

Life-Cycle Models with Stock and Labor Market Cointegration: Insights from Analytical Solutions*

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Comments welcome.

Abstract

Two recent articles Lynch and Tan (2011) and Benzoni et al. (2007) show that optimal policies can be on reasonable level over the life-cycle if one takes into account dynamic labor income. This paper examines the economic framework of LT and BCG in complete markets. Based on our analytic results we show that the two models are closely related and that the BCG model can be considered as a special case of the LT model. Moreover, we introduce a two factor model with a cointegration factor that guarantees a long-run equilibrium between the stock and the labor market. We show that under the presence of a cointegration factor, the importance of a second factor decreases.

Key Words: consumption/portfolio optimization, labor income, cointegration, time-varying investment opportunities, time-varying labor income growth, HARA-utility, HJB-equation, closed-form solutions

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Modeling optimal consumption and portfolio choice in the presence of a non-financial income stream such as labor income is a challenging task. In general, a future income stream implies additional wealth for an individual and hence, her behavior becomes unrealistically extreme, i.e. current consumption is higher than current income and risky investment is a multiple of financial wealth. Moreover, the models generally state that risky investment should be higher for young individuals¹. However, empirical studies do not show such clear age effects². In contrast, Benzoni et al. (2007) even state that empirical studies suggest that risky asset holdings are typically low for young individuals, and then either are increasing or hump-shaped over the life-cycle.

Furthermore, the importance of life-cycle investment is already high and is likely to rise over the coming years. In fact, recent years have shown a shift from defined-benefit (DB) plans to defined-contribution (DC) plans³. As a consequence, individuals must increasingly rely on their own saving and investment decisions to fund their life-time consumption.

Two recent articles Lynch and Tan (2011, hereafter LT) and Benzoni et al. (2007, hereafter BCG) show that consumption and risky investment can be on reasonable level over the life-cycle if one takes into account dynamic labor income⁴. LT consider time variation in the moments of labor income while BCG consider cointegration between the labor and the stock market. These article have in common that they focus on rather complicated life-cycle models with multiple realistic features and have to rely

¹For an early example with deterministic labor income, see Merton (1971). For more recent examples, see Viceira (2001) or Huang et al. (2008).

²See Campbell (2006) or Curcuru et al. (2009).

³An empirical survey by Tower Watson shows that DC assets accounted for 44% of global pension assets, compared with 35% in 1999. <http://www.towerswatson.com/assets/pdf/3761/Global-Pensions-Asset-Study-2011.pdf>

⁴Another related paper is Munk and Sørensen (2010). They model labor income dynamics that depend on the short interest rate and their focus is on combined long-term bond-stock holdings.

on numerical methods. In this paper we will approach these models with analytical methods. The purpose of the first part of this paper is not to study every multiple feature of the BCG and the LT model, but to study their basic frameworks in detail. We will be able to reproduce results from the aforementioned studies and offer considerably more theoretical insights and implications for numerical calibration based on empirical research. However, for the sake of analytical traceability we have to consider more restrictive assumptions, i.e. complete markets.

Our main contributions are as follows: (i) We show that the model of BCG and LT are closely related. In fact, the model of BCG can be considered as a special case of the more general model by LT. (ii) We find a closed-form solution for the future income stream. This formula allows to interpret the value of future income over states and a detailed interpretation of the optimal policies over time and states. (iii) The two studies contain some critical issues that are not discovered and discussed, probably, because of the complexity of their models. Most notably, (a) in the LT model only labor income growth converges to a long-run equilibrium. In that sense, a shock to the economy has a permanent impact on labor income and the development of the labor and the stock market can be very different, which seems not in line with economic theory⁵. (b) In the BCG model, labor income and the stock market have a long-run equilibrium, but their model implies strong restrictions on the parameter values of the different processes in the economy. We show that only strong cointegration is able to generate low risky investment for young individuals but implies unreasonably strong variation in labor income growth. On the other hand, weak cointegration implies more realistic variation in income growth but leads to high risky investment for young individuals.

In the second part of this paper, we combine the models of LT and BCG in a two

⁵For example, it is well-known that under a Cobb-Douglas production function, the share of labor income and capital income on total output are constant.

factor model that includes desirable properties of both models. Specifically, the model includes an economically meaningful long-run equilibrium between the labor and the stock market as in BCG and, in addition, has the higher flexibility of the LT model. In contrast to models that have to be solved by numerical methods, the inclusion of additional state variables does not lead to computational difficulties and the precision of the results are not affected. Our main contribution for the second part is as follows: We show that for long-run horizon the presence of cointegration between the labor and the stock market decreases the importance of the second factor. The statement remains valid even if cointegration is weak. Hence, we believe that ignoring possible cointegration and focusing on factors with only temporary effect could lead to a misspecification of the economy and misleading optimal policies would result. For these reasons, we stress the importance of future research in the direction of financial and stock market cointegration and/or a clear macroeconomic theory in order to reliably study consumption and portfolio decisions over the life-cycle.

The remainder of this paper is as follows. In Section 1, we present the basics of the simplified LT and BCG model in complete markets. In Section 2, we compare the analytical valuation of human capital in both models and extend the model to a two factor model. The analytical properties of the optimal policies are discussed in Section 3. The final Section concludes. Mathematical derivations as the solution of the HJB-equations and other non-trivial derivations are provided in the Appendices A.1 - A.3.

1 Model Basics

In the first part we review the model of LT and BCG in continuous time under complete markets. The financial market consists of a risky and a riskless asset. Specifically, the

riskless asset follows

$$\frac{dB(t)}{B(t)} = r_0 dt \quad (1)$$

The risky assets dynamics are given by

$$\frac{dS(t)}{S(t)} = (r_0 + \pi_0) dt + \sigma_s dZ(t) \quad (2)$$

with $r_0 \geq 0$, $\pi_0 > 0$ and $\sigma_s > 0$. It should be noticed that the financial market is one-to-one similar to the classical Merton (1969) model and does not account for time-varying investment opportunities. BCG consider the same financial market model and LT consider this specification as a modification of their basic model. In the basic model, they assume that the risky assets premium varies with the same state variable as labor income. Including a term $\pi_1 X(t)$ in the drift term of (2) would not affect the analytical solvability but makes the model more tedious to solve and additional insights are limited⁶. Moreover, Moos (2011b) shows that the sensitivity of the optimal policies under time-varying income growth and expected returns is highly unrealistic unless time variation in expected returns is low. For these reasons, we skip this issue. In addition, this specification can be justified from an empirical point of view as well. In fact, the empirical results of Table 1 in LT show that the coefficient for return predictability of the dividend yield is not even statistically significant at the 10 percent level⁷.

1.1 The LT Economy in Complete Markets

In the LT model, the individual faces an income stream with the following dynamics

$$\frac{dY(t)}{Y(t)} = (y_0 + y_1 X(t)) dt \quad (3)$$

where $X(t)$ is a state variable, which is specified below. For the sake of analytical solvability we assume that the income stream is locally riskfree, i.e. there is no diffusion risk

⁶See Moos (2011b).

⁷For a critical overview of the empirical evidence for return predictability in stock markets, see Goyal and Welch (2008). Based on this work, it seems reasonable to neglect stock return predictability.

but the growth rate varies over time. As an alternative and in line with the assumption of complete markets, we could include a diffusion term $\rho_{sy}\sigma_y dZ(t)$ with $\rho_{sy} \in \{-1, 1\}$. We skip this issue for two reason. Firstly, empirical surveys do not show such clear short-run correlations between the stock market and labor income⁸. Secondly, the effect of state variable risk on the optimal policies is more clear without this extension. In other words, without this diffusion term, the optimal policies are only affected by state variable risk of the labor income stream⁹. Finally, LT model also state dependent labor income volatility and find that this channel is of high importance for their result. Nevertheless, the main result of their paper, lower risky investment for young individuals, is valid without this extension. For this reason and since this extension is not part of the BCG model, we neglect this issue. Finally, including stochastic volatility would make interpretation of the results more difficult¹⁰.

There is one single state variable $X(t)$ with the following dynamics

$$dX(t) = -\kappa(X(t) - \bar{X})dt + \rho_{sx}\sigma_x dZ(t) \quad (4)$$

with $\kappa \geq 0$, $\bar{X} \geq 0$, $\sigma_x > 0$ and $\rho_{sx} \in \{-1, 1\}$. The notation of (4) is standard in the dynamic portfolio choice literature¹¹.

Admittedly, the complete market assumption does not match reality one-to-one. Nevertheless, several papers have shown that the results of exactly solvable special cases are qualitatively similar to cases with non-perfect correlation¹². Hence, we expect that the qualitative results hold for more general cases as described in LT and BCG.

Furthermore, LT chose the dividend yield as the state variable and the dividend yield

⁸See, for example, Fama and Schwert (1977) or Bottazzi et al. (1996).

⁹For further details, the reader is referred to Moos (2011b).

¹⁰A continuous time version of this model in complete markets and a detailed discussion can be found in Moos (2011a) Chapter 3.

¹¹See, for example, Kim and Omberg (1996), Wachter (2002) or Campbell et al. (2004).

¹²See Cocco et al. (2005), Huang et al. (2008), Huang and Milevsky (2008), Bick et al. (2009) and Dybvig and Liu (2010).

has a correlation to equity close to -1 . Thus, the assumption $\rho_{sx} = -1$ is, in this case, not too restrictive. Indeed, the reported correlation of the state variable and the risky asset in Table 1 of LT is given by -0.8835 . Moreover, the regression of the dividend yield on labor income growth in Table 1 of LT is highly significant. However, we want to stress that for the remainder of this paper we do not interpret the factor of the LT model as the dividend yield but simply as a factor that relates labor income growth to the stock market.

In addition, if labor income volatility is rather low, the locally riskfree labor income case $\sigma_y = 0$ is certainly a reasonable approximation. Alternatively, the model can be thought of a representative agent model where the short-run labor income risks of the economy are low. Hence, it can be stated that despite these assumptions, the results of the model are not only of theoretical interest, but have implications for realistic cases.

Finally, these assumptions come with an advantage besides the interpretability of closed-form solutions. In fact, in the case of $\rho_{sy} \notin \{-1, 1\}$ and $\sigma_y > 0$, current financial wealth has to be higher than the reserves for the future subsistence consumption. This would be an unrealistic assumption, especially for young individuals who generally have a low financial wealth.

1.2 The BCG model in complete markets

BCG deduce the dynamics of the stock as a claim to the following dividend stream

$$\frac{dD(t)}{D(t)} = g_d dt + \sigma_d dZ(t)$$

Furthermore, it is assumed that the market price of risk for $Z(t)$ is given by $\theta \equiv \frac{\pi_0}{\sigma_d}$.

Thus, the dynamics of the pricing kernel for the dividend stream follows

$$\frac{d\Lambda(t)}{\Lambda(t)} = -r_0 dt - \theta dZ(t)$$

The price of the risky asset can be derived with the general asset pricing formula in continuous time¹³

$$P(t) = \frac{1}{\Lambda(t)} E_t \left[\int_t^\infty \Lambda(s) D(s) ds \right] \quad (5)$$

The solution of (5) is given by

$$P(t) = \frac{D(t)}{r_0 + \sigma_d \theta - g_d}$$

The constancy of the dividend-price ratio $\phi \equiv D(t)/P(t) = r_0 + \sigma_d \theta - g_d = r_0 + \pi_0 - g_d$ implies that the stock price dynamics and dividend dynamics are identical. In fact, the constancy of the dividend-price ratio implies cointegration not only between labor income and but cointegration between the gain process (capital gain plus dividends) with an additional deterministic drift. The gain process follows

$$\begin{aligned} \frac{dS(t)}{S(t)} = \frac{dP(t) + D(t) dt}{P(t)} &= (g_d + r_0 + \sigma_d \theta - g_d) dt + \sigma_d dZ(t) \\ &= (r_0 + \pi_0) dt + \sigma_s dZ(t) \end{aligned} \quad (6)$$

where $\sigma_s = \sigma_d$ and the dynamics are identical to (2). In contrast to LT, instead of specifying the dynamics of labor income directly, BCG start with the specification of the cointegration between labor income and the stock market. BCG define the log-income-dividend ratio as

$$\tilde{X}(t) \equiv \ln \left[\frac{Y(t)}{D(t)} \right] - \overline{y\bar{d}}$$

where $\overline{y\bar{d}}$ is the long-run mean of the log-income-dividend ratio. Then, BCG assume that the dynamics of $\tilde{X}(t)$ are given by

$$d\tilde{X}(t) = -\kappa \tilde{X}(t) dt + \rho_{sx} \sigma_x dZ(t) \quad (7)$$

where we have to assume $\rho_{sx} \in \{-1, 1\}$ because of the complete market assumption. It should be noticed that in the base case of the numerical calibration of BCG¹⁴ $\rho_{sx} =$

¹³See, for example, Cochrane (2005).

¹⁴From BCG p. 2145: $\rho_{sx} = -\sigma v_3 / (\sqrt{\sigma^2 v_1^2 + v_3^2}) = -v_3 / \sqrt{v_1^2 + v_3^2} = -0.16 / \sqrt{0.05^2 + 0.16^2} = -0.9545$.

-0.9545 , which is close to -1 and hence, this assumption is not too restrictive.

The dynamics of labor income can be found by the application of the Itô-lemma

$$\frac{dY(t)}{Y(t)} = \left(r_0 + \pi_0 - \phi + \frac{1}{2}\sigma_x^2 + \rho_{sx}\sigma_s\sigma_x - \kappa\tilde{X}(t) \right) dt + (\sigma_s + \rho_{sx}\sigma_x) dZ(t) \quad (8)$$

Two points should be noted from (8). Firstly, (8) has the same structure as (3) with

$$X(t) = \tilde{X}(t), \quad y_0 \equiv r_0 + \pi_0 - \phi + \frac{1}{2}\sigma_x^2 + \rho_{sx}\sigma_s\sigma_x, \quad y_1 \equiv -\kappa$$

Hence, it is straightforward that the models can be solved by the same methods. This statement will even become more clear in the next section, where we adapt the BCG notation to the notation of LT. Secondly, in the cointegration model of BCG in complete markets, labor income is locally riskfree under the additional assumption¹⁵ $\sigma_s = -\rho_{sx}\sigma_x$.

Finally, it should be noticed that the simple dividend-stock price model is clearly not in line with the empirical literature. BCG are aware of this issue and in return, BCG state (p. 2130):

“We note that this simple model predicts the counterfactual result that the volatility of the dividend growth rate, σ , is identical to the stock return volatility. We emphasize, however, that only the stock return volatility is relevant for the agent’s portfolio decision. As such, we fix σ to match historical stock return volatility in our calibration below.”

However, this statement is only partially correct. Indeed, it is well known that the unconditional volatility of (7) is given by

$$\sigma_{\tilde{X}} \equiv \sqrt{\frac{\sigma_x^2}{2\kappa}}$$

Under the additional assumption of locally riskfree labor income (in complete markets) or no contemporaneous correlation between labor income and the stock (in incomplete

¹⁵This is exactly the assumption of BCG in their basic model parametrization in order to ensure that labor income and stock returns are contemporaneously uncorrelated.

markets) as assumed by BCG, $\sigma_x = \sigma_s$. Thus, the state variable has a variation comparable to stock variation and not comparable to dividend variation. As a consequence, the time-varying part of labor income $\kappa \tilde{X}(t)$ becomes high in magnitude. In the numerical calibration of BCG, the time-varying part of labor income growth varies from $(-0.0459, 0.0459)$ within one standard deviation of the state variable¹⁶. More precisely, the time-varying part of labor income growth measured by

$$\kappa \sigma_{\tilde{X}} = \sqrt{\kappa \frac{\sigma_x^2}{2}}$$

grows in the square root of κ and it becomes clear that variation in income growth can only be low if cointegration is weak. As will be shown below, this property implies that if cointegration is weak (low κ), the model is not able to generate low risky investment for young individuals and if cointegration is strong (high κ), variation in income growth is unreasonably high. This property is clearly a drawback of the BCG model. On the other hand, the basic cointegration framework is theoretically appealing.

1.3 The Connection Between the LT and the BCG Model

In this step we unify the notation for the gain process and labor income of both models. For this purpose, we derive the BCG model a second time from an other direction. Instead of working with the income-dividend ratio, we will work with the income-gain process ratio

$$\tilde{X}(t) = \ln \left[e^{\eta t} \frac{Y(t)}{S(t)} \right]$$

¹⁶For the numerical calibration of BCG $\sigma_{\tilde{X}} = \sqrt{(v_1^2 + v_3^2) / (2\kappa)} = \sqrt{(0.05^2 + 0.16^2) / (2 \cdot 0.15)} = 0.3061 \Rightarrow \pm \kappa \sigma_{\tilde{X}} = \pm 0.0459$.

where η can be chosen that $\tilde{X}(t)$ represents still the income-dividend ratio.

$$\begin{aligned}
&= \eta t + \ln \left[\frac{Y(t)}{D(t)} \frac{D(t)}{S(t)} \right] \\
&= \eta t + \ln \left[\frac{Y(t)}{D(t)} \right] - \ln \left[\frac{D(0)}{S(0)} \right] + (g_d - r_0 - \pi_0) t \\
&= \eta t + \ln \left[\frac{Y(t)}{D(t)} \right] - \ln \left[\frac{D(0)}{S(0)} \right] - \phi t
\end{aligned}$$

It should be noticed that compared to the income-dividend ratio, the income-gain process ratio has an additional drift with the size of the dividend yield, which is intuitive from (6). Without loss of generality, we can assume that $D(0) = S(0)$ and hence, for $\eta = \phi$, $\tilde{X}(t)$ is the income-dividend ratio¹⁷. The advantage of this notation is that we get rid of one variable (dividends) for optimization. However, it should be noticed that η can be chosen freely and the link via dividends is not necessary and we can interpret $\tilde{X}(t)$ simply as a variable that measures ratio between the gain process and labor income in the presence of a deterministic drift, which could represent a risk premium. In analogy to LT, we assume that labor income follows

$$\frac{dY(t)}{Y(t)} = (y_0 + y_1 \tilde{X}(t)) dt \quad (9)$$

From the dynamics of the gain process (2) and the dynamics of labor income (9), we derive the dynamics of the state variable $\tilde{X}(t)$ with the application of the Itô-lemma

$$\begin{aligned}
d\tilde{X}(t) &= \left(y_0 + \eta - r_0 - \pi_0 + \frac{1}{2} \sigma_s^2 + y_1 \tilde{X}(t) \right) dt - \sigma_s dZ(t) \\
&= y_1 \left(\tilde{X}(t) - \frac{r_0 + \pi_0 - \frac{1}{2} \sigma_s^2 - y_0 - \eta}{y_1} \right) dt - \sigma_s dZ(t)
\end{aligned} \quad (10)$$

It should be noticed that there is only mean-reversion if $y_1 < 0$. This is intuitive, as the model implies that cointegration stems from the correction of labor income to the stock market development. A stock market downturn leads to a higher log-income-gain process ratio and income growth must decline in order to restore equilibrium.

¹⁷It should be noticed that BCG take implicitly also into account the possibility of a deterministic drift between labor income and dividends. This can be seen from Table 1 in BCG.

More importantly, the model of BCG can be fitted into the model of LT with some constraints on the parameter. In fact, if

$$\kappa = -y_1, \quad \bar{X} = \frac{r_0 + \pi_0 - \frac{1}{2}\sigma_s^2 - y_0 - \eta}{y_1}, \quad \sigma_x = -\sigma_s$$

the LT model captures the BCG model.

Hence, despite the different economic motivation of the two models, both models share the same types of stochastic processes and can be solved by the same method. From this perspective, it does not come as a surprise that both models generate similar results. Moreover, we can conclude that the basic model of BCG is a special case of the basic model of LT with some stronger restrictions on the parameters of the system risky asset, labor income and state variable.

2 The Value of Human Capital

We assume that consumption $c(t)$ at time t leads to an utility

$$u(c(t)) = \frac{e^{-\delta t}}{1-\gamma} (c(t) - \bar{c})^{1-\gamma}, \quad \gamma > 1 \quad (11)$$

The utility function is well known and belongs to the class of HARA utility. Both LT and BCG consider only power utility, which is covered by setting $\bar{c} = 0$. The financial market and the dynamics of labor income are chosen in analogy to LT and BCG and are given as in Section 1.

We derive the optimal policies and the value of one unit of income in a infinite horizon framework with life-time uncertainty¹⁸. Moreover, since the one factor models are special cases of the more general two factor model, we derive the solution of the life-cycle model for the two factor model.

¹⁸For the sake of readability the derivations are done in Appendix A.1.

2.1 One Factor Model

Because of the assumption of complete market, the optimization problem of the more general LT system (1), (2), (3) and (4) can be solved analytically and the value of one unit of labor income is given as in Proposition 1:

Proposition 1 *The value of one unit of labor income $Y(t)$ is given by*

$$k(\tau, X) = \int_0^\tau e^{d_0(s)+d_1(s)X} ds \quad (12)$$

where $d_1(s)$ and $d_0(s)$ are the solution to the following system of ordinary differential equations

$$\frac{\partial d_1(s)}{\partial s} = l_0 + l_1 d_1(s) \quad (13)$$

$$\frac{\partial d_0(s)}{\partial s} = l_2 + l_3 d_1(s) + l_4 d_1(s)^2 \quad (14)$$

with initial condition $d_0(0) = d_1(0) = 0$ and where

$$l_0 \equiv y_1, \quad l_1 \equiv -\kappa$$

$$l_2 \equiv y_0 - r_0, \quad l_3 \equiv \kappa \bar{X} - \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_0, \quad l_4 \equiv \frac{1}{2} \sigma_x^2$$

The solution of equation (13) with initial condition $d_1(0) = 0$ is given by

$$\begin{aligned} d_1(s) &= \frac{l_0}{l_1} (e^{l_1 s} - 1) \\ &= -\frac{y_1}{\kappa} (e^{-\kappa s} - 1) \end{aligned} \quad (15)$$

Because of the simple form of $d_1(s)$ and equation (14), the solution of $d_0(s)$ is also available in closed-form. Simple integration yields

$$d_0(s) = \left(l_2 - l_3 \frac{l_0}{l_1} + l_4 \frac{l_0^2}{l_1^2} \right) s + \left(l_3 \frac{l_0}{l_1^2} - 2l_4 \frac{l_0^2}{l_1^3} \right) (e^{l_1 s} - 1) + \frac{1}{2} l_4 \frac{l_0^2}{l_1^3} (e^{2l_1 s} - 1) \quad (16)$$

The sensitivity of one unit of income across states is measured by

$$\frac{\partial k(\tau, X)}{\partial X} = \int_0^\tau d_1(s) e^{d_0(s)+d_1(s)X} ds \quad (17)$$

Furthermore, this expression will also be important for hedging demand of state variable risk in labor income. From (17) it should be recognized that $d_1(s)$ is the key for this measure and thus it should be understood in detail. The following important properties of the LT model should be noted¹⁹:

1. The solution to the linear differential equation (13) converges to a finite solution $\bar{d}_1 = -l_0/l_1$. Nevertheless, as shown in Figure 1, for low values of κ , which means high persistency of labor income growth, $|\bar{d}_1| = |\frac{y_1}{\kappa}|$ can become high in magnitude.
2. $d_1(s)$ has the same sign as l_0 .
3. $d_1(s)$ is monotone.

The proof for the properties can be found in Appendix A.3.

Insert Figure 1 about here.

The properties have an economic interpretation. In fact, the mean reversion property of the state variable implies that labor income growth will return to its long-run growth path. $d_1(s)$ measures the sensitivity of the valuation of the income stream across states and a bounded $d_1(s)$ means that deviations are not valued too strongly. Furthermore, in case of low mean reversion, $d_1(s)$ becomes larger in magnitude, which is intuitive since the individual anticipates that it will take longer until labor income growth returns to the long-run growth path. The second property simply states that the valuation of labor income increases (decreases) with the state variable if labor income growth increases (decreases) with the state variable. The third property implies that the time horizon matters, i.e. if the time horizon is low, $d_1(s)$ is close to zero as the individual knows that the short remaining time horizon implies only a limited time frame to exploit the deviations from the long-run growth path.

¹⁹More details can be found in Moos (2011b).

The additional parameter restriction of the BCG model imply

$$l_0 \equiv y_1, \quad l_1 \equiv y_1$$

$$l_2 \equiv y_0 - r_0, \quad l_3 \equiv y_0 + \eta - r_0 - \pi_0 + \frac{1}{2}\sigma_s^2 - \rho_{sx}\lambda_0, \quad l_4 \equiv \frac{1}{2}\sigma_s^2$$

Plugging in these parameter into the general solution leads to even simpler solutions for $d_1(s)$ and $d_0(s)$ of the BCG model. Specifically,

$$\begin{aligned} d_1(s) &= e^{l_1 s} - 1 \\ &= e^{y_1 s} - 1 \end{aligned} \tag{18}$$

and

$$d_0(s) = (l_2 - l_3 + l_4)s + (l_3 - 2l_4)\frac{1}{l_1}(e^{l_1 s} - 1) + l_4\frac{1}{2l_1}(e^{2l_1 s} - 1) \tag{19}$$

Compared to the LT model, the BCG model has the following properties:

1. The solution to the linear differential equation (13) converges to an unambiguous finite solution $\bar{d}_1 = -l_0/l_1 = -1$.
2. $d_1(s) \leq 0$.
3. $d_1(s)$ is monotone.

It should be noticed that for values of y_1 close to zero, $d_1(s)$ is close to zero unless the time horizon is long. This is intuitive because a low y_1 implies only a weak correction of labor income growth on the stock market development.

One desirable property of the BCG model is that the stock and labor income are cointegrated in the sense that the ratio has a stable long-run mean. In the more general case of LT, the log-income-gain process ratio follows

$$d\tilde{X}(t) = y_1 \left(X(t) - \frac{r_0 + \pi_0 - \frac{1}{2}\sigma_s^2 - y_0 - \eta}{y_1} \right) dt - \sigma_s dZ(t)$$

and for the stability of this ratio, one has look at the system of two stochastic linear differential equations

$$\begin{pmatrix} d\tilde{X}(t) \\ dX(t) \end{pmatrix} = \left[\underbrace{\begin{pmatrix} a_1 \\ \kappa\tilde{X} \end{pmatrix}}_{\equiv a} + \underbrace{\begin{pmatrix} 0 & y_1 \\ 0 & -\kappa \end{pmatrix}}_{\equiv A} \begin{pmatrix} \tilde{X}(t) \\ X(t) \end{pmatrix} \right] dt + \underbrace{\begin{pmatrix} -\sigma_s \\ \rho_{sx}\sigma_x \end{pmatrix}}_{\equiv b} dZ(t) \quad (20)$$

where

$$a_1 \equiv y_0 + \eta - r_0 - \pi_0 + \frac{1}{2}\sigma_s^2$$

It is well-known that the mean of

$$\mathbf{m}(s) \equiv E_t \left[\begin{pmatrix} \tilde{X}(s) & X(s) \end{pmatrix}' \right]$$

is the solution to the following system of deterministic differential equations and the following initial condition²⁰

$$\frac{d\mathbf{m}(s)}{ds} = a + A\mathbf{m}(s), \quad \mathbf{m}(t) = \begin{pmatrix} \tilde{X}(t) & X(t) \end{pmatrix}'$$

For the stability of this system, we have to check the eigenvalues²¹ of A . Since the second column is zero, one of the two eigenvalues is zero. The second eigenvalue is equal to y_1 .

Although that the state variable is mean-reverting and converges to its long-run, the log-income-gain process ratio does not have a long-run mean. The lack of an long-run equilibrium is at odds with economic theory and we suggest that an economically meaningful model should capture this issue.

Insert Figure 2 about here.

The result is illustrated in the phase plan analysis in the panel to the left of Figure 2. As can be seen, only $X(t)$ will converge and $\tilde{X}(t)$ meanders without a long-run equilibrium. Summing up, in the LT model only income growth returns to a long-run

²⁰See, for example, Duffie (2001) Appendix E.

²¹See, for example, Simon and Blume (1994).

mean, while in the BCG model labor income growth and the log-income-gain process ratio return to a long-run mean but there is a rather restrictive dependence between income growth and the long-run equilibrium ratio.

2.2 Two Factor Model

A resolution to this issue is to combine the approaches of LT and BCG. Specifically, we introduce a two factor model into the dynamics of labor income. It should be noticed that the application of matrix notation implies that the solution is not restricted to the two factor case. However, for the sake of simplicity we interpret the results in a two factor framework with a cointegration factor similar to the factor in the BCG model, which ensures an equilibrium between the stock and the labor market. The other factor is a free factor, which is related to the risky asset²².

Admittedly, the assumption of complete markets implies a strong dependence of the two factors and a direct calibration of the model on empirical data is not reasonable. Nevertheless, we suggest that our results are of importance. In fact, several studies showed that the results of models in incomplete markets are qualitatively similar to the results of corresponding complete market models²³. Moreover, numerical solution approaches come along with several critical issues. For example, the boundary conditions applied in the numerical solution of the BCG model are unknown and were found by experimentation²⁴. Moreover, optimization problem with multiple state variables suffer from the curse of dimensionality, i.e. computational time is fast rising with the number

²²It should be noticed that the model could easily be extended to a multi asset model with, for example, an additional long-term bond as in Munk and Sørensen (2010). In such a framework, the free factor must clearly not be a stock market factor but could be a bond market factor as, for example, the short interest rate or a term spread. Hence, the model can be applied to a wide scope of optimization problems.

²³See, for example, Cocco et al. (2005), Huang et al. (2008), Huang and Milevsky (2008), Bick et al. (2009), Munk and Sørensen (2010) and Dybvig and Liu (2010).

²⁴See Appendix A in BCG.

of state variables and numerical stability of the results becomes more difficult. For these reasons, we believe that well understood benchmark models are of high importance.

We assume that labor income follows

$$\frac{dY(t)}{Y(t)} = [y_0 + \mathbf{y}'_1 \mathbf{X}(s)] dt \quad (21)$$

where $\mathbf{y}_1 \equiv \begin{pmatrix} y_{11} & y_{12} \end{pmatrix}'$ and $\mathbf{X}(t) \equiv \begin{pmatrix} \tilde{X}(s) & X(s) \end{pmatrix}'$. This model is able to combine the good properties of both models presented above. In fact, the cointegration factor ensures that the model has an economically meaningful background, while the free factor allows for more flexibility.

In the two factor case, the log-income-gain process ratio follows

$$d\tilde{X}(t) = \left(y_0 + \eta - r_0 - \pi_0 + \frac{1}{2}\sigma_s^2 + \mathbf{y}'_1 \mathbf{X}(s) \right) dt - \sigma_s dZ(t)$$

and the system (20) adapts to

$$\begin{pmatrix} d\tilde{X}(t) \\ dX(t) \end{pmatrix} = \left[\underbrace{\begin{pmatrix} a_1 \\ \kappa \bar{X} \end{pmatrix}}_{\equiv a} + \underbrace{\begin{pmatrix} y_{11} & y_{12} \\ 0 & -\kappa \end{pmatrix}}_{\equiv A} \begin{pmatrix} \tilde{X}(t) \\ X(t) \end{pmatrix} \right] dt + \underbrace{\begin{pmatrix} -\sigma_s \\ \rho_{sx}\sigma_x \end{pmatrix}}_{\equiv b} dZ(t) \quad (22)$$

In analogy to above, we check the stability of the mean vector $\mathbf{m}(s)$. Since the eigenvalues of a diagonal matrix are equal to the diagonal elements of the matrix, the mean of the system is stable if $y_{11} < 0$, which is exactly the assumption of BCG in their one factor model. Of course, if y_{11} is close to zero mean reversion of the stock-income ratio is low. The right panel of Figure 2 illustrates that $\tilde{X}(t)$ will converge if $y_{11} < 0$.

For the remainder of this paper we assume that

$$y_{11} \leq 0 \quad (\text{a.1})$$

As shown in Appendix A.1, in the two factor case, the value of the future income stream is given by²⁵.

²⁵It should be noticed that the diagonal form of matrix A is not crucial for the results of Proposition

Proposition 2 *The value of one unit of labor income $Y(t)$ is given by*

$$k(\tau, \mathbf{X}) \equiv \int_0^\tau e^{d_0(s) + \mathbf{X}(s)' \mathbf{d}_1(s)} ds \quad (23)$$

where $\mathbf{d}_1(s)$ and $d_0(s)$ are the solution to the following system of ordinary differential equations

$$\frac{\partial \mathbf{d}_1(s)}{\partial s} = \mathbf{y}_1 + \mathbf{A}' \mathbf{d}_1(s) \quad (24)$$

$$\frac{\partial d_0(s)}{\partial s} = y_0 - r_0 + \left(\mathbf{a}' - \frac{\pi_0}{\sigma_s} \mathbf{b}' \right) \mathbf{d}_1(s) + \frac{1}{2} \mathbf{b}' \mathbf{d}_1(s) \mathbf{d}_1(s)' \mathbf{b} \quad (25)$$

with $d_0(0) = 0$ and $\mathbf{d}_1(0) = \mathbf{0}$ and where $\mathbf{a}$, $\mathbf{A}$ and $\mathbf{b}$ are defined in equation (22).

The first equation (24) is a system of two linear differential equations, the second equation (25) can be solved by integration. Since the first equation has the same eigenvalues as the system that is defining the mean vector, the solution converges under the same conditions.

$\mathbf{d}_1(s)$ can be solved in closed-form, which is done in Appendix A.2. Specifically, the solution is²⁶

$$\mathbf{d}_1(s) = \begin{pmatrix} e^{y_{11}s} - 1 \\ -\frac{y_{12}}{y_{11} + \kappa} (1 - e^{(y_{11} + \kappa)s}) e^{-\kappa s} \end{pmatrix} \quad (26)$$

It should be noticed that if either y_{11} or y_{12} is equal to zero, the solution collapses to the solution of the aforementioned one factor models. Furthermore, the solution of the part for the cointegration factor is one-to-one identical to above.

Because of the simple form of $\mathbf{d}_1(s)$, $d_0(s)$ can be found in closed-form. Nevertheless, the resulting formula is long and tedious. Moreover, for the sensitivity of the value of human capital and the optimal policies on the state variable, $d_0(s)$ is not of importance.

2. Hence, even more general system of state variables could be analyzed in this framework (with possibly complex eigenvalues). For the sake of brevity we leave this issue for future research.

²⁶It should be noticed that the case $y_{11} + \kappa = 0$ is well defined. For the sake of brevity, details for this special case are omitted.

For these reasons, we do not explicitly show the closed-form solution of $d_0(s)$ and simply state²⁷

$$d_0(s) = \int_0^\tau \left(y_0 - r_0 + \left(\mathbf{a}' - \frac{\pi_0}{\sigma_s} \mathbf{b}' \right) \mathbf{d}_1(s) + \frac{1}{2} \mathbf{b}' \mathbf{d}_1(s) \mathbf{d}_1(s)' \mathbf{b} \right) ds$$

Since the solution of the cointegration factor is unchanged, the same properties as in the BCG model are valid. For the free factor the evolution of $\mathbf{d}_1(s)$ is more interesting:

1. The solution of $d_{12}(s)$ converges to zero.
2. $d_{12}(s)$ has the same sign as y_{12} .
3. $d_{12}(s)$ is hump-shaped and reaches its minimum ($y_{12} < 0$) or maximum ($y_{12} > 0$) at $s^* = \frac{\ln(-y_{11}/\kappa)}{y_{11} + \kappa}$.
4. Given the sensitivity of labor income growth on the free factor y_{12} , the presence of a cointegration factor decreases the importance of the free factor. In fact,

$$\left| -\frac{y_{12}}{y_{11} + \kappa} \left(1 - e^{(y_{11} + \kappa)s} \right) e^{-\kappa s} \right| \leq \left| -\frac{y_{12}}{\kappa} (e^{-\kappa s} - 1) \right|$$

The LHS (RHS) corresponds to the solution of the two (one) factor model.

The economic interpretation of the properties of the free factor is as follows. As shown in the phase plan analysis, the first property takes into account that in the long-run the level of labor income is exclusively determined by the cointegration factor. In other words, for the long-run level of labor income the free factor is not of importance and the convergence of $d_{12}(s)$ to zero mirrors this property of the economy. As before, the second property states that the valuation of future income varies directly with the impact of the free factor on labor income growth. Indeed, although that the free factor does not have an impact on the long-run level of labor income, the free factor has an impact on the path of labor income to its long-run level. The third property shows that

²⁷The closed-form solution is available from the author upon request.

the free factor is most important for intermediate horizon. In fact, as in the one factor model a change in $X(t)$ has an immediate impact on the growth rate of labor income and thus, $d_{12}(s)$ rises in absolute terms for short horizons. On the other hand, as stated in the first property, in the long-run only the cointegration factor is important for the evolution of labor income and $d_{12}(s)$ fades for long horizons. The combination of these two effects lead to an extrema at an intermediate horizon. The forth property states that the presence of a cointegration factor leads to a lower importance of the value of human capital on the free factor for all horizons. This property takes into account that even weak cointegration has an immediate impact on the growth path of labor income, which relatively lowers the effect of the free factor on income growth.

The valuation of one unit of human capital shows the high importance of cointegration factors relative to other factors. In fact, the two factor model shows that in the presence of cointegration between the stock and labor market, the valuation of one unit of income depends considerably on the cointegration factor. Moreover, ignoring a cointegration factor and focusing on a factor with only temporary effect could lead to an overestimation of the importance of this factor, which could lead to severe utility losses. Based on these results, we suggest that further empirical research in the direction of financial and stock market cointegration and/or a clear macroeconomic theory are of high importance for life-cycle investment decisions.

3 Optimal Policies

3.1 One Factor Model

The optimal policies of the one factor models are stated in Proposition 3.

Proposition 3 *Investment into the risky asset is given by*

$$\alpha_t^* A(t) = \underbrace{\frac{1}{\gamma} \frac{\pi_0}{\sigma_s^2} \hat{A}(t)}_{\alpha_t^m: \text{ "myopic" }} - \underbrace{\frac{\rho_{sx} \sigma_x}{\sigma_s} \left(\int_0^\tau d_1(s) e^{d_0(s)+d_1(s)X} ds \right) Y}_{\alpha_t^h: \text{ "hedging" }}$$

for the LT model and given by

$$\alpha_t^* A(t) = \underbrace{\frac{1}{\gamma} \frac{\pi_0}{\sigma_s^2} \hat{A}(t)}_{\alpha_t^m: \text{ "myopic" }} + \underbrace{\left(\int_0^\tau d_1(s) e^{d_0(s)+d_1(s)X} ds \right) Y}_{\alpha_t^h: \text{ "hedging" }}$$

in the BCG model, respectively.

Total wealth $\hat{A}(X, t) = A(t) + k(\tau, X) Y(t) + R$ is the sum of financial wealth, human capital and the reserves that are necessary to cover future subsistence consumption, which are given by

$$R = -\frac{\bar{c}}{r_0}$$

Optimal consumption is given by

$$c_t^* = \frac{\hat{A}(t, X)}{h(\tau)} + \bar{c}$$

where

$$h(\tau) = \gamma \left\{ \left(\frac{1}{\Delta_r} - \frac{1}{\Delta_e} \right) e^{-\frac{\Delta_e}{\gamma} \tau} + \frac{1}{\Delta_e} \right\}$$

and

$$\Delta_i \equiv (\delta + \lambda_i) - (1 - \gamma) \left[r_0 + \frac{1}{2\gamma} \frac{\pi_0^2}{\sigma_s^2} \right], \quad i \in \{e, r\}$$

The proof is a special case of the proof of Proposition 4 and therefore omitted.

It should be noticed that the second part of the RHS of risky investment is hedging demand for state variable risk in the future income stream. Indeed, the second part is the first derivative of the value of one unit of human capital with respect to the state variable

$$\frac{\partial k(\tau, X)}{\partial X} = \int_0^\tau d_1(s) e^{d_0(s)+d_1(s)X} ds$$

Hence, the following properties result

1. In the BCG model, hedging demand is negative and reduces the demand for the risky asset.
2. In the LT model, the sign of $-\rho_{sx}d_1(s)$ determines the sign of hedging demand.

This follows from the fact that the sign of the first derivative of one unit of labor income with respect to $X(t)$ is equal to the sign of $d_1(s)$, which is equal to the sign of the sensitivity of labor income growth (y_1) on X . Since the discussion from above revealed that in the LT model $d_1(s)$ can become high in magnitude if the state variable is persistent, risky investment can be reduced substantially for long horizon (young individuals), which may lead to no or even negative investment in the risky asset. Hence, it can be stated that the analytical results confirm the main results of LT²⁸ and BCG.

For the BCG model an even more precise statement with respect to the optimal risky investment strategy can be made.

1. Compared to a model with no labor income, risky investment can only be reduced if

$$\frac{1}{\gamma} \frac{\pi_0}{\sigma_s^2} < 1$$

This property explains the high sensitivity of the BCG model towards risk aversion²⁹. In fact, for low risk aversion parameter γ , $\frac{1}{\gamma} \frac{\pi_0}{\sigma_s^2}$ becomes greater than one and the positive impact of additional income on risky investment is stronger than the negative hedging desire. Moreover, from (18) it should be recognized that given a horizon, the closer y_1 is to zero, the closer $d_1(s)$ will be to zero as well. Hence, the model is not able to generate low or even negative equity exposure if the cointegration parameter y_1 is close to zero³⁰.

Moreover, for the sensitivity of myopic and hedging demand across states, we find the following properties

²⁸In the parameter set of LT $y_1 < 0$ and thus $d_1(s)$ is negative.

²⁹See, Figure 10 in BCG and the statement in Benzoni and Goldstein (2010, p. 3).

³⁰See, Figure 7 in BCG.

1. In the BCG model, myopic demand unambiguously decreases with $X(t)$ and hedging demand unambiguously increases with $X(t)$.
2. In the LT model, the sign of $d_1(s)$ determines the sign of the sensitivity of myopic demand and the sign of $-\rho_{sx}$ determines the sign of the sensitivity of hedging demand.

Finally, for both models it should be noticed that

1. Consumption increases (decreases) if total wealth increases (decreases).

This property follows directly from the sensitivity of consumption and total wealth on the state variable

$$\frac{\partial c_t^*}{\partial X} = \frac{1}{h(\tau)} \frac{\partial \hat{A}(t, X)}{\partial X} = \frac{1}{h(\tau)} \frac{\partial k(\tau, X)}{\partial X(t)} Y(t)$$

and $h(\tau) > 0$.

3.2 Two Factor Model

The optimal policies of the two factor models are stated in Proposition 4.

Proposition 4 *Investment into the risky asset is given by*

$$A(t) \alpha_t^* = \underbrace{\frac{1}{\gamma} \frac{\pi_0}{\sigma_s^2} \hat{A}(t, \mathbf{X})}_{\alpha_t^m: \text{ "myopic" }} + \underbrace{\left(\begin{matrix} 1 & -\frac{\rho_{sx}\sigma_x}{\sigma_s} \end{matrix} \right) \left(\int_0^\tau \mathbf{d}_1(s) e^{d_0(s) + \mathbf{X}(s)' \mathbf{d}_1(s)} ds \right) Y}_{\alpha_t^h: \text{ "hedging" }}$$

Total wealth $\hat{A}(t, \mathbf{X}) = A(t) + k(\tau, \mathbf{X}) Y(t) + R$ is the sum of financial wealth, human capital and the reserves that are necessary to cover future subsistence consumption, which are given as in Proposition 3.

Optimal consumption is given by

$$c_t^* = \frac{\hat{A}(t, \mathbf{X})}{h(\tau)} + \bar{c}$$

where $h(\tau)$ is defined as in Proposition 3.

It should be noticed that the qualitative properties of the two parts of optimal risky investment are similar to the properties of the corresponding factor of the one factor model. However, as shown in the discussion of Section 2.2 the impact of the free factor is reduced because of the presence of the cointegration factor. As a consequence the possibly high magnitude of $d_1(s)$ is reduced in the two factor model. Thus, since $d_{12}(s)$ is of lower magnitude, both the level of hedging demand and the sensitivity of hedging demand for this state variable is reduced and extreme results are less likely to occur.

As in the one factor model, optimal consumption depends directly on total wealth. Hence, the properties are similar to the properties of total wealth and a discussion is omitted.

It becomes clear that the close relation of the optimal policies on the value and the sensitivity of one unit of income suggest that the statement at the end of Section 2.2 is even more important. In fact, our results suggest further empirical research in the direction of financial and stock market cointegration and/or a clear macroeconomic theory are of high importance in order to examine the life-cycle decisions of an individual. Finally, our results also imply that not only cointegration between the labor and the stock market but cointegration in other markets should be taken into account. For example, Quan and Titman (1997) provide some evidence for cointegration between the stock and real estate market. In combination with the high importance of real estate in wealth accumulation over the life-cycle such consideration could also have a strong impact on the optimal decisions of an life-cycle investor. We leave this issue for future research.

4 Conclusion

This paper examines the economic framework of LT and BCG in complete markets. Based on our analytic results we show that the two models are closely related and that the BCG model can be considered as a special case of the LT model. For these reasons,

it seems natural that both models generate similar results.

The model of BCG includes a long-run equilibrium between the stock and the labor market, which seems appealing from a theoretical point of view. We introduce a two factor model with a cointegration factor similar to BCG and an additional factor with the higher flexibility of the LT framework. We show that under the presence of a cointegration factor, the importance of the second factor decreases. We conclude that long-run equilibria are important for the life-cycle decisions. In fact, ignoring a cointegration factor and focusing on a factor with only temporary effect could lead to a misspecification of the economy. As a consequence, recommendation based on such models could lead to severe utility losses. In this sense, our results suggest that further empirical research in the direction of financial and stock market cointegration and/or a clear macroeconomic theory are of high importance in order to thoroughly study the life-cycle consumption and investment decisions.

A Appendix

A.1 Solution of the Life Cycle Model

Since the one factor models are special cases of the more general two factor model, we derive the solution of the life-cycle model for the two factor model.

The life of an individual consists of a phase of employment and a phase of retirement.

Insert Figure 3 about here.

As illustrated in Figure 3 the individual is active in the labor market from time 0 till time T . Afterwards she retires. During the phase of employment the individual earns a dynamic labor income (21), consumes and invests in a riskless and a risky asset. During retirement, the individual has no non-financial income and has to ensure consumption from accumulated financial wealth. During the phase of employment (retirement), there

is a constant force of mortality λ_e (λ_r) and we assume $\lambda_r \geq \lambda_e \geq 0$.

The solution approach asks to work backwards from the end of the planning horizon, i.e. it is necessary to solve the HJB-equation of the phase of retirement first. In a second step, the problem of the phase of employment is solved and linked to the retirement phase.

A.1.1 Phase of Retirement

For $t \geq T$, it has to be noticed that during retirement consumption $c(t)$ must exceed the subsistence level $\bar{c}$. For this reason, feasible consumption plans over the phase of retirement exist under the assumption

$$\hat{A}(T) = A(T) - \frac{\bar{c}}{r_0} \geq 0 \quad (\text{a.2})$$

As will become more clear below, (a.2) ensures that at the beginning of the phase of retirement financial wealth and pension benefits are sufficient to afford the future subsistence consumption. The standard transversality condition

$$\Delta_r \equiv (\delta + \lambda_r) - (1 - \gamma) \left[r_0 + \frac{1}{2\gamma} \frac{\pi_0^2}{\sigma_s^2} \right] > 0 \quad (\text{t.1})$$

that is known from Merton (1971) adjusted for the hazard rate is obviously satisfied since we assume $\gamma > 1$.

During the phase of retirement, the dynamics of financial wealth are given by

$$dA(t) = [(r_0 + \alpha(t) \pi_0) A(t) - c(t)] dt + \alpha(t) \sigma_s A(t) dZ(t)$$

The corresponding HJB-equation is given by³¹

$$\begin{aligned} 0 = & J_t + \sup_{c_t} \left[\frac{e^{-\delta t}}{1 - \gamma} (c_t - \bar{c})^{1-\gamma} - J_A c_t \right] \\ & + \sup_{\alpha_t} \left[J_A A \alpha_t \pi_0 + \frac{1}{2} J_{AA} A^2 \alpha_t^2 \sigma_s^2 \right] \\ & + J_A A r_0 - \lambda_r J \end{aligned} \quad (27)$$

³¹For the sake of brevity, time subscripts of financial wealth are neglected.

In the presence of a constant force of mortality, the adaption of the HJB-equation by the term $-\lambda_r J$ is already shown by Merton (1971). The candidate for the value function is

$$J(t, A) = \frac{\bar{h}^\gamma e^{-\delta t}}{1 - \gamma} (A(t) + R)^{1-\gamma} \quad (28)$$

where $\bar{h}$ and R are constants that have to be determined.

Plugging in the relevant partial derivatives of (28) into the first order conditions and into the HJB-equation (27) leads to

$$\begin{aligned} 0 = & -\frac{\delta}{1-\gamma} (A + R) + \frac{\gamma}{1-\gamma} \bar{h} (A + R) + r_0 (A + R) - r_0 R - \bar{c} + \frac{1}{2\gamma} \frac{\pi_0^2}{\sigma_s^2} \\ & - \frac{\lambda_r}{1-\gamma} (A + R) \end{aligned} \quad (29)$$

Equation (29) can be separated into terms linear in A , which yields

$$\bar{h} = \frac{\gamma}{\Delta_r}, \quad \Delta_r \equiv (\delta + \lambda_r) - (1 - \gamma) \left[r_0 + \frac{1}{2\gamma} \frac{\pi_0^2}{\sigma_s^2} \right]$$

and constant terms³², which leads to

$$R = -\frac{\bar{c}}{r_0}$$

A.1.2 Phase of Employment

For $t \leq T$, we shall need the following assumption

$$\hat{A}(0) = A(0) + K(0) + R(0) \geq 0 \quad (\text{a.3})$$

Assumption (a.3) guarantees the existence of feasible consumption plans. The assumption is intuitive and states that the combination of initial financial wealth and income has to be sufficiently large to afford the subsistence level of consumption³³.

The solution for the phase of employment is more complicated and shown in detail.

Dynamics of financial wealth are given by

$$dA(t) = [(r_0 + \alpha(t) \pi_0) A(t) + Y(t) - c(t)] dt + \alpha(t) \sigma_s A(t) dZ(t)$$

³²Terms similar to A are directly set to zero.

³³Moreover, it is easy to show that given (r, T) , for $\bar{c} > 0$ the $(M(0), Y(0))$ set is restricted but not empty. If $\bar{c} = 0$, (a.3) is always fulfilled.

The HJB-equation is given by³⁴

$$\begin{aligned}
0 = & J_t + \sup_{c_t} \left[\frac{e^{-\delta t}}{1-\gamma} (c_t - \bar{c})^{1-\gamma} - J_A c_t \right] \\
& + \sup_{\alpha_t} \left[J_A A \alpha_t \pi_0 + \frac{1}{2} J_{AA} A^2 \alpha_t^2 \sigma_s^2 + \alpha_t A \sigma_s J_{A\mathbf{X}'} \mathbf{b} \right] \\
& + J_A A r_0 + J_A Y - \lambda_r J + J_Y (y_0 + y_1' \mathbf{X}) + J_{\mathbf{X}'} (\mathbf{a} + \mathbf{A} \mathbf{X}) + \frac{1}{2} \mathbf{b}' J_{\mathbf{X}\mathbf{X}'} \mathbf{b} \quad (30)
\end{aligned}$$

The first order conditions are given by

$$c_t^* = \left(e^{-\delta t} J_A \right)^{-\frac{1}{\gamma}} + \bar{c} \quad (31)$$

$$\alpha_t^* A = -\frac{J_A}{J_{AA}} \frac{\pi_0}{\sigma_s^2} - \frac{J_{A\mathbf{X}'}}{J_{AA}} \frac{\mathbf{b}}{\sigma_s} \quad (32)$$

The candidate for the value function is given by

$$J(t, \mathbf{X}, A, Y) = \frac{h(\tau)^\gamma e^{-\delta t}}{1-\gamma} (A(t) + k(\tau, \mathbf{X}) Y(t) + R)^{1-\gamma}$$

where we made a change in the time domain from t to $\tau \equiv T - t$. R is a constant, $h(\tau)$ is a function of time and $k(\tau, \mathbf{X})$ is a function of time and the two state variables that have to be determined. The relevant partial derivatives are given by

$$\begin{aligned}
J_\tau &= e^{-\delta(T-\tau)} \left\{ \frac{\gamma}{1-\gamma} h^{\gamma-1} \frac{\partial h}{\partial \tau} \hat{A}^{1-\gamma} + \frac{\delta}{1-\gamma} h^\gamma \hat{A}^{1-\gamma} + h^\gamma \hat{A}^{-\gamma} \frac{\partial k}{\partial \tau} Y \right\} \\
J_A &= e^{-\delta(T-\tau)} h^\gamma \hat{A}^{-\gamma}, \quad J_{AA} = -\gamma e^{-\delta(T-\tau)} h^\gamma \hat{A}^{-\gamma-1} \\
J_Y &= e^{-\delta(T-\tau)} h^\gamma \hat{A}^{-\gamma} k \\
J_{\mathbf{X}} &= e^{-\delta(T-\tau)} h^\gamma \hat{A}^{-\gamma} \frac{\partial k}{\partial \mathbf{X}} Y \\
J_{\mathbf{X}\mathbf{X}'} &= e^{-\delta(T-\tau)} \left\{ -\gamma h^\gamma \hat{A}^{-\gamma-1} \frac{\partial k}{\partial \mathbf{X}} \frac{\partial k}{\partial \mathbf{X}'} Y^2 + h^\gamma \hat{A}^{-\gamma} \frac{\partial^2 k}{\partial \mathbf{X} \partial \mathbf{X}'} Y \right\} \\
J_{A\mathbf{X}} &= -\gamma e^{-\delta(T-\tau)} h^\gamma \hat{A}^{-\gamma-1} \frac{\partial k}{\partial \mathbf{X}} Y
\end{aligned}$$

where $\hat{A} \equiv A + kY + R$. Plugging in into the FOC (31) and (32) yields

$$c_t^* = \frac{\hat{A}}{h} + \bar{c}$$

³⁴For the sake of brevity, time subscripts of financial wealth, labor income and the state variables are neglected.

$$\alpha_t^* A = \frac{1}{\gamma} \frac{\pi_0}{\sigma_s^2} \hat{A} - \frac{\partial k}{\partial \mathbf{X}'} \frac{\mathbf{b}}{\sigma_s} Y$$

Plugging in into the HJB-equation (30) and multiplying by $e^{\delta(T-\tau)} h^{-(\gamma-1)} \hat{A}^\gamma$ leads to

$$\begin{aligned} 0 = & -\frac{\gamma}{1-\gamma} \frac{\partial h}{\partial \tau} \hat{A} - \frac{\delta}{1-\gamma} h \hat{A} - h \frac{\partial k}{\partial \tau} Y + \frac{\gamma}{1-\gamma} \hat{A} - h \bar{c} + \frac{1}{2\gamma} \frac{\pi_0^2}{\sigma_s^2} h \hat{A} - h \frac{\partial k}{\partial \mathbf{X}'} \frac{\mathbf{b}}{\sigma_s} \pi_0 Y \\ & + h (y_0 + y_1' \mathbf{X}) + h \frac{\partial k}{\partial \mathbf{X}'} (\mathbf{a} + \mathbf{A} \mathbf{X}) Y + \frac{1}{2} h \mathbf{b}' \frac{\partial^2 k}{\partial \mathbf{X} \partial \mathbf{X}'} \mathbf{b} \\ & + h r_0 \hat{A} - h r_0 Y - h r_0 R + h Y - \frac{\lambda_e}{1-\gamma} h \hat{A} \end{aligned} \quad (33)$$

Separating (33) by $\hat{A}$ yields

$$\frac{\partial h}{\partial \tau} = -\frac{\Delta_e}{\gamma} h + 1 \quad (34)$$

where $\Delta_e \equiv (\delta + \lambda_e) - (1 - \gamma) \left[r_0 + \frac{1}{2\gamma} \frac{\pi_0^2}{\sigma_s^2} \right]$. (34) is a linear differential equation and the general solution is well known³⁵ with initial condition $h(\tau = 0) = \bar{h} = \frac{\gamma}{\Delta_r}$.

$$h(\tau) = \gamma \left\{ \left(\frac{1}{\Delta_r} - \frac{1}{\Delta_e} \right) e^{-\frac{\Delta_e}{\gamma} \tau} + \frac{1}{\Delta_e} \right\}$$

Separating (33) by Y yields the following partial differential equation

$$0 = -\frac{\partial k}{\partial \tau} + (y_0 + y_1' \mathbf{X}) k - r_0 k + \frac{\partial k}{\partial \mathbf{X}'} (\mathbf{a} + \mathbf{A} \mathbf{X}) - \frac{\partial k}{\partial \mathbf{X}'} \frac{\mathbf{b}}{\sigma_s} \pi_0 + \frac{1}{2} \mathbf{b}' \frac{\partial^2 k}{\partial \mathbf{X} \partial \mathbf{X}'} \mathbf{b} + 1 \quad (35)$$

A candidate for the solution of (35) is

$$k(\tau, \mathbf{X}) \equiv \int_0^\tau e^{d_0(s) + \mathbf{X}(s)' \mathbf{d}_1(s)} ds$$

where $\tau \equiv T - t$ with initial conditions³⁶ $d_0(s = 0) = 0$ and $\mathbf{d}_1(s = 0) = \mathbf{0}$. The relevant

³⁵See, for example, Zwillinger (1998).

³⁶These are the only initial conditions that solve (35).

partial derivatives of $k(t)$ are given by³⁷.

$$\begin{aligned} k_\tau &= \int_0^\tau \left(\frac{\partial d_0(s)}{\partial s} + \mathbf{X}(s)' \frac{\partial \mathbf{d}_1(s)}{\partial s} \right) e^{d_0(s) + \mathbf{X}(s)' \mathbf{d}_1(s)} ds + 1 \\ k_{\mathbf{X}} &= \int_0^\tau \mathbf{d}_1(s) e^{d_0(s) + \mathbf{X}(s)' \mathbf{d}_1(s)} ds \\ k_{\mathbf{X}\mathbf{X}'} &= \int_0^\tau \mathbf{d}_1(s) \mathbf{d}_1(s)' e^{d_0(s) + \mathbf{X}(s)' \mathbf{d}_1(s)} ds \end{aligned}$$

Plugging in the relevant partial derivatives into (35) yields

$$\int_0^\tau e^{d_0(s) + \mathbf{X}(s)' \mathbf{d}_1(s)} \left\{ \begin{array}{l} -\frac{\partial d_0(s)}{\partial s} - \mathbf{X}(s)' \frac{\partial \mathbf{d}_1(s)}{\partial s} + y_0 + \mathbf{X}(s)' \mathbf{y}_1 \\ -r_0 + \mathbf{a}' \mathbf{d}_1(s) + \mathbf{X}(s)' \mathbf{A}' \mathbf{d}_1(s) \\ -\frac{\pi_0}{\sigma_s} \mathbf{b}' \mathbf{d}_1(s) + \frac{1}{2} \mathbf{b}' \mathbf{d}_1(s) \mathbf{d}_1(s)' \mathbf{b} \end{array} \right\} = 0 \quad (36)$$

It should be noticed that (36) can be separated into terms linear in $\mathbf{X}(t)$ and constant terms and the following system of ordinary equations results

$$\begin{aligned} \frac{\partial \mathbf{d}_1(s)}{\partial s} &= \mathbf{y}_1 + \mathbf{A}' \mathbf{d}_1(s) \\ \frac{\partial d_0(s)}{\partial s} &= y_0 - r_0 + \left(\mathbf{a}' - \frac{\pi_0}{\sigma_s} \mathbf{b}' \right) \mathbf{d}_1(s) + \frac{1}{2} \mathbf{b}' \mathbf{d}_1(s) \mathbf{d}_1(s)' \mathbf{b} \end{aligned}$$

with $d_0(0) = 0$ and $\mathbf{d}_1(0) = \mathbf{0}$. The first equation is a system of two linear differential equations, the second can be solved by integration. Since the first equation has the same eigenvalues as the system that is defining the mean vector, the solution converges under the same conditions.

Finally, separating (33) by the constant terms gives

$$R = -\frac{\bar{c}}{r_0}$$

³⁷It should be noted that for k_τ , the following rule was applied:

$$f(a, b) = \int_b^a g(x) dx = G(a) - G(b)$$

$\Rightarrow$

$$\frac{\partial f(a, b)}{\partial a} = \frac{\partial G(a)}{\partial a} = g(a) - g(b) + g(b) = \int_b^a \frac{\partial g(x)}{\partial x} dx + g(b)$$

The results of Proposition 4 follow by plugging in the solutions into the FOCs (31) and (32).

A.2 Solution of the SODE of the Two Factor Model

We look at a diagonal system of two linear ordinary differential equation

$$\begin{pmatrix} \frac{\partial g_{11}(s)}{\partial s} \\ \frac{\partial g_{12}(s)}{\partial s} \end{pmatrix} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} + \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix} \begin{pmatrix} g_{11}(s) \\ g_{12}(s) \end{pmatrix}$$

with initial conditions $g_{11}(0) = g_{12}(0) = 0$. The second equation is an univariate linear differential equation. The derivation of the solution is well-known and omitted. We simply state

$$g_{12}(s) = \frac{m_2}{M_{22}} (e^{M_{22}s} - 1)$$

Plugging in the solution for $g_{12}(s)$ into the first equation yields

$$\frac{\partial g_{11}(s)}{\partial s} = \underbrace{m_1 + m_2 \frac{M_{12}}{M_{22}} (e^{M_{22}s} - 1)}_{\equiv f_0(s)} + \underbrace{M_{11}g_{11}(s)}_{\equiv f_1} \quad (37)$$

The general formula for the solution of a linear differential equation with time-varying coefficients

$$\frac{\partial d(s)}{\partial s} = f_0(s) + f_1(s) d(s)$$

is given by³⁸

$$d(s) = e^{F(s)} \int f_0(s) e^{-F(s)} ds + e^{F(s)} K \quad (38)$$

where

$$F(s) \equiv \int f_1(s) ds$$

and K is the constant of integration.

For the specific equation (37) we get

$$F(s) = M_{11}s \quad (39)$$

³⁸See, for example, Polyanin and Zaitsev (1995, p. 1).

and

$$\int f_0(s) e^{-F(s)} ds = -\frac{1}{M_{11}} \left(m_1 - m_2 \frac{M_{12}}{M_{22}} \right) e^{-M_{11}s} + m_2 \frac{M_{12}}{M_{22}(M_{22} - M_{11})} e^{(M_{22} - M_{11})s} \quad (40)$$

Plugging in (39) and (40) into (38) and setting taking into account the boundary condition $g_{12}(0) = 0$ leads to

$$K = \frac{1}{M_{11}} \left(m_1 - m_2 \frac{M_{12}}{M_{22}} \right) - m_2 \frac{M_{12}}{M_{22}(M_{22} - M_{11})} \quad (41)$$

Plugging in (41), leads to.

$$\mathbf{d}_1(s) = \begin{pmatrix} \frac{1}{M_{11}} \left(m_1 - m_2 \frac{M_{12}}{M_{22}} \right) (e^{M_{11}s} - 1) - m_2 \frac{M_{12}}{M_{22}(M_{22} - M_{11})} (e^{M_{11}s} - e^{M_{22}s}) \\ \frac{m_2}{M_{22}} (e^{M_{22}s} - 1) \end{pmatrix}$$

By adapting equation (24) to the notation used in this Appendix and plugging in the coefficients of equation (22), (26) can be found.

A.3 Proofs of Properties

A.3.1 One Unit of Income in the LT Model

Proof: The first property is obvious and follows from $l_1 = -\kappa < 0$. For the second property, it has to be noticed that in equation (15)

$$l_1^{-1} (e^{l_1 s} - 1) > 0$$

The relation between $d_1(s)$ and l_0 follows immediately. For the proof of the third property we need

$$\partial d_1(s) / \partial s = l_0 e^{l_1 s}$$

In connection with the second property it becomes clear that $\partial d_1(s) / \partial s$ and $d_1(s)$ have the same sign. ■

A.3.2 One Unit of Income in the BCG Model

Proof: The first property is clear. For the second property, it has to be noticed that in equation (18)

$$e^{l_1 s} - 1 \leq 0$$

since $l_1 = y_1$ has to be negative in order to ensure cointegration. For the third property

$$\partial d_1(s) / \partial s = l_1 e^{l_1 s} < 0$$

■

A.3.3 One Unit of Income in the 2 Factor Model

Proof:

1. The second line of equation (26) can be rewritten as

$$-\frac{y_{12}}{y_{11} + \kappa} (e^{-\kappa s} - e^{y_{11} s})$$

because $-\kappa < 0$ and $y_{11} < 0$, the term in the brackets converges to zero.

2. The second property follows from the fact that $\frac{1}{y_{11} + \kappa} (1 - e^{(y_{11} + \kappa)s})$ and $-e^{-\kappa s}$ are unambiguously negative.

3. Setting the first derivative with respect to s

$$\frac{\partial d_{12}(s)}{\partial s} = -\frac{y_{12}}{y_{11} + \kappa} (-\kappa e^{-\kappa s} - y_{11} e^{y_{11} s})$$

to zero yields $s^* = -\frac{\ln(-y_{11}/\kappa)}{y_{11} + \kappa} > 0$. In order to show the hump-shaped pattern, we draw from the intermediate value theorem. Firstly, it has to be noticed that $\partial d_{12}(s) / \partial s$ is continuous. Secondly, at $s = 0$ the first derivative has the same sign as y_{12} , for finite s the only point with $\partial d_{12}(s) / \partial s = 0$ is $s = s^*$ and $\lim_{s \rightarrow \infty} \partial d_{12}(s) / \partial s$ has the opposite sign of y_{12} and converges to zero.

4. This property can be found by comparing the first derivative with respect to s of the two factor model (LHS) and the one factor model ($y_{11} = 0$). From the second property it is known that $d_{12}(s)$ divided by y_{12} is non-negative. In fact,

$$\tilde{d}_{12}(s) \equiv -\frac{1}{y_{11} + \kappa} \left(1 - e^{(y_{11} + \kappa)s}\right) e^{-\kappa s} \geq 0$$

Now, we have to show that the first derivative of this expression of the one factor model grows faster than the two factor counterpart. After simple algebraic manipulations, we find that $\tilde{d}_{12}^{2F}(s) - \tilde{d}_{12}^{1F}(s)$ has the same sign as

$$y_{11} \frac{(e^{(y_{11} + \kappa)s} - 1)}{y_{11} + \kappa} \leq 0 \quad (42)$$

It is crucial to notice that $y_{11} + \kappa$ switches sign at $y_{11} = \kappa$, thus the second part of the RHS of (42) is always non-negative. From $y_{11} < 0$ in the two factor model, the result in (42) follows.

■

A.3.4 Optimal Policies in the 1 Factor Model

Proof: Since $-1 < d_1(s) \leq 0$,

$$-\frac{\partial k(\tau, X)}{\partial X} < k(\tau, X), \quad \tau > 0$$

Optimal risky investment can be rewritten as

$$A(t) \alpha_t^* = \frac{1}{\gamma} \frac{\pi_0}{\sigma_s^2} (A(t) + R) + \int_0^\tau \left(\frac{1}{\gamma} \frac{\pi_0}{\sigma_s^2} + d_1(s) \right) e^{d_0(s) + d_1(s)X} ds Y$$

The first part corresponds to the case without labor income. The second part cannot be negative if the term in the brackets is not negative.

■

A.3.5 Optimal Policies in the 2 Factor Model

Proof: For myopic demand of the BCG model

$$\frac{\partial \alpha_t^m}{\partial X} = \frac{1}{\gamma} \frac{\pi_0}{\sigma_s^2} \left(\int_0^\tau d_1(s) e^{d_0(s)+d_1(s)X} ds \right) Y(t) < 0 \quad (43)$$

because of the unambiguous negativity of $d_1(s)$ in the BCG model. For hedging demand

$$\frac{\partial \alpha_t^h}{\partial X} = \left(\int_0^\tau d_1(s)^2 e^{d_0(s)+d_1(s)X} ds \right) Y(t) > 0$$

because of the unambiguous positivity of $d_1(s)^2$.

For the sensitivity of myopic demand in the LT model the same formula as in (43) can be applied. However, the sign of $d_1(s)$ is ambiguous. For hedging demand

$$\frac{\partial \alpha_t^h}{\partial X} = -\frac{\rho_{sx}\sigma_x}{\sigma_s} \left(\int_0^\tau d_1(s)^2 e^{d_0(s)+d_1(s)X} ds \right) Y(t)$$

In analogy to above, the sign of $-\rho_{sx}$ determines the sensitivity of hedging demand in the LT model. ■

References

- L. Benzoni and R. Goldstein. Lifecycle investment decisions and labor income risk. *FRBSF Economic Letter*, 2010:21, 2010.
- L. Benzoni, P. Collin-Dufresne, and R.S. Goldstein. Portfolio Choice over the Life-Cycle when the Stock and Labor Markets Are Cointegrated. *The Journal of Finance*, 62(5): 2123–2167, 2007. ISSN 1540-6261.
- B. Bick, H. Kraft, and C. Munk. Investment, Income, and Incompleteness. *Working Paper*, 2009.
- L. Bottazzi, P. Pesenti, and E. Van Wincoop. Wages, profits and the international portfolio puzzle. *European Economic Review*, 40(2):219–254, 1996.

- J.Y. Campbell. Household Finance. *The Journal of Finance*, 61(4):1553–1604, 2006. ISSN 1540-6261.
- J.Y. Campbell, G. Chacko, J. Rodriguez, and L.M. Viceira. Strategic Asset Allocation in a Continuous-Time VAR Model. *Journal of Economic Dynamics and Control*, 28(11):2195–2214, 2004. ISSN 0165-1889.
- J.F. Cocco, F.J. Gomes, and P.J. Maenhout. Consumption and Portfolio Choice over the Life Cycle. *Review of Financial Studies*, 18(2):491, 2005. ISSN 0893-9454.
- J.H. Cochrane. *Asset Pricing*. Princeton Univ Pr, 2005. ISBN 0691121370.
- S. Curcucu, J. Heaton, D. Lucas, and D. Moore. Heterogeneity and portfolio choice: Theory and evidence. In Y. Ait-Sahalia and L.P. Hansen, editors, *Handbook of Financial Econometrics*. North Holland, Netherlands, 2009.
- D. Duffie. *Dynamic Asset Pricing Theory*. Princeton Univ Pr, 2001. ISBN 069109022X.
- P.H. Dybvig and H. Liu. Lifetime Consumption and Investment: Retirement and Constrained Borrowing. *Journal of Economic Theory*, 145(3):885–907, 2010. ISSN 0022-0531.
- E.F. Fama and G.W. Schwert. Human capital and capital market equilibrium. *Journal of Financial Economics*, 4(1):95–125, 1977.
- A. Goyal and I. Welch. A Comprehensive Look at the Empirical Performance of Equity Premium Prediction. *Review of Financial Studies*, 21(4):1455, 2008. ISSN 0893-9454.
- H. Huang and M.A. Milevsky. Portfolio Choice and Mortality-Contingent Claims: The General HARA Case. *Journal of Banking & Finance*, 32(11):2444–2452, 2008. ISSN 0378-4266.

- H. Huang, M.A. Milevsky, and J. Wang. Portfolio Choice and Life Insurance: The CRRA Case. *Journal of Risk and Insurance*, 75(4):847–872, 2008. ISSN 1539-6975.
- T.S. Kim and E. Omberg. Dynamic Nonmyopic Portfolio Behavior. *Review of Financial Studies*, 9(1):141–161, 1996. ISSN 0893-9454.
- A.W. Lynch and S. Tan. Labor Income Dynamics at Business-Cycle Frequencies: Implications for Portfolio Choice. *Journal of Financial Economics*, 101:333–359, 2011. ISSN 0304-405X.
- R.C. Merton. Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case. *The Review of Economics and Statistics*, 51(3):247–257, 1969. ISSN 0034-6535.
- R.C. Merton. Optimum Consumption and Portfolio Rules in a Continuous-Time Model. *Journal of Economic Theory*, 3(4):373–413, 1971. ISSN 0022-0531.
- D. Moos. *Optimal Consumption and Portfolio Choice with Dynamic Labor Income*. Dissertation University of St. Gallen. Difo-Druck, Bamberg, 2011a.
- D. Moos. Portfolio and Consumption Decisions under Mean-Reverting Returns and Mean-Reverting Labor Income Growth. *Working Paper University of St. Gallen*, 2011b.
- C. Munk and C. Sørensen. Dynamic Asset Allocation with Stochastic Income and Interest Rates. *Journal of Financial Economics*, 96(3):433–462, 2010. ISSN 0304-405X.
- AD Polyanin and VF Zaitsev. *Handbook of Exact Solutions for Differential Equations*. CRC Press, Boca Raton, FL, 1995.
- D.C. Quan and S. Titman. Commercial real estate prices and stock market returns: An international analysis. *Financial Analysts Journal*, pages 21–34, 1997.
- C.P. Simon and L. Blume. *Mathematics for Economists*, volume 7. Norton, 1994.

L.M. Viceira. Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income. *The Journal of Finance*, 56(2):433–470, 2001. ISSN 1540-6261.

J.A. Wachter. Portfolio and Consumption Decisions under Mean-Reverting Returns: An Exact Solution for Complete Markets. *Journal of Financial and Quantitative Analysis*, 37(01):63–91, 2002. ISSN 0022-1090.

Daniel Zwillinger. *Handbook of Differential Equations*. Academic Pr, 3rd edition, 1998.

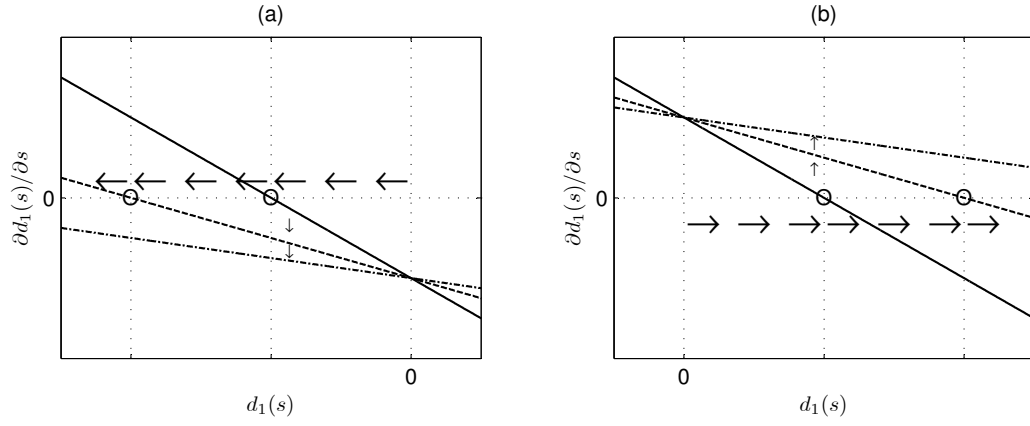


Figure 1: Phase Plane Analysis of $d_1(s)$

This Figure shows a phase plane analysis of the linear differential equation $\frac{\partial d_1(s)}{\partial s} = l_0 + l_1 d_1(s)$. In the panel to the left (right) $l_1 < 0$ ($l_1 > 0$). $d_1(s)$ converges to stable solutions $\bar{d}_1$ marked by circles. The solid line shows the base case and the dash (dash-dotted) line show the cases where l_1 approaches zero and $\bar{d}_1$ become large in magnitude.

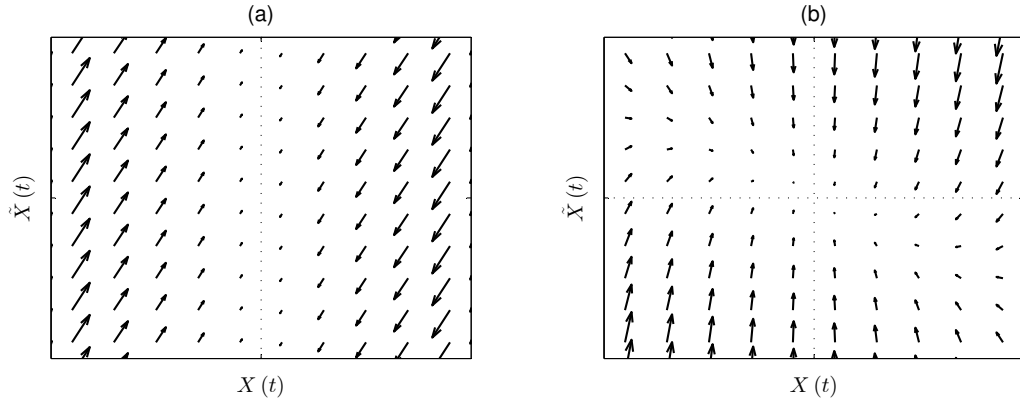


Figure 2: Phase Plane Analysis of $X(t)$ and $\tilde{X}(t)$

This Figure shows a phase plane analysis of the system of two stochastic linear differential equations (20) (left panel) and (22) (right panel). In the panel to the left (right), the log-income-dividend ratio $\tilde{X}(t)$ does not converge (does converge) to a long-run mean.

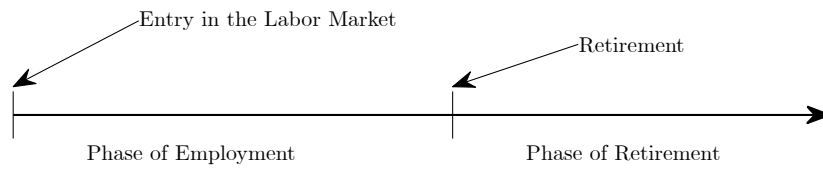


Figure 3: Life-Cycle

This Figure shows the life-cycle of an individual. The individual enters the labor market at time t and retires at time T .