14 & 14 & 17 & 17 & 17 & 17 & 24 & 24 & 24 & 24 & 15 & 15 & 15 & 15 & 9 & 9 & 9 & 9 \\ 34 & 34 & 34 & 34 & 12 & 12 & 12 & 12 & 33 & 33 & 33 & 33 & 34 & 34 & 34 & 34 & 16 & 16 & 16 & 16 & 34 & 34 & 34 & 34 & 26 & 26 & 26 & 26 & 1 & 1 & 1 & 1 & 33 & 33 & 33 & 33 \\ 8 & 11 & 29 & 26 & 1 & 6 & 36 & 31 & 10 & 23 & 27 & 14 & 1 & 6 & 36 & 31 & 1 & 6 & 36 & 31 & 8 & 11 & 29 & 26 & 10 & 23 & 27 & 14 & 8 & 11 & 29 & 26 & 8 & 11 & 29 & 26 \\ 35 & 25 & 2 & 12 & 22 & 21 & 15 & 16 & 8 & 11 & 29 & 26 & 4 & 24 & 33 & 13 & 17 & 28 & 20 & 9 & 16 & 22 & 21 & 15 & 17 & 28 & 20 & 9 & 3 & 18 & 34 & 19 & 17 & 28 & 20 & 9 \\ 19 & 3 & 18 & 34 & 3 & 18 & 34 & 19 & 36 & 31 & 1 & 6 & 16 & 22 & 21 & 15 & 30 & 32 & 7 & 5 & 32 & 7 & 5 & 30 & 3 & 18 & 34 & 19 & 15 & 16 & 22 & 21 & 13 & 4 & 24 & 33 \\ 23 & 27 & 14 & 10 & 29 & 26 & 8 & 11 & 14 & 10 & 23 & 27 & 27 & 14 & 10 & 23 & 29 & 26 & 8 & 11 & 27 & 14 & 10 & 23 & 31 & 1 & 6 & 36 & 1 & 6 & 36 & 31 & 23 & 27 & 14 & 10 \\ 3 & 34 & 3 & 34 & 10 & 27 & 10 & 27 & 9 & 28 & 9 & 28 & 1 & 36 & 1 & 36 & 10 & 27 & 10 & 27 & 4 & 33 & 4 & 33 & 16 & 21 & 16 & 21 & 3 & 34 & 3 & 34 & 4 & 33 & 4 & 33 \\ 27 & 10 & 27 & 10 & 35 & 2 & 35 & 2 & 22 & 15 & 22 & 15 & 4 & 33 & 4 & 33 & 22 & 15 & 22 & 15 & 8 & 29 & 8 & 29 & 5 & 32 & 5 & 32 & 15 & 22 & 15 & 22 & 27 & 10 & 27 & 10 \\ 21 & 16 & 21 & 16 & 30 & 7 & 30 & 7 & 25 & 12 & 25 & 12 & 16 & 21 & 16 & 21 & 4 & 33 & 4 & 33 & 16 & 21 & 16 & 21 & 27 & 10 & 27 & 10 & 1 & 36 & 1 & 36 & 25 & 12 & 25 & 12 \\ 15 & 21 & 22 & 16 & 26 & 29 & 11 & 8 & 34 & 18 & 3 & 19 & 1 & 31 & 36 & 6 & 26 & 29 & 11 & 8 & 2 & 25 & 35 & 12 & 33 & 24 & 4 & 13 & 15 & 21 & 22 & 16 & 2 & 25 & 35 & 12 \\ 24 & 4 & 13 & 33 & 17 & 9 & 20 & 28 & 5 & 7 & 32 & 30 & 4 & 13 & 33 & 24 & 35 & 12 & 2 & 25 & 4 & 13 & 33 & 24 & 8 & 26 & 29 & 11 & 1 & 31 & 36 & 6 & 32 & 30 & 5 & 7 \end{pmatrix}$$

Staring at this generator matrix of an $[36, 14]_{37}$ -code

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Staring at this generator matrix of an $[36, 14]_{37}$ -code

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Staring at this generator matrix of an $[36, 14]_{37}$ -code

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Staring at this generator matrix of an $[36, 14]_{37}$ -code

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Staring at this generator matrix of an $[18, 8]_{37}$ -code

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34	34	34	34	12	12	12	12	33	33	33	33	34	34	34	34	16	16	16	16	34	34	34	34	26	26	26	26	1	1	1	1	33	33	33	33
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27	10	27	10	35	2	35	2	22	15	22	15	4	33	4	33	22	15	22	15	8	29	8	29	5	32	5	32	15	22	15	22	27	10	27	10
21	16	21	16	30	7	30	7	25	12	25	12	16	21	16	21	4	33	4	33	16	21	16	21	27	10	27	10	1	36	1	36	25	12	25	12
15	21	22	16	26	29	11	8	34	18	3	19	1	31	36	6	26	29	11	8	2	25	35	12	33	24	4	13	15	21	22	16	2	25	35	12
24	4	13	33	17	9	20	28	5	7	32	30	4	13	33	24	35	12	2	25	4	13	33	24	8	26	29	11	1	31	36	6	32	30	5	7

Commutator Relation

Let $G \in \mathbb{F}_{37}^{14 \times 36}$ be the generator matrix from before.

What information can we get from relations of the form $LG = GR$ for $(L, R) \in \text{GL}_k(\mathbb{F}_q) \times \text{GL}_n(\mathbb{F}_q)$?

Commutator Relation

$$L_2 G = G R_2$$

for

[illegible]

[illegible]

As left-representation?...

$$L_2 = P_2 D_2 P_2^{-1}$$

[illegible]

... as right-representation?...

$$R_2 = Q_2 E_2 Q_2^{-1}$$

...and as a pair of representations!

$$P_2^{-1}GQ_2 = 2 \cdot \begin{pmatrix} 8 & 11 & 1 & 6 & 10 & 23 & 1 & 6 & 1 & 6 & 8 & 11 & 10 & 23 & 8 & 11 & 8 & 11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 35 & 25 & 22 & 21 & 8 & 11 & 4 & 24 & 17 & 28 & 16 & 22 & 17 & 28 & 3 & 18 & 17 & 28 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 19 & 3 & 3 & 18 & 36 & 31 & 16 & 22 & 30 & 32 & 32 & 7 & 3 & 18 & 15 & 16 & 13 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 23 & 27 & 29 & 26 & 14 & 10 & 27 & 14 & 29 & 26 & 27 & 14 & 31 & 1 & 1 & 6 & 23 & 27 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 15 & 21 & 26 & 29 &quad 34 & 18 & 1 & 31 & 26 &quad 29 & 2 & 25 &quad 33 & 24 & 15 & 21 &quad 2 & 25 &quad 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 24 & 4 & 17 &quad 9 & 5 & 7 & 4 & 13 &quad 35 & 12 & 4 & 13 &quad 8 & 26 & 1 & 31 &quad 32 &quad 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 9 & 26 &quad 26 &quad 7 &quad 7 &quad 1 &quad 1 &quad 26 &quad 26 &quad 16 &quad 16 &quad 34 &quad 34 &quad 9 &quad 9 &quad 16 &quad 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 &quad 7 &quad 17 &quad 17 &quad 13 &quad 13 &quad 4 &quad 4 &quad 35 &quad 35 &quad 32 &quad 32 &quad 6 &quad 6 &quad 8 &quad 8 &quad 34 &quad 34 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 26 &quad 26 &quad 4 &quad 4 &quad 3 &quad 3 &quad 16 &quad 16 &quad 3 &quad 3 &quad 27 &quad 27 &quad 25 &quad 25 &quad 3 &quad 3 &quad 26 &quad 26 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 &quad 12 &quad 14 &quad 14 &quad 32 &quad 32 &quad 27 &quad 27 &quad 14 &quad 14 &quad 17 &quad 17 &quad 24 &quad 24 &quad 15 &quad 15 &quad 9 &quad 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 34 &quad 34 &quad 12 &quad 12 &quad 33 &quad 33 &quad 34 &quad 34 &quad 16 &quad 16 &quad 34 &quad 34 &quad 26 &quad 26 &quad 1 &quad 1 &quad 33 &quad 33 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 &quad 34 &quad 10 &quad 27 &quad 9 &quad 28 &quad 1 &quad 36 &quad 10 &quad 27 &quad 4 &quad 33 &quad 16 &quad 21 &quad 3 &quad 34 &quad 4 &quad 33 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 27 &quad 10 &quad 35 &quad 2 &quad 22 &quad 15 &quad 4 &quad 33 &quad 22 &quad 15 &quad 8 &quad 29 &quad 5 &quad 32 &quad 15 &quad 22 &quad 27 &quad 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21 &quad 16 &quad 30 &quad 7 &quad 25 &quad 12 &quad 16 &quad 21 &quad 4 &quad 33 &quad 16 &quad 21 &quad 27 &quad 10 &quad 1 &quad 36 &quad 25 &quad 12 \end{pmatrix}.$$

...and as a pair of representations!

$$P_2^{-1}GQ_2 = 2 \cdot \begin{pmatrix} 8 & 11 & 1 & 6 & 10 & 23 & 1 & 6 & 1 & 6 & 8 & 11 & 10 & 23 & 8 & 11 & 8 & 11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 35 & 25 & 22 & 21 & 8 & 11 & 4 & 24 & 17 & 28 & 16 & 22 & 17 & 28 & 3 & 18 & 17 & 28 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 19 & 3 & 3 & 18 & 36 & 31 & 16 & 22 & 30 & 32 & 32 & 7 & 3 & 18 & 15 & 16 & 13 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 23 & 27 & 29 & 26 & 14 & 10 & 27 & 14 & 29 & 26 & 27 & 14 & 31 & 1 & 1 & 6 & 23 & 27 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 15 & 21 & 26 & 29 & 34 & 18 & 1 & 31 & 26 & 29 & 2 & 25 & 33 & 24 & 15 & 21 & 2 & 25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 24 & 4 & 17 & 9 & 5 & 7 & 4 & 13 & 35 & 12 & 4 & 13 & 8 & 26 & 1 & 31 & 32 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 9 & 26 & 26 & 7 & 7 & 1 & 1 & 26 & 26 & 16 & 16 & 34 & 34 & 9 & 9 & 16 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 7 & 17 & 17 & 13 & 13 & 4 & 4 & 35 & 35 & 32 & 32 & 6 & 6 & 8 & 8 & 34 & 34 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 26 & 26 & 4 & 4 & 3 & 3 & 16 & 16 & 3 & 3 & 27 & 27 & 25 & 25 & 3 & 3 & 26 & 26 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 12 & 14 & 14 & 32 & 32 & 27 & 27 & 14 & 14 & 17 & 17 & 24 & 24 & 15 & 15 & 9 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 34 & 34 & 12 & 12 & 9 & 33 & 34 & 34 & 16 & 16 & 34 & 34 & 26 & 26 & 1 & 1 & 33 & 33 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 34 & 10 & 27 & 9 & 28 & 1 & 36 & 10 & 27 & 4 & 33 & 16 & 21 & 3 & 34 & 4 & 33 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 27 & 10 & 35 & 2 & 22 & 15 & 4 & 33 & 22 & 15 & 8 & 29 & 5 & 32 & 15 & 22 & 27 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21 & 16 & 30 & 7 & 25 & 12 & 16 & 21 & 4 & 33 & 16 & 21 & 27 & 10 & 1 & 36 & 25 & 12 \end{pmatrix}.$$

A Decomposition Lemma to Compute Folded Codes

Lemma (simplified) [P.-Horlemann '24]

Let $G \in \mathbb{F}_q^{k \times n}$ be the generator matrix of a linear code. Suppose $LG = GR$ for some $L \in \text{GL}_k(\mathbb{F}_q)$ and $R \in \text{GL}_n(\mathbb{F}_q)$. If L and R are diagonalizable matrices, then there exist a pair of invertible matrices $(P, Q) \in \text{GL}_k(\mathbb{F}_q) \times \text{GL}_n(\mathbb{F}_q)$, such that

$$P^{-1}GQ = \begin{pmatrix} G_{i_1, j_1} & 0 & \dots & 0 \\ 0 & G_{i_2, j_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & G_{i_t, j_t} \end{pmatrix},$$

where $t = |\{\lambda_i : 1 \leq i \leq r\} \cap \{\mu_j : 1 \leq j \leq s\}|$ is the size of the intersection between the spectra of L and R , $G_{i_u, j_u} \in \mathbb{F}_q^{k_{i_u} \times n_{j_u}}$. What's more, k_{i_u} and n_{j_u} are the respective geometric dimensions of the left and right eigenspaces associated to the same eigenvalue λ_u in the intersection.

In particular there exists a family $\{Q_u\}_{1 \leq u \leq t}$ of projections $Q_u : C \rightarrow C'_u$, where C is the code generated by G , and C'_u is an $[n_{j_u}, k_{i_u}, d_u]$ -code, such that

$$\forall \mathbf{x} \in \mathbb{F}_q^n, (\mathbf{x} \in C) \iff (\forall u \in \{1, \dots, t\}, \mathbf{x}Q_u \in C'_u).$$

Recovering the FRI's folding map

$$\begin{aligned}\Phi &: \begin{cases} \mathbb{F}_q^\times & \hookrightarrow & \text{Aut}(\mathbb{P}^1) \\ \alpha & \mapsto & \varphi_\alpha: (x:y) \mapsto (\alpha x:y) \end{cases} \\ \mathbf{L} &: \begin{cases} \text{Aut}(\mathbb{P}^1) & \rightarrow & \text{GL}(L_{\mathbb{P}^1}(D)) \cong \text{GL}_k(\mathbb{F}_q) \\ \varphi & \mapsto & \varphi^*: f \mapsto f \circ \varphi \end{cases} \\ \mathbf{R} &: \begin{cases} \text{Aut}(\mathbb{P}^1) & \rightarrow & \text{GL}(\mathbb{F}_q[\Gamma]) \cong \text{GL}_n(\mathbb{F}_q) \\ \varphi & \mapsto & \sum_x a_{(x:1)} \mathbf{e}_{(x:1)} \mapsto \sum_x a_{(x:1)} \mathbf{e}_{\varphi(x:1)} \end{cases}\end{aligned}$$

with D a divisor of degree $0 \leq k-1 \leq q-1$ supported by $\mathbb{F}_q^\times$ -inv. points.

For a primitive element $\alpha \in \mathbb{F}_q^\times$, $\mathbf{L}(\Phi(\alpha))G = \mathbf{GR}(\Phi(\alpha))$. If k and $n = q-1$ are divisible by 2, applying the lemma to $L = \mathbf{L}(\Phi(\alpha^{\frac{q-1}{2}}))$, $R = \mathbf{R}(\Phi(\alpha^{\frac{q-1}{2}}))$, induces a folding map $\mathbf{Fold}[\mathbf{c}, z] = \mathbf{c}(Q_1 + zQ_2)$.

Conclusion: What does this framework bring?

- *Generality*: focus on generator matrices and code automorphism groups.
- *Coding Theory Techniques*: study efficiency and soundness using coding theory tools.
- *Step-by-Step Approach*: a class of code does not need to be 'closed' under folding.
- *Folding Codes OR Folding Structure*: free to either fix G or fix (L, R) .
- *Other Metrics*: consider different classes of isometries.
- *Code Automorphism*: computing a "good" (L, R) -pair is still difficult.
- *Control of Sufficient Conditions*: less guarantees that the folded codes are balanced or have good minimal distance.
- *Code Equivalence*: can be computationally difficult to merge the projectors.

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Thank you!



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