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Optimal Strategies during Retirement

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OPTIMAL STRATEGIES DURING RETIREMENT

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Abstract

The present paper studies a pensioner deriving utility from a stream of consumption or an annuity and from bequeathing wealth to his heirs in a continuous-time framework. The task of finding the pensioner's optimal consumption, asset allocation and annuity decision rule leads to the interesting interplay of optimal control theory, optimal stopping theory and mortality issues or, technically speaking, to a combined optimal stopping and optimal control problem (COSOCP). [18] solved this problem in an all-or-nothing framework assuming exponential mortality and power utility functions. In this paper we extend his model in several dimensions: We contribute the essential inclusion of a bequest motive, we additionally study the economically interesting range of relative risk aversion levels greater than one and we provide a new solution method for the COSOCP via duality arguments. For identical risk aversion levels [18] finds that the pensioner either annuitises immediately or never which means that COSOCP reduces to a trivial or to a pure optimal control problem. In contrast to this the annuitisation decision rule can become wealth-dependent in our more general model and consequently, a real COSOCP has to be dealt with. The main result is that longevity risk matters very much (quite attractive annuity market) even if we allow for a bequest motive.

Keywords: optimal control; optimal stopping; annuitisation; consumption; asset allocation; bequest motive.

1 Introduction

Consumption and portfolio optimisation during retirement has not received as much attention in financial research as optimisation prior to retirement. However, retirement planning is becoming more and more relevant because of rising conditional life expectancies, growing number of defined contribution pension plans in many countries and continuing wealth concentration among pensioners. Thus, the demand for financial planning advice at the end of the life-cycle is already huge and it will still increase. In addition to the afore-mentioned statistical reasons there is of course academic interest in a good understanding of the end of the life-cycle because this can be used as an input to economic models focusing on the labour phase.

We therefore seek to answer the question of how much a pensioner should consume and how he should invest his wealth in the financial market at each point in time (optimal control part). Besides financial market risk, the pensioner is essentially confronted with longevity risk. Thus, we plan to determine the time when it is optimal for the pensioner to annuitise his wealth (optimal stopping part). Technically speaking, we have to solve the pensioner's combined optimal stopping and optimal control problem (COSOCP). Summarising, the task of finding the pensioner's optimal consumption, asset allocation and annuity decision rule leads to the interesting interplay of optimal control theory, optimal stopping theory and mortality issues. This is the natural consequence of identifying the pensioner's main sources of uncertainty: Longevity risk and investment risk.

The pioneering work of [10] led to a vast literature on continuous-time models of the optimal consumption and investment problem. However, most of it assumes either a fixed or an infinite planning horizon and/or does not include an annuity market. Clearly, neither of the approaches is adequate if one wants to compute the optimal consumption and investment rule of a pensioner. The pensioner's remaining lifetime should obviously be treated as a random variable. [19] was the first to include uncertain lifetimes. He showed that a utility maximising individual with mortality risk but no bequest motive should annuitise his entire wealth given that annuities are fair. The work of [19] was followed by [8], [11] and [16]. [11] obtained the result that an individual with exponential mortality behaves as if he were going to live forever with an adjusted discounting parameter. [16] extended the framework in [10] by additionally including freely reversible annuities. However, it is crucial to model the annuitisation decision as irreversible because of adverse selection issues. There are only a few authors providing normative continuous-time models for the retirement phase until now which include both the afore-mentioned uncertainty of remaining lifetime and the irreversibility of annuitisation. Concerning probability minimisation of retirement ruin ths branch of the literature started with [14] and [12]. Concerning utility maximisation this branch of the literature essentially started with [4] and [13]. The main shortcoming of the latter paper is that the pensioner has to adhere to a predetermined

annuitisation time no matter what happens after the time of optimisation. [18] has overcome this unrealistic feature by modelling the annuitisation decision as a stopping time controlled by the agent. He provides an appropriate continuous-time model to find the pensioner's optimal consumption, asset allocation and annuitisation decision rule assuming the exponential mortality law and power utility functions. In this paper we extend the model of [18] in several dimensions. First, we account for the fact that people leave a substantial part of their wealth at death. Thus, we contribute the essential inclusion of a bequest motive (annuitisation is in conflict with a potential bequest motive). Second, [18] only studies power utility functions with relative risk aversion levels less than one. Hence, we additionally study the economically interesting range of relative risk aversions greater than one which complicates the analysis. Third, as a consequence of allowing for a bequest motive, we explicitly account for a prior life insurance and a corresponding subsistence level of bequests. Lastly, we provide a new solution method for the COSOCP via duality arguments.

The studied pensioner derives utility from a stream of consumption or an annuity and from bequeathing wealth to his heirs. All these utility functions are modelled as power utility functions with identical relative risk aversion or as subsistence level utility functions. The individual of interest is a pensioner in the sense that he does not receive any stochastic income such as labour income in the private economy.[1] Thus, the individual can potentially be younger or, of course, older than some retirement age. In order to exclude any control part in the post-annuitisation phase we have to assume that the pensioner annuitises his entire remaining wealth and that he subsequently consumes his entire annuity. We will refer to this as an all-or-nothing framework. Lastly, we will only consider one riskless asset and one risky asset, with the latter modelled by a geometric Brownian motion with constant parameters. The risky asset can of course be thought of to be a broad index.

For identical risk aversion levels [18] finds that the pensioner either annuitises immediately or never which means that the COSOCP either reduces to a trivial or to a pure optimal control problem. In contrast to this the annuitisation decision rule can become wealth-dependent in our more general model and consequently, a real COSOCP has to be dealt with. We provide a new solution method via duality arguments to solve this COSOCP. The main result is that longevity risk matters very much (quite attractive annuity market) even if we allow for a bequest motive. This gives rise to another legitimation of pension funds besides the very important argument that pension funds can achieve more than the free market because they can exploit risk transfer across generations as described in [3].

The paper is organised as follows. Section 2 presents the model and formalises the pensioner's COSOCP with general strictly increasing and concave utility functions that satisfy the Inada conditions. We adjust the verification theorem of [18] to our aforementioned extensions of his model. This verification theorem reduces the COSOCP to a variational inequality which contains the HJB-equation of pure optimal control as a special

[1] This assumption simplifies the wealth dynamics considerably.

case. Moreover, some important properties of the crucial continuation region (where it is not optimal to stop) are derived. In Section 3 we exclude any bequest motive and specialise to power utility functions. Depending on the sign of some crucial quantity the pensioner either annuitises immediately (trivial case) or never (pure optimal control case). This is in accordance with the findings of [18] what concerns relative risk aversion levels smaller than one. Apart from very extreme settings, annuitisation is always optimal for the pensioner. This is not very surprising, since we have not taken into account an important model element: A bequest motive (which is in conflict with annuitisation). Consequently, we then adress the essential inclusion of a bequest motive in Section 4. The previous derivation in the no-bequest case can be applied in the bequest case for relative risk aversions less than one (Subsection 4.1) if we assume that prior life insurance equals the subsistence level of bequest. Thus, the annuitisation decision is again of the now-or-never-type. The crucial quantity however, now contains an additional bequest term. Most importantly, the strong tendency for the annuity market in the no-bequest case now turns into a slight tendency for the financial market in the bequest case. Lastly, the annuitisation decision rule becomes wealth-dependent for relative risk aversions greater than one (Subsection 4.2) which means that a real COSOCP (no reduction to a pure optimal control problem) has to be solved which is done by exploiting some duality arguments. Lastly, Section 5 draws the most important conclusions.

This paper is mainly a part of the author's doctoral thesis [17].

2 The Model

2.1 Model basics

The pensioner derives utility from a stream of consumption or an annuity and from bequeathing wealth to his heirs. We denote the corresponding utility functions by $U_i : (0,\infty) \to \mathbb{R}$ for $i = 1,2,3$ where $U_i \in C^2$, U_i is strictly increasing and strictly concave, and U_i satisfies the Inada-conditions

$$\frac{\partial U_i}{\partial \theta}(0) := \lim_{\theta \downarrow 0} \frac{\partial U_i}{\partial \theta}(\theta) = \infty \quad \text{and} \quad \frac{\partial U_i}{\partial \theta}(\infty) := \lim_{\theta \to \infty} \frac{\partial U_i}{\partial \theta}(\theta) = 0.$$

We define the financial market as consisting of one riskless asset ('bond' or money market account), which we call $S_0(t)$ and one risky asset ('stock') $S_1(t)$ which depends on the standard Brownian motion $B(t)$ on a given probability measure space $(\Omega, \mathcal{F}, P)$. The price processes of these two assets are given by

$$dS_0(t) = S_0(t)\, r dt, \quad S_0(0) = s_0 \tag{1}$$

and

$$dS_1(t) = S_1(t)\left(\mu dt + \sigma dB(t)\right), \quad S_1(0) = s_1 \tag{2}$$

with the constant positive real market coefficients r, μ and σ. We get a complete and arbitrage-free market by imposing $\mu > r$. Using (1) and (2) the pensioner's wealth

evolution can be shown to be

$$dW(t) = W(t)\left[r + \pi(t)(\mu - r)\right]dt + W(t)\,\pi(t)\,\sigma dB(t) - c(t)\,dt \tag{3}$$

with initial wealth $W(0) = w > 0$ and the consumption and portfolio process $c(t)$ and $\pi(t)$, respectively. Let $\mathcal{D}(w)$ denote the set of pairs $(c(t), \pi(t))$ that satisfy the usual admissibility requirements that guarantee the existence and uniqueness of a solution to (3) as well as $W(t) \geq 0$ a.s. $\forall t \geq 0$.

At time zero the pensioner is aged x and his remaining lifetime (from time zero onwards) is represented by the random variable T_x. Throughout the entire paper we assume the exponential mortality law. Moreover, we allow for the circumstance that the (subjective) pensioner and the (objective) insurance company will generally have different perceptions towards the pensioner's remaining lifetime. We denote the corresponding constant mortality rates by λ_x^S and λ_x^O, respectively,[2] and introduce the mortality rate transformation

$$\lambda_x^S = \lambda_x^O(1 + l) \tag{4}$$

with $l \in [-1, \infty)$. Discussing subjective mortality, $l = -1$ obviously corresponds to living forever, $l = 0$ to average health and finally, $l \to \infty$ brings the individual to death's door.

We next define the *annuity* $\overline{a}_x$ as the value of a claim to one unit of currency at each point in time conditional on survival of the x year-old individual. Imposing the exponential mortality law we get

$$\overline{a}_x = \int_0^\infty e^{-rt}\left({}_tp_x^O\right)dt = \int_0^\infty e^{-rt}e^{-\lambda_x^O t}dt = \frac{1}{r + \lambda_x^O}$$

where ${}_tp_x^O$ denotes the objective survival probability of an individual of age x. Annuitisation is modelled as an all-or-nothing irreversible decision. The irreversibility of the annuity purchase is due to adverse selection issues.

Moreover, we get a handy expression for expected remaining lifetime, too. We have

$$E^S[T_x] = \int_0^\infty \left({}_tp_x^S\right)dt = \int_0^\infty e^{-\lambda_x^S t}dt = \frac{1}{\lambda_x^S} \text{ and } E^O[T_x] = \frac{1}{\lambda_x^O},$$

respectively. Clearly, an insurance company will consider a great deal of information about an individual applying for an annuity.[3] Yet, there will always be some informational asymmetry: The individual will always know more about himself than the insurance company.[4] However, we will not pursue this issue of informational asymmetry and hence, adverse selection any further.[5] Instead, we just note that subjective life expectancy can

[2] We will often skip any subjective or objective declaration when the meaning of a quantity becomes clear from the context.

[3] The most important factors affecting an individual's life expectancy are gender, existing diseases, obesity, smoking, drinking, ethnicity, religion, education, physical activity and marital status.

[4] See [6] and [7] for some evidence that individuals are likely to be able to estimate their life expectancy beyond the observable mortality risk factors.

[5] See [1] for an early contribution to the field of adverse selection.

differ from the objective life expectancy and that the objective life expectancy is likely to be higher to incorporate profits and administrative costs.[6]

Lastly, the random time of the annuity purchase will be represented by τ (measured from time zero onwards). We naturally postulate stochastic independence between T_x and any financial variable such as $W(t)$ or τ. Moreover, we will mainly drop the age dependence of quantities for notational ease in the following, since x is given at the time of optimisation.

2.2 Optimisation problem

The pensioner is confronted with two kinds of problems. On the one hand, he has to find the optimal time to annuitise his entire wealth which creates an optimal stopping problem. On the other hand, the pensioner faces an optimal control problem as he has to determine the optimal consumption and investment rule in the pre-annuitisation phase. Furthermore, these two problems are tightly connected via wealth and hence, we have to solve the following combined optimal stopping and optimal control problem (COSOCP). We have

$$V(w) = \sup_{(c,\pi,\tau)\in\mathcal{G}(w)} J_{c,\pi,\tau}(w) \qquad \text{for all } w > 0 \tag{5}$$

with the total expected utility function

$$\begin{aligned} J_{c,\pi,\tau}(w) &= E\left[\int_0^T e^{-\delta^S t}\left\{U_1(c(t))\,1_{\{t\le\tau\}} + U_2\left(\frac{W(\tau)}{\bar{a}_{x+\tau}}\right)1_{\{t>\tau\}}\right\}dt\right. \\ &\quad \left. +\eta e^{-\delta^S T}\left\{U_3(W(T)+Z^s)\,1_{\{T\le\tau\}} + U_3(Z^s)\,1_{\{T>\tau\}}\right\}\right] \end{aligned} \tag{6}$$

with the subjective discount rate δ^S, the indicator function $1_{\{.\}}$, the bequest parameter $\eta \geq 0$ and where Z^s denotes the difference between the prior decision on life insurance and the subsistence level of bequest, i.e. $Z^s = Z^{prior} - \underline{Z}$. The set $\mathcal{G}(w)$ of admissible strategies (c,π,τ) in the COSOCP is the natural extension to the set $\mathcal{D}(w)$ of admissible strategies (c,π) in the pure optimal control problem with $\tau < \tau_G$ where $G = (0,\infty)$ and $\tau_G = \inf\{t > 0 | W(t) \notin G\}$.

Remark: 2.1

1. Note that we can always recover the no-bequest case by setting $\eta = 0$.
2. Obviously, we have to consider four different sources of utility.
   (a) The most natural is the utility from consumption when the individual is alive, which is captured in the first term.
   (b) The second term is made up of the utility the agent derives when still alive but already having annuitised his complete wealth.

[6] According to [13] profits and administrative costs can account for a markup in the life expectancy between 1 and 5 percent.

(c) Should the individual die before annuitisation, then his wealth at death will be bequeathed to his heirs. On top of this remaining wealth the heirs receive the life insurance amount Z^{prior}. The utility derived from such a bequest is captured in the third term.

(d) Finally, there is the case that the agent dies after having annuitised his entire wealth where only the life insurance will be bequeathed to the heirs.

3. In the bequest case we assume that the pensioner previously bought a life insurance Z^{prior} and that he has a subsistence level of bequests $\underline{Z}$. If the pensioner dies after annuitising his wealth, then he gets the utility $U_3(Z^s)$. Later, we will replace U_3 with a power utility function (see (32)).

(a) Note that we therefore have to assume[7] $Z^s > 0$ for risk aversion coefficients greater than one to keep the model well defined.

(b) For risk aversion coefficients less than one, however, we can simplify the analysis by setting $Z^s = 0$ without making the model ill defined.

□

LEMMA 2.2 *Under the exponential mortality law, the total expected discounted utility in (6) can be written as*

$$\begin{aligned} J_{c,\pi,\tau}(w) &= E^w\left[\int_0^\tau e^{-\beta^S t}\left\{U_1(c(t)) + \lambda^S \eta U_3(W(t) + Z^s)\right\} dt\right. \\ &\quad \left. + e^{-\beta^S \tau}\frac{1}{\beta^S}\left\{U_2\left(W(\tau)\left(r+\lambda^O\right)\right) + \lambda^S \eta U_3(Z^s)\right\}\right] \end{aligned} \tag{7}$$

with

$$\beta^S = \delta^S + \lambda^S. \tag{8}$$

The proof mainly exploits Fubini's theorem, the exponential mortality law and the assumed stochastic independence of T and τ. We omit it for the sake of brevity.

To simplify the exposition we will sometimes use the following general indirect utility function

$$J_{c,\pi,\tau}(w) = E^w\left[\int_0^\tau e^{-\beta^S t} f(c(t), W(t))\, dt + e^{-\beta^S \tau} g(W(\tau))\right]. \tag{9}$$

Clearly, we can recover our current indirect utility function (7) by specifying the running reward

$$f(c(t), W(t)) = U_1(c(t)) + \lambda^S \eta U_3(W(t) + Z^s) \tag{10}$$

and the terminal reward

$$g(W(t)) = \frac{1}{\beta^S}\left\{U_2\left(W(t)\left(r+\lambda^O\right)\right) + \lambda^S \eta U_3(Z^s)\right\}. \tag{11}$$

[7] Imposing $Z^s > 0$ is equivalent to assuming that the previously bought life insurance Z^{prior} exceeds the subsistence level of bequests $\underline{Z}$ which seems very natural.

2.3 Verification theorem and some general results

The HJB equation of pure optimal control inspires us to define the operator L^{com} as

$$L^{\text{com}}v\left(W\left(t\right)\right)=\sup_{\left(c,\pi\right)\in\mathcal{G}^{\tau}\left(W\left(t\right)\right)}\left\{f\left(c\left(t\right),W\left(t\right)\right)-\beta^{S}v\left(W\left(t\right)\right)+Lv\left(W\left(t\right)\right)\right\} \tag{12}$$

for all $W\left(t\right)>0$ where $\mathcal{G}^{\tau}\left(W\left(t\right)\right)$ denotes the set of admissible strategies for a given stopping time τ and where L denotes the partial differential operator

$$Lv\left(W\left(t\right)\right)=\left[W\left(t\right)\left(r+\pi\left(t\right)\left(\mu-r\right)\right)-c\left(t\right)\right]v_{W}\left(W\left(t\right)\right)+\frac{1}{2}\sigma^{2}\pi\left(t\right)^{2}W\left(t\right)^{2}v_{WW}\left(W\left(t\right)\right) \tag{13}$$

for all $W\left(t\right)>0$.

Inspired by results from pure optimal stopping theory (see f.e. [15]), we give the following theorem, which is crucial to our COSOCP. It is an extension to theorem 2.1 in [18]. As mentioned previously, [18] only studies risk aversion levels less than one.[8] Consequently, he solely has to deal with nonnegative power utility functions. Since we will additionally consider the economically interesting range of risk aversions greater than one (see (32)), we have to adjust the verification theorem of [18]. In accordance with future needs we therefore also deal with the case where the running and terminal reward function f and g are both negative.

Theorem 2.3 (COSOCP verification theorem)

a) Assume that the following conditions are satisfied.

(i) $v\in C^{1}\left(\mathbb{R}^{+}\right)$ *with continuous second order derivatives a.e. in* $\mathbb{R}^{+}$.

(ii)

$$v\left(W\left(t\right)\right)\geq g\left(W\left(t\right)\right)\quad \textit{on } G. \tag{14}$$

(iii)

$$L^{com}v\left(W\left(t\right)\right)\leq 0\quad \textit{on } G. \tag{15}$$

(iv) For all $\left(c,\pi,\tau\right)\in\mathcal{G}\left(w\right)$ *with* $w>0$ *the process* $\int_{0}^{t}e^{-\beta^{S}s}\sigma\pi\left(s\right)W\left(s\right)v_{W}\left(W\left(s\right)\right)dB\left(s\right)$ *is a martingale for all* $t\geq 0$.

Then, we have

$$V\left(w\right)=\sup_{\left(c,\pi,\tau\right)\in\mathcal{G}\left(w\right)}J_{c,\pi,\tau}\left(w\right)\leq v\left(w\right)\quad \textit{for all } w>0.$$

b) Furthermore, we define the continuation region D *as*

$$D=\left\{W\left(t\right)\in G\,|\,v\left(W\left(t\right)\right)>g\left(W\left(t\right)\right)\right\}. \tag{16}$$

[8] [18] uses the more restrictive power utility functions $U_{i}\left(\theta\right)=\theta^{\gamma}$ which implies that the relative risk aversion must be between zero and one.

(v) We assume that

$$L^{com}v\left(W\left(t\right)\right)=0 \quad on\ D. \tag{17}$$

(vi) Moreover, we make the assumption that $v\left(W\left(t\right)\right)$ is strictly concave on D and that $f\left(c\left(t\right),W\left(t\right)\right)$ is strictly concave in its first argument everywhere.[9]
Define the strategies[10]

$$\tau^{*}=\inf\left\{t\geq 0|W^{*}\left(t\right)\notin D\right\}, \tag{18}$$

$$c^{*}=I\left(v_{W}\left(W^{*}\left(t\right)\right)\right)1_{\{t\leq\tau^{*}\}} \tag{19}$$

and

$$\pi^{*}=-\frac{\mu-r}{\sigma^{2}}\frac{v_{W}\left(W^{*}\left(t\right)\right)}{W^{*}\left(t\right)v_{WW}\left(W^{*}\left(t\right)\right)}1_{\{t\leq\tau^{*}\}}, \tag{20}$$

where $W^{*}\left(t\right)$ is the solution of (3) corresponding to the control pair (c^{*},π^{*}) and $I\left(.\right)$ denotes the inverse function of $\frac{\partial f}{\partial c}\left(c\left(t\right),W\left(t\right)\right)$ w.r.t. the first argument.

(vii) We further assume that the transversality condition

$$\lim_{t\rightarrow\infty}e^{-\beta^{S}t}E^{w}\left[v\left(W^{*}\left(t\right)\right)1_{\{t<\tau^{*}\}}\right]=0 \tag{21}$$

holds.

Then, we have

$$V\left(w\right)=\sup_{(c,\pi,\tau)\in\mathcal{G}(w)}J_{c,\pi,\tau}\left(w\right)=v\left(w\right)\quad for\ all\ w>0$$

and (c^{*},π^{*},τ^{*}) are the optimal strategies.

The proof is given in A.

The optimal strategies of theorem 2.3 are very intuitive and easy to understand. We discuss them in the following remark.

Remark: 2.4 (Optimal strategies of theorem 2.3)

1. The optimal stopping time (18) is the first exit time of the crucial continuation region D. This is very intuitive, since the value function strictly exceeds the terminal reward on D.

2. The optimal consumption level (19) ensures that the marginal instantaneous utility from consumption equals the marginal value of wealth, which is a well-known intertemporal optimality result.

[9] Note that this assumption will be met when we apply the theorem to the indirect utility function (7), since we then have $f\left(c\left(t\right),W\left(t\right)\right)=U_{1}\left(c\left(t\right)\right)+\lambda^{S}\eta U_{3}\left(W\left(t\right)+Z^{s}\right)$ and because U_{1} is strictly concave.

[10] The rules for consumption and investment of course only apply to the pre-annuitisation phase, since we have assumed that the pensioner consumes the entire annuity and as there is no wealth left to be optimally allocated after the annuitisation.

3. The optimal portfolio rule (20) is given by the equity premium times the reciprocal of the relative risk aversion of the value function.

4. Lastly, the optimal strategies (18)-(20) are obviously given in terms of the yet unknown value function. We will need to specialise the utility functions to obtain explicit expressions. Before doing so we give some more general results.

□

We next give an important remark on the nature of theorem 2.3 that simplifies its application.

REMARK: 2.5 (ESSENCE OF THEOREM 2.3)

1. The strategy will be to find a candidate solution to the COSOCP given in (5) and (7) and then to use the verification theorem 2.3 to see whether we have indeed found a solution.

2. Exploiting the assumptions (ii), (iii) and (v) of theorem 2.3 as well as the definition of the continuation region D in (16), we establish that

$$\max\left\{L^{\text{com}}v\left(W\left(t\right)\right), g\left(W\left(t\right)\right) - v\left(W\left(t\right)\right)\right\} = 0 \quad \text{for } W\left(t\right) > 0. \tag{22}$$

Thus, theorem 2.3 essentially reduces the COSOCP to the variational inequality (22). Using the definition of the continuation region D, we can rewrite it as

$$L^{\text{com}}v\left(W\left(t\right)\right) = 0 \text{ for } W\left(t\right) \in D \text{ and } v\left(W\left(t\right)\right) = g\left(W\left(t\right)\right) \text{ for } W\left(t\right) \in \mathbb{R}^{+}\backslash D. \tag{23}$$

3. With the definition of the set D and the the regularity assumed in theorem 2.3 for the solution v we can state the 'smooth paste condition'

$$v\left(W\left(t\right)\right) = g\left(W\left(t\right)\right) \quad \text{for all } W\left(t\right) \in \partial D \tag{24}$$

as well as the 'smooth fit condition'

$$v_{W}\left(W\left(t\right)\right) = g_{W}\left(W\left(t\right)\right) \quad \text{for all } W\left(t\right) \in \partial D. \tag{25}$$

□

The following two lemmas taken from [18] will help us to characterise the continuation region D.

LEMMA 2.6 *The value function $V\left(w\right)$ is monotone non-decreasing and lower semicontinuous.*

The proof is omitted for the sake of brevity.

LEMMA 2.7 *The continuation region D is an open and connected set.*

Proof:
Because our value function is lower semicontinuous according to lemma 2.6 and due to the strict concavity of $g(W(t))$ (see (11)), we conclude that D is an open connected subset of $\mathbb{R}^+$.[11] ■

In order to further characterise the crucial set D, we give the following lemma.

LEMMA 2.8 *Using the operator L^{com} given in (12) we define the set U as*

$$U = \left\{ W(t) \in \mathbb{R}^+ \mid L^{com} g(W(t)) > 0 \right\}. \tag{26}$$

We then have

$$U \subset D. \tag{27}$$

The proof is given in B.

REMARK: 2.9 (U VERSUS D)
From (18) and (27) we conclude that it is never optimal to annuitise before wealth exits from the set U. Clearly, we have to consider two cases.

1. If $U \subsetneq D$ then it is optimal to continue if wealth falls out of the set U.
2. If $U = D$ then it is optimal to annuitise immediately if wealth exits from U.

However, even in the former case it is often possible to infer important information about the form of the crucial continuation region D by studying the set U. □

The following lemma provides an explicit expression for $L^{\text{com}} g(W(t))$ which we need to determine the important set U defined in (26).

LEMMA 2.10 *In the introduced model we have*

$$L^{com} g(W(t)) = \frac{W(t)^{1-\gamma}}{1-\gamma} \left[\gamma K_0^{-\frac{1-\gamma}{\gamma}} - \gamma K_0 K_2^{-1} \right] + \frac{\lambda^S \eta}{1-\gamma} \left[(W(t) + Z^s)^{1-\gamma} - (Z^s)^{1-\gamma} \right] \tag{28}$$

with the constants

$$K_0 = \frac{\left(r + \lambda^O\right)^{1-\gamma}}{\beta^S} > 0, \tag{29}$$

$$K_2 = \frac{\gamma}{\beta^S - (1-\gamma)\left(r + \frac{\kappa}{\gamma}\right)} \tag{30}$$

and

$$\kappa = \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 > 0. \tag{31}$$

The proof is omitted for the sake of brevity.

[11] The interested reader is referred to [9].

3 Solving the No-Bequest Case

From now on we specialise $U_i(.)$ with $i = 1, 2, 3$ to the power utility functions

$$U_i(\theta) = \frac{\theta^{1-\gamma}}{1-\gamma} \quad \text{with } \gamma > 0 \text{ and } \gamma \neq 1. \tag{32}$$

Setting the bequest parameter η at zero simplifies the derivation of the set U considerably. From (28) we see that U is completely determined with the signs of $1-\gamma$ and of

$$M^{nb} = \gamma K_0^{-\frac{1-\gamma}{\gamma}} - \gamma K_0 K_2^{-1}. \tag{33}$$

We get

$$U = \begin{cases} \mathbb{R}^+ & \text{if } \gamma < 1 \text{ and } M^{nb} > 0 \text{ (case 1)} \quad \text{or} \quad \text{if } \gamma > 1 \text{ and } M^{nb} < 0 \text{ (case 2)} \\ \emptyset & \text{if } \gamma < 1 \text{ and } M^{nb} \leq 0 \text{ (case 3)} \quad \text{or} \quad \text{if } \gamma > 1 \text{ and } M^{nb} \geq 0 \text{ (case 4).} \end{cases} \tag{34}$$

THEOREM 3.1 *In the no-bequest case we have*

$$D = U. \tag{35}$$

The proof is given in C.

REMARK: 3.2

1. Theorem 3.1 shows that the annuitisation decision is independent of wealth. Depending on the parameters of the model, annuitisation either occurs immediately or never: The annuitisation rule is of the now-or-never type. This result seems to stem from our exponential mortality law assumption, where we have constant mortality rates.

2. [13] show that the annuitisation decision is not of the now-or-never-type using a more general mortality law. Instead they prove that annuitisation takes place at a prescribed future time. The main shortcoming of their model is that the pensioner has to adhere to this predetermined annuitisation time. This is clearly a highly unrealistic feature of their model which we have overcome by modelling the annuitisation decision as a stopping time controlled by the pensioner.

3. Our now-or-never annuitisation rule is in line with the results of [18] regarding risk aversion levels less than one (cases 1 and 3). Yet, our findings in the economically interesting case with risk aversion levels greater than one (cases 2 and 4) are novel.

4. Unfortunately, the crucial quantity regarding the annuitisation decision M^{nb} depends on as many as seven parameters. In detail, we have

$$\begin{aligned} M^{nb} &= \gamma \left(\lambda^S + \delta^S\right)^{\frac{1-\gamma}{\gamma}} \left(r + \lambda^O\right)^{-\frac{(1-\gamma)^2}{\gamma}} - \left(r + \lambda^O\right)^{1-\gamma} \\ &\quad + \left(r + \frac{1}{\gamma}\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right) \frac{\left(r + \lambda^O\right)^{1-\gamma}}{\lambda^S + \delta^S} (1-\gamma). \end{aligned} \tag{36}$$

The constant M^{nb} depends on the subjective and objective discounting parameters δ^S and r, the subjective and objective mortality rate parameters λ^S and λ^O, the financial parameters μ and σ (and the already mentioned r) and finally, the risk aversion coefficient γ. Moreover, all parameters appear in a non-linear fashion. Hence, we mainly have to refuge to numerical analysis.

□

We get only one unambiguous effect, i.e. the effect of the Sharpe ratio $SR = \frac{\mu - r}{\sigma}$ as a measure for the attractiveness of the financial market. Using (36), we obtain

$$\frac{\partial M^{nb}}{\partial SR} = \frac{1-\gamma}{\gamma} K_0 SR. \tag{37}$$

We obviously have $\frac{\partial M^{nb}}{\partial SR} > 0$ for $\gamma < 1$ and $\frac{\partial M^{nb}}{\partial SR} < 0$ for $\gamma > 1$. Hence, the pensioner will never annuitise but always stay in the attractive financial market if the Sharpe ratio is high enough (cases 1 and 3 of (34)).[12]

We can confirm the intuitive qualitative findings of [18] what concerns risk aversion levels less than one: While a higher objective and identical life expectancy decrease the attractiveness of the annuity market, a higher subjective life expectancy increases it. We contribute that these effects hold for risk aversions greater than one, too. Moreover, we get the intuitive result that a higher relative risk aversion level makes annuitisation more likely whereas a higher effective time horizon is an argument for the financial market regardless of whether we assume cheap ($l < 0$) or expensive annuities ($l > 0$).[13]

However, apart from very extreme settings, annuitisation is always optimal for the pensioner. Let us show this main result by studying the effect of the Sharpe ratio on the annuitisation decision while fixing all other model parameters at realistic values. To isolate the effect of the Sharpe ratio we assume the identical mortality rates $\lambda^S = \lambda^O = 0.05$ (which imply an identical expected remaining lifetime of 20 years) and identical subjective and objective discount parameters, i.e. $\delta^S = r = 0.035$.[14] By (37) we can always compute the minimum Sharpe ratio such that the pensioner stays in the financial market.

Table 1 shows that a higher risk aversion increases the attractiveness of the annuity market[15] while a higher identical life expectancy is an argument for the financial market as claimed. If we assume $\mu = 0.08$ and $\sigma = 0.2$ we get a Sharpe ratio of 0.225. Thus, Table 1 clearly demonstrates that annuitisation is chosen in almost all situations.[16] This very strong tendency for the annuity market will change considerably with the important inclusion of a bequest motive in Section 4.

[12] Note that we could proceed in a more detailed manner and study the effects of the three elements of the Sharpe ratio individually: μ, r and σ. A higher μ and a lower σ, ceteris paribus, naturally increase the attractiveness of the financial market. The effect of the interest rate r on M^{nb} however, is more complicated as r affects both the financial and the annuity market.

[13] The interested reader is referred to [17].

[14] Note that a higher μ and a lower σ now translate into a higher Sharpe ratio.

[15] Note that annuitisation eliminates all longevity risk.

[16] The financial market is chosen only if $E[T] = 30$ and $\gamma = 0.7$ (marked bold in Table 1).

Table 1: Minimum Sharpe ratio for staying in the financial market in the no-bequest case for some identical life expectancy and risk aversion combinations.

	$\gamma = 0.7$	$\gamma = 0.9$	$\gamma = 1.2$	$\gamma = 1.6$	$\gamma = 2$	$\gamma = 4$
$E[T] = 5$	0.5290	0.6000	0.6928	0.8000	0.8944	1.2649
$E[T] = 10$	0.3746	0.4242	0.4898	0.5656	0.6324	0.8944
$E[T] = 15$	0.3055	0.3464	0.4000	0.4618	0.5163	0.7302
$E[T] = 20$	0.2645	0.3000	0.3464	0.4000	0.4472	0.6324
$E[T] = 25$	0.2366	0.2683	0.3098	0.3577	0.4000	0.5656
$E[T] = 30$	**0.2160**	0.2449	0.2828	0.3265	0.3651	0.5163

As discussed in remark 3.2, annuitisation either occurs immediately or never. Thus, the COSOCP reduces either to a trivial or to a pure optimal control problem. In the latter case we have to solve the infinite Merton problem with the HJB equation $L^{\text{com}}V(W(t)) = 0$. The corresponding results are summarised in the following theorem.

THEOREM 3.3 *In case 1 and 2 where annuitisation never occurs we get*

$$V^{nb}\left(W^{nb}(t)\right) = H^{nb}\frac{W^{nb}(t)^{1-\gamma}}{1-\gamma} \text{ with } H^{nb} = K_2^{\gamma}, \tag{38}$$

$$c^{nb}(t) = \left(H^{nb}\right)^{-\frac{1}{\gamma}} W^{nb}(t) = K_2^{-1}W^{nb}(t), \tag{39}$$

$$\pi^{nb}(t) = \frac{\mu - r}{\sigma^2}\frac{1}{\gamma} \tag{40}$$

and

$$W^{nb}(t) = w\exp\left[\left(r + \frac{\kappa}{\gamma}\left(2 - \frac{1}{\gamma}\right) - \left(H^{nb}\right)^{-\frac{1}{\gamma}}\right)t + \frac{\mu - r}{\sigma}\frac{1}{\gamma}B(t)\right]. \tag{41}$$

The proof is omitted for the sake of brevity. REMARK: 3.4

1. The portfolio rule is independent of time and wealth. The consumption rule is a linear function of wealth.

2. As consumption has to be nonnegative, we impose the natural parameter restriction

$$\beta^S > (1-\gamma)\left(r + \frac{\kappa}{\gamma}\right). \tag{42}$$

   Inequality (42) is trivially satisfied for $\gamma > 1$. Yet, it does not hold for very low levels of the risk aversion. However, such low risk aversion levels are not very plausible[17] and [17] shows that (42) is not restrictive for $\gamma > 0.5625$.

[17] See f.e. [5] and [2].

3. Exploiting (41) one can show that

$$\beta^S > r + \kappa \tag{43}$$

is a necessary but not sufficient condition to ensure that expected wealth falls. [17] demonstrates that inequality (43) is very likely to be satisfied for a retiree.

4. The inequalities (42) and (43) are assumed to hold throughout the entire paper. They ensure the validity of our results.

□

4 Solving the Bequest Case

The additional term in (28) in the bequest case obviously complicates the derivation of the set U and therefore the determination of the continuation region D.

4.1 Risk aversion $\gamma < 1$ without life insurance

To keep the analysis simple, we set $Z^s = 0$ in the current subsection.[18] This is equivalent to assuming that the previously bought life insurance matches the subsistence level of bequests. From (28) we then see that the set U is completely determined with the signs of $1-\gamma$ and of

$$M^b = M^{nb} + \lambda^S \eta. \tag{44}$$

The only difference to the no-bequest case lies in the additional term $\lambda^S \eta$. Thus, we can exploit the analysis of the no-bequest case to conclude that[19]

$$D = \begin{cases} \mathbb{R}^+ & \text{if } M^b > 0 \quad \text{case I} \\ \emptyset & \text{if } M^b \leq 0 \quad \text{case III.} \end{cases} \tag{45}$$

Clearly, the annuitisation decision is again of the now-or-never type. The bequest parameter solely enters in the additive term $\lambda^S \eta$ where it is directly coupled with the subjective mortality rate. A higher λ^S - meaning that death is closer - increases the importance of the bequest motive, as expected. The inclusion of a bequest motive obviously makes annuitisation less likely. This a very natural result because there will be no wealth left after the annuitisation that can be bequeathed in addition to $Z^{prior} = \underline{Z}$.

The nature of the effects of the objective life expectancy and the Sharpe ratio on the annuitisation decision clearly remain the same as in the no-bequest case. In contrast, the influence of the subjective and the identical life expectancy as well as the effect of the markup parameter l could potentially be different. However, [17] shows that the nature of these effects does not change either compared to the no-bequest case. We therefore

[18] Note that we have seen in remark 2.1 that our model allows for $Z^s = 0$ if $\gamma < 1$.

[19] Obviously, case I in the bequest case corresponds to case 1 in the no-bequest case, while bequest case III corresponds to no-bequest case 3.

concentrate on demonstrating that the absurdly strong tendency for the annuity market, which we discovered in the no-bequest case of Section 3, now - with the important inclusion of a bequest motive - turns into a slight tendency toward the financial market.

Table 2: Minimum Sharpe ratio for staying in the financial market in the bequest case for some identical life expectancy and bequest motive combinations assuming $\gamma = 0.8$.

	$\eta = 0$	$\eta = 0.25$	$\eta = 0.5$	$\eta = 0.75$	$\eta = 1$
$E[T] = 5$	0.5656	0.4409	0.2623	**0.0000**	**0.0000**
$E[T] = 10$	0.4000	0.3459	0.2817	**0.1977**	**0.0000**
$E[T] = 15$	0.3265	0.2919	0.2526	**0.2059**	**0.1449**
$E[T] = 20$	0.2828	0.2570	0.2284	**0.1955**	**0.1559**
$E[T] = 25$	0.2529	0.2322	**0.2094**	**0.1838**	**0.1539**
$E[T] = 30$	0.2309	**0.2134**	**0.1942**	**0.1730**	**0.1488**
$E[T] = 35$	**0.2138**	**0.1985**	**0.1819**	**0.1637**	**0.1431**

Obviously, we can compute the minimum Sharpe ratio such that the pensioner stays in the financial market as in the no-bequest case. Table 2 shows these minimum Sharpe ratios for different combinations of the identical life expectancy and the bequest motive for $\gamma = 0.8$ only, as the effects are qualitatively the same for other levels of the risk aversion. The displayed minimum Sharpe ratios decrease with the bequest motive as expected. Moreover, they tend to fall in the identical life expectancy. Note that we have established in the no-bequest case that a higher identical life expectancy (effective time horizon) makes the financial market more attractive. This explains why the minimum Sharpe ratio normally falls when we increase the life expectancy. Yet, for low life expectancies this is not necessarily true. Increasing the life expectancy makes the bequest term less important (through a lower λ^S). This can obviously outweigh the above-mentioned effect. Using the same parameter values as in the no-bequest case, we get a Sharpe ratio of 0.225. We obtain a slight tendency for the financial market, since we still have quite realistic settings where immediate annuitisation is optimal for the pensioner.[20]

Lastly, we again fix $\gamma = 0.8$ and display the minimum markup parameter[21] l such that the pensioner stays in the financial market. Intuitively, a stronger bequest motive and a higher markup parameter l both increase the attractiveness of the financial market. Table 3 confirms this intuition. Furthermore, we even get negative values for the minimum markup parameter. These are marked bold in Table 3 and represent all situations where the pensioner rejects fair and even some favourable annuities due to a strong enough bequest motive and/or a high enough effective time horizon.

We know from the continuation region (45) that annuitisation either occurs immediately or never. We skip the trivial case III and give the results to case I where we have to solve a similar infinite Merton problem as in the no-bequest case 1 and 2.

[20] Situations where the pensioner chooses the financial market are marked in bold face in Table 2.

[21] See the mortality rate transformation given in (4).

Table 3: Minimum markup parameter l for staying in the financial market in the bequest case for some objective life expectancy and bequest motive combinations assuming $\gamma = 0.8$.

	$\eta = 0$	$\eta = 0.25$	$\eta = 0.5$	$\eta = 0.75$	$\eta = 1$
$E^O[T] = 5$	1.4089	0.4191	0.0379	**−0.1760**	**−0.3161**
$E^O[T] = 10$	1.3395	0.5557	0.1760	**−0.0573**	**−0.2179**
$E^O[T] = 15$	1.2292	0.5359	0.1606	**−0.0826**	**−0.2557**
$E^O[T] = 20$	1.0681	0.4167	0.0337	**−0.2308**	**−0.4348**
$E^O[T] = 25$	0.8316	0.1666	**−0.3529**	**−0.5449**	**−0.6189**
$E^O[T] = 30$	0.4180	**−0.2325**	**−0.3708**	**−0.4780**	**−0.5646**
$E^O[T] = 35$	0.0855	**−0.1321**	**−0.2901**	**−0.4126**	**−0.5117**

THEOREM 4.1 *In case I where annuitisation never occurs we get*

$$V^b\left(W^b(t)\right) = H^b \frac{W^b(t)^{1-\gamma}}{1-\gamma}, \tag{46}$$

$$c^b(t) = \left(H^b\right)^{-\frac{1}{\gamma}} W^b(t), \tag{47}$$

$$\pi^b(t) = \frac{\mu - r}{\sigma^2}\frac{1}{\gamma} \tag{48}$$

and

$$W^b(t) = w \exp\left[\left(r + \frac{\kappa}{\gamma}\left(2 - \frac{1}{\gamma}\right) - \left(H^b\right)^{-\frac{1}{\gamma}}\right)t + \frac{\mu - r}{\sigma}\frac{1}{\gamma}B(t)\right]. \tag{49}$$

The proof is omitted for the sake of brevity.

REMARK: 4.2

1. The investment rule is the same whether we include a bequest motive or not because we have assumed identical risk aversion levels over consumption and over bequests.

2. In contrast, the optimal consumption rule, the wealth evolution and the value function are different in the current bequest case, since they all depend on the constant H^b. [17] proves that $H^b > H^{nb}$ holds. Exploiting (39), (41), (38), (47), (49) and (46), we therefore have

$$\frac{c^{nb}(t)}{W^{nb}(t)} > \frac{c^b(t)}{W^b(t)}, \tag{50}$$

$$W^{nb}(t) < W^b(t) \tag{51}$$

and

$$V^{nb}\left(W^{nb}(t)\right) < V^b\left(W^b(t)\right). \tag{52}$$

Thus, a pensioner with a bequest motive consumes a lower fraction of his wealth which allows him to bequeath more wealth to his heirs. Including a bequest motive clearly creates a trade-off between consumption and bequests. Furthermore, a

pensioner with a bequest motive simply has an additional source of utility which explains inequality (52).

3. Lastly, Table 4 gives the optimal consumption rule as a percentage of wealth for some combinations of the risk aversion level and the bequest motive assuming the usual parameter values. The consumption fraction naturally decreases in the bequest motive for all studied levels of the risk aversion.

Table 4: Optimal consumption in percentages of wealth in the bequest case I for some combinations of the relative risk aversion level and the bequest motive.

	$\eta = 0$	$\eta = 0.25$	$\eta = 0.5$	$\eta = 0.75$	$\eta = 1$
$\gamma = 0.6$	9.0208	8.5446	8.0986	7.6807	7.2893
$\gamma = 0.7$	9.0931	8.7681	8.4592	8.1655	7.8861
$\gamma = 0.8$	8.9589	8.7367	8.5231	8.3178	8.1204
$\gamma = 0.9$	8.7430	8.5905	8.4427	8.2995	8.1605

□

4.2 Risk aversion $\gamma > 1$ including life insurance

Remember from our discussion in remark 2.1 that we need a positive life insurance net of subsistence ($Z^s > 0$) if we study $\gamma > 1$.

THEOREM 4.3 *Let $\gamma > 1$, $Z^s > 0$. The continuation region D is then given by*

(i) $D = (0, \infty)$ *if* $M^{nb} \leq 0$ *Case II*

(ii) $D = (\underline{W}, \infty)$ *with* $\underline{W} \leq W^U$ *if* $M^{nb} > 0$ *Case IV*

where $\underline{W}^U$ represents the smallest root of $L^{com} g(W(t))$.

The proof is given in D.

In the following remark we compare the continuation region in the no-bequest case in theorem 3.1 with the bequest case analogue in the above theorem 4.3. Moreover, we characterise case II and IV.

REMARK: 4.4

1. If $M^{nb} < 0$, then the pensioner never annuitises his wealth regardless of whether we include a bequest motive or not (see case 2 in theorem 3.1 and case II in theorem 4.3).

2. If $M^{nb} > 0$, then a pensioner without a bequest motive annuitises his wealth immediately (case 4 in theorem 3.1). In contrast, a pensioner with a bequest motive annuitises only as soon as wealth falls below the threshold $\underline{W}$ (case IV in theorem 4.3). Hence, the inclusion of a bequest motive makes annuitisation less likely as expected.

3. Our COSOCP reduces to a pure optimal control problem in case II. We therefore concentrate on case IV where we have to solve a real COSOCP.

□

The pensioner will exit the financial market and annuitise as soon as his wealth falls below the yet unknown threshold $\underline{W}$. We exploit the important verification theorem 2.3, which reduces the current real COSOCP to the problem of finding a solution v to the variational inequality (23) subject to the smooth paste and smooth fit condition (24) and (25), respectively. Thus, we want to find a function $u(W(t))$ that satisfies

$$L^{\text{com}}u(W(t)) = 0, \tag{53}$$

subject to the 'smooth paste condition'

$$u(\underline{W}) = \frac{\left(r+\lambda^O\right)^{1-\gamma}}{\beta^S}\frac{\underline{W}^{1-\gamma}}{1-\gamma} + \frac{1}{\beta^S}\lambda^S\eta\frac{(Z^s)^{1-\gamma}}{1-\gamma} \tag{54}$$

and to the 'smooth fit condition'

$$u_W(\underline{W}) = \frac{\left(r+\lambda^O\right)^{1-\gamma}}{\beta^S}\underline{W}^{-\gamma}. \tag{55}$$

Finally, we can set

$$v(W(t)) = \begin{cases} \frac{\left(r+\lambda^O\right)^{1-\gamma}}{\beta^S}\frac{W(t)^{1-\gamma}}{1-\gamma} + \frac{1}{\beta^S}\lambda^S\eta\frac{(Z^s)^{1-\gamma}}{1-\gamma} & \text{if } W(t) \leq \underline{W} \\ u(W(t)) & \text{if } W(t) > \underline{W}. \end{cases}$$

LEMMA 4.5 *The equation $L^{com}u(W(t)) = 0$ can be written as*

$$\begin{aligned} 0 &= \frac{\gamma}{1-\gamma}\left[u_W(W(t))\right]^{-\frac{1-\gamma}{\gamma}} + \lambda^S\eta\frac{(W(t)+Z^s)^{1-\gamma}}{1-\gamma} \\ &\quad -\beta^S u(W(t)) + rW(t)\,u_W(W(t)) - \kappa\frac{\left[u_W(W(t))\right]^2}{u_{WW}(W(t))}. \end{aligned} \tag{56}$$

Furthermore, the optimal strategies are given by

$$c^*(t) = \left[u_W(W(t))\right]^{-\frac{1}{\gamma}} =: C(W(t)) \tag{57}$$

and

$$\pi^*(t) = -\frac{\mu-r}{\sigma^2}\frac{u_W(W(t))}{W(t)\,u_{WW}(W(t))} := \Pi(W(t)). \tag{58}$$

The proof is omitted for the sake of brevity.

Clearly, we face a highly non-linear second order ordinary differential equation (ODE) for u. We use some duality arguments to simplify this ODE.

LEMMA 4.6 *Let the convex dual of u be defined as*

$$\tilde{u}(y(t)) = \max_{W(t)>0}\left[u(W(t)) - y(t)\,W(t)\right]. \tag{59}$$

The ODE for u in (56) can then be written as

$$\begin{aligned} 0 &= \frac{\gamma}{1-\gamma} y(t)^{-\frac{1-\gamma}{\gamma}} + \lambda^S \eta \frac{[-\tilde{u}_y(y(t)) + Z^s]^{1-\gamma}}{1-\gamma} \\ &\quad -\beta^S \tilde{u}(y(t)) + \tilde{u}_y(y(t)) y(t) \left(\beta^S - r\right) + \kappa \tilde{u}_{yy}(y(t)) y(t)^2 . \end{aligned} \tag{60}$$

Furthermore, the optimal strategies are given by

$$c^*(t) = y(t)^{-\frac{1}{\gamma}} =: \tilde{C}(y(t)) \tag{61}$$

and

$$\pi^*(t) = -\frac{\mu - r}{\sigma^2} \frac{\tilde{u}_{yy}(y(t)) y(t)}{\tilde{u}_y(y(t))} =: \tilde{\Pi}(y(t)) . \tag{62}$$

Proof:
We conclude from (59) that the optimal value $W^*(t)$ solves the first-order condition[22]

$$u_W(W^*(t)) - y(t) = 0, \tag{63}$$

which implies

$$W^*(t) = I(y(t)), \tag{64}$$

where I is the inverse function of $u_W(W(t))$. We can therefore give the following expression for the convex dual

$$\tilde{u}(y(t)) = u(W^*(t)) - y(t) W^*(t) = u(I(y(t))) - y(t) I(y(t)) . \tag{65}$$

We next differentiate the dual function w.r.t. y and get

$$\begin{aligned} \tilde{u}_y(y(t)) &= u_W(I(y(t))) I_y(y(t)) - I(y(t)) - y(t) I_y(y(t)) \\ &= y(t) I_y(y(t)) - I(y(t)) - y(t) I_y(y(t)) \\ &= -I(y(t)) < 0. \end{aligned} \tag{66}$$

Exploiting (64) and (66) gives

$$W^*(t) = -\tilde{u}_y(y(t)) . \tag{67}$$

We next compute the second derivative of the dual function $\tilde{u}$ w.r.t. y using the fact that I and u_W are inverse functions. We then get

$$\tilde{u}_{yy}(y(t)) = -I_y(y(t)) = -\frac{1}{u_{WW}(I(y(t)))} > 0. \tag{68}$$

Hence, we have established the strict convexity of the dual function $\tilde{u}$ with the strict concavity of the primal function u. Now, we are ready to rewrite the ODE for the primal function u in (56) in terms of the dual function $\tilde{u}$ using (67), (65), (63) and (68). After some manipulations we arrive at (60). Lastly, the optimal strategies in (57) and (58) can be written in terms of the dual with (67), (63) and (68). ■

[22] This first-order condition is also sufficent with the strict concavity of u.

Note that we have transformed the highly non-linear ODE (56) for u into the slightly non-linear ODE (60) for $\tilde{u}$. The only non-linear term showing up in the ODE for $\tilde{u}$ involves the parameter η. Unfortunately, we cannot solve this ODE for $\tilde{u}$ symbolically on the dual continuation region $\tilde{D} = \left(0, \underline{y}\right)$. We therefore refuge to a numerical solution algorithm which means that we must have explicit boundary value conditions. Using the smooth paste condition for u in (54) and exploiting (65) at the boundary, we have

$$\tilde{u}\left(\underline{y}\right) = u\left(\underline{W}\right) - \underline{y}\underline{W} = K_0 \frac{\underline{W}^{1-\gamma}}{1-\gamma} + \frac{1}{\beta^S}\lambda^S \eta \frac{\left(Z^s\right)^{1-\gamma}}{1-\gamma} - \underline{y}\underline{W}. \tag{69}$$

Moreover, we use (67) at the threshold $\underline{y}$ and get

$$\tilde{u}_y\left(\underline{y}\right) = -\underline{W}. \tag{70}$$

The main problem is that we do not know the thresholds $\underline{y}$ and $\underline{W}$. However, we will construct an algorithm that simultaneously solves for the dual function $\tilde{u}$ as well as for the boundaries $\underline{y}$ and $\underline{W}$. The central idea of the algorithm is that the probability of annuitisation (wealth falling below $\underline{W}$) becomes zero as wealth approaches infinity. Thus, we impose that we get the constant Merton investment rule in the limit as wealth approaches infinity. Exploiting the optimal investment rule in terms of the dual function $\tilde{u}$ in (62), we therefore get the condition

$$\lim_{\substack{y(t)\downarrow 0 \\ n\to\infty}} -\frac{\tilde{u}_{yy}\left(\underline{W}_n; y\left(t\right)\right) y\left(t\right)}{\tilde{u}_y\left(\underline{W}_n; y\left(t\right)\right)} = \frac{1}{\gamma} \tag{71}$$

for the series $\tilde{u}\left(\underline{W}_n; y\left(t\right)\right)$ of dual functions with $n = 0, 1, 2...$ The series of dual functions $\tilde{u}\left(\underline{W}_n; y\left(t\right)\right)$ is constructed with the bisection method. We start with $\tilde{u}\left(\underline{W}_0^{low}; y\left(t\right)\right)$ and $\tilde{u}\left(\underline{W}_0^{up}; y\left(t\right)\right)$ satisfying $\lim\limits_{y(t)\downarrow 0} -\frac{\tilde{u}_{yy}\left(\underline{W}_0^{low}; y(t)\right) y(t)}{\tilde{u}_y\left(\underline{W}_0^{low}; y(t)\right)} - \frac{1}{\gamma} < 0$ and $\lim\limits_{y(t)\downarrow 0} -\frac{\tilde{u}_{yy}\left(\underline{W}_0^{up}; y(t)\right) y(t)}{\tilde{u}_y\left(\underline{W}_0^{up}; y(t)\right)} - \frac{1}{\gamma} > 0$, respectively,[23] and set $\underline{W}_0 = \frac{\underline{W}_0^{low} + \underline{W}_0^{up}}{2}$. Then we continue to update $\underline{W}_n^{low}$, $\underline{W}_n^{up}$ and $\underline{W}_n$ such that we keep the sign change for $n = 0, 1, 2...$ until some convergence criterion is met. At all stages of the algorithm we can exploit (63) and (55) to establish $\underline{y}_n = u_W\left(\underline{W}_n\right) = K_0 \underline{W}_n^{-\gamma}$. Thus, we can always solve the boundary value problem given by (60), (69) and (70) with $\underline{y}_n$ and $\underline{W}_n$ and by fixing the model parameters at the usual realistic values. Lastly, we naturally stop the algorithm as soon as we have

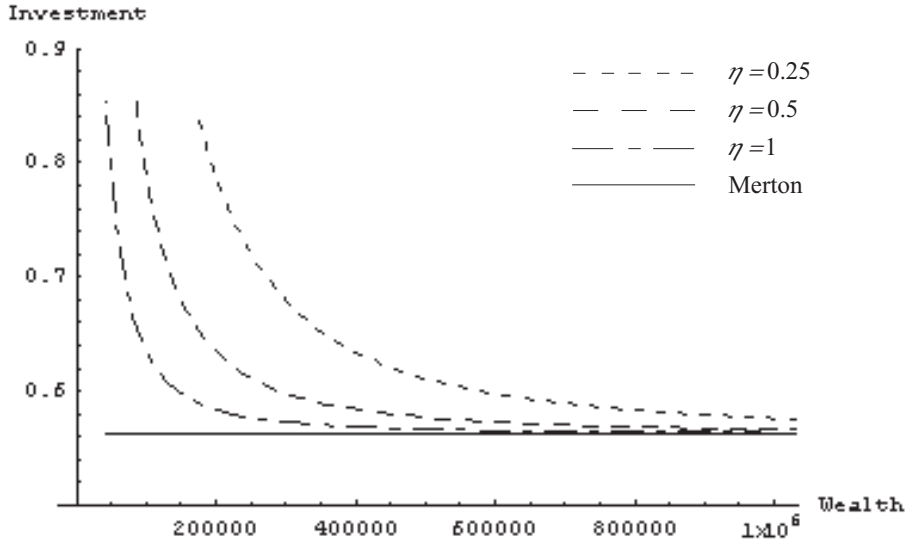
$$\left| \lim_{y(t)\downarrow 0} \tilde{\Pi}\left(\underline{W}_N; y\left(t\right)\right) - \pi^{Merton} \right| < \varepsilon \quad \text{for } \varepsilon > 0 \text{ and } N \in \mathbb{N}. \tag{72}$$

Once the above algorithm has converged after N steps, we can use $\tilde{u}\left(\underline{W}_N; y\left(t\right)\right)$ to compute the optimal consumption and investment strategy with (61) and (62). Moreover, we will naturally display the optimal strategies in terms of the primal wealth by exploiting $W\left(t\right) = -\tilde{u}_y\left(\underline{W}_N; y\left(t\right)\right)$ from (67).

[23] We narrow the range containing $\underline{W}_0^{low}$ and $\underline{W}_0^{up}$ by exploiting the fact that $\underline{W} \leq W^U$ where W^U denotes the smallest root of $L^{\text{com}} g\left(W\left(t\right)\right)$.

We next apply our numerical solution algorithm using the precision parameter $\varepsilon = 10^{-5}$ and the usual model parameters. We concentrate on the effects of the new parameters in the bequest case, i.e. the bequest motive η and prior life insurance net of subsistence Z^s. Figure 4.2 shows the investment rule of a pensioner with $Z^s = 500$ and $\gamma = 2$ for different values of the bequest motive. The additional option of the annuity market makes the pensioner more aggressive compared to the Merton investor for whom no annuity market exists at all ($\pi^{Merton} = \frac{\mu - r}{\sigma^2}\frac{1}{\gamma} = 0.5625$). Furthermore, we get the intuitive result that a stronger bequest motive decreases the threshold and therefore the probability of annuitisation. Figure 4.2 displays the corresponding optimal consumption fractions. These fractions have to be interpreted relative to the annuity of 11.76 which the pensioner gets if his wealth reaches the threshold. The option of annuitisation clearly leads to heavy consumption smoothing which is in contrast to the constant Merton consumption fraction.

Figure 1: Investment for different values of the bequest motive assuming $Z^s = 500$ and $\gamma = 2$.



Lastly, Figure 4.2 shows how the optimal portfolio strategy of a pensioner with $\eta = 0.75$ and $\gamma = 2$ depends on prior life insurance net of subsistence. Intuitively, a higher life insurance increases the attractiveness of the annuity market as there will be more wealth to bequeath to the heirs even if the pensioner annuitises. Figure 4.2 confirms this intuition by demonstrating that a higher life insurance increases the threshold and therefore the probability of annuitisation.

Figure 2: Consumption for different values of the bequest motive assuming $Z^s = 500$ and $\gamma = 2$.

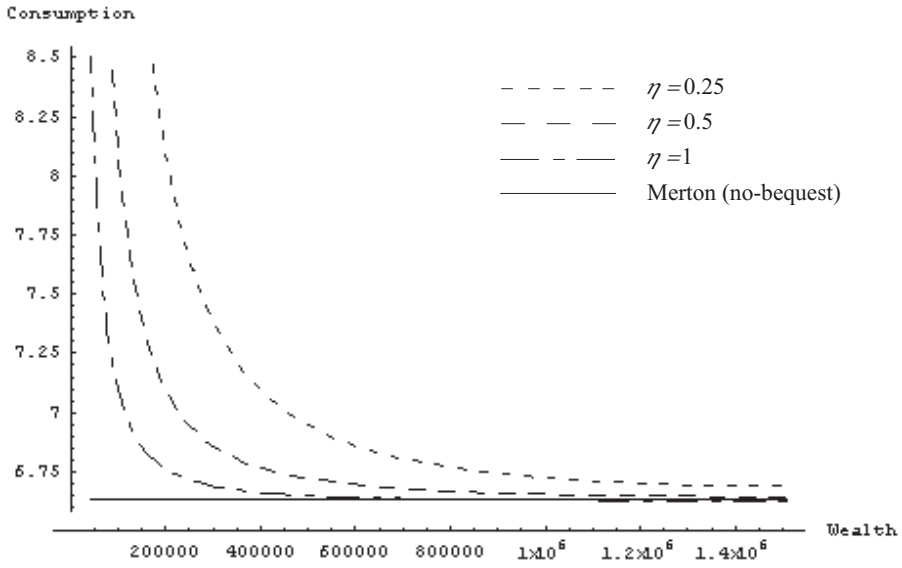
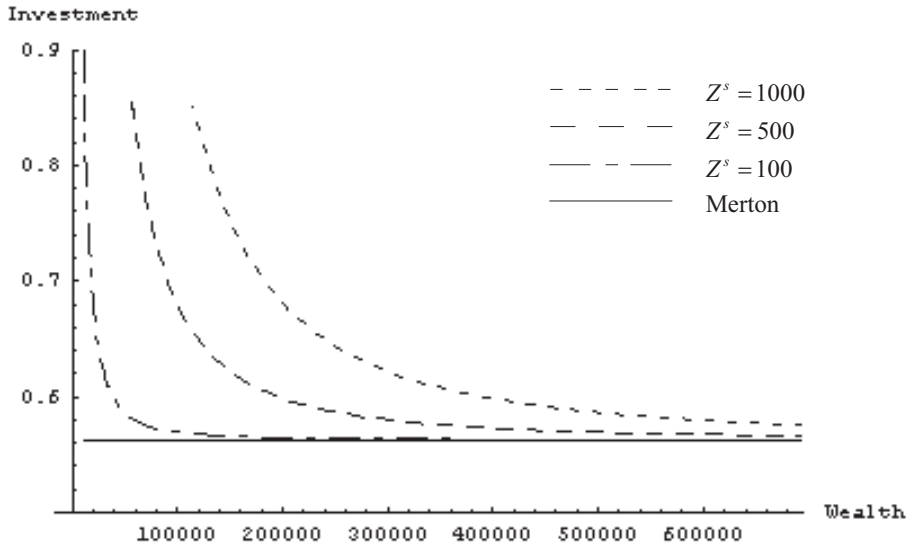


Figure 3: Investment for different values of prior life insurance net of subsistence assuming $\eta = 0.75$ and $\gamma = 2$.



5 Conclusions

The task of finding the optimal asset allocation, consumption and annuitisation time for a pensioner leads to a combined optimal stopping and optimal control problem (COSOCP). Assuming power utility with identical relative risk aversion and the exponential mortality law, the COSOCP normally reduces to a trivial or to a pure optimal control problem as in [18]. We extended the model of [18] by additionally studying the economically interesting range of relative risk aversion levels greater than one and by introducing a bequest motive and consequently, prior life insurance and a subsistence level of bequests. Normally, annuitisation is of the now-or-never type: Depending on the sign of some quantity the pensioner either annuitises immediately or never. Moreover, we found that the model parameters affect this decision in an intuitive manner. In contrast to [18] we also get a wealth-dependent annuitisation rule without imposing a higher relative risk aversion coefficient in the post-annuitisation phase. This real COSOCP occurs for relative risk aversion levels greater than one when we include a bequest motive. We solved this COSOCP via duality arguments and studied the effects of the bequest parameter and life insurance. The main result is that the essential inclusion of a bequest motive turns the very strong tendency for the annuity market into a slight tendency for the financial market. But even in the bequest case we find many realistic situations where the pensioner chooses the annuity market. This highlights the importance of longevity risk and therefore provides an additional legitimation for pension funds besides intergenerational risk transfer.

A Proof of Theorem 2.3

Proof:

a) We consider an arbitrary admissible consumption/investment/annuitisation time triple $(c, \pi, \tau) \in \mathcal{G}(w)$ with corresponding wealth $W(t)$. Applying the Itô formula to the function $e^{-\beta^S t} v(W(t))$, we get for $t \geq 0$

$$\begin{aligned} e^{-\beta^S t} v(W(t)) &= v(w) + \int_0^t e^{-\beta^S s} \left[Lv(W(s)) - \beta^S v(W(s)) \right] ds \\ &\quad + \int_0^t e^{-\beta^S s} \sigma \pi(s) W(s) v_W(W(s)) \, dB(s). \end{aligned}$$

Note that since τ is a stopping time, we know that $t \wedge \tau$ is a stopping time, too.

Hence, we can state the tautology

$$\begin{aligned}
&\int_0^{t\wedge\tau} e^{-\beta^S s} f\left(c\left(s\right), W\left(s\right)\right) ds + e^{-\beta^S (t\wedge\tau)} g\left(W\left(t\wedge\tau\right)\right) \\
= \;& v\left(w\right) + e^{-\beta^S (t\wedge\tau)} \left[g\left(W\left(t\wedge\tau\right)\right) - v\left(W\left(t\wedge\tau\right)\right)\right] \\
&+ \int_0^{t\wedge\tau} e^{-\beta^S s} \left[Lv\left(W\left(s\right)\right) - \beta^S v\left(W\left(s\right)\right) + f\left(c\left(s\right), W\left(s\right)\right)\right] ds \\
&+ \int_0^{t\wedge\tau} e^{-\beta^S s} \sigma\pi\left(s\right) W\left(s\right) v_W\left(W\left(s\right)\right) dB\left(s\right).
\end{aligned}$$

By the fact that $v\left(W\left(t\right)\right)$ satisfies (14), the definition of $L^{\mathrm{com}} v\left(W\left(t\right)\right)$ in (12) and (15), we have

$$\begin{aligned}
&\int_0^{t\wedge\tau} e^{-\beta^S s} f\left(c\left(s\right), W\left(s\right)\right) ds + e^{-\beta^S (t\wedge\tau)} g\left(W\left(t\wedge\tau\right)\right) \\
\leq \;& v\left(w\right) + \int_0^{t\wedge\tau} e^{-\beta^S s} L^{\mathrm{com}} v\left(W\left(s\right)\right) ds + \int_0^{t\wedge\tau} e^{-\beta^S s} \sigma\pi\left(s\right) W\left(s\right) v_W\left(W\left(s\right)\right) dB\left(s\right) \\
\leq \;& v\left(w\right) + \int_0^{t\wedge\tau} e^{-\beta^S s} \sigma\pi\left(s\right) W\left(s\right) v_W\left(W\left(s\right)\right) dB\left(s\right).
\end{aligned}$$

Taking the expectation yields due to (iv)

$$E^w \left[\int_0^{t\wedge\tau} e^{-\beta^S s} f\left(c\left(s\right), W\left(s\right)\right) ds + e^{-\beta^S (t\wedge\tau)} g\left(W\left(t\wedge\tau\right)\right)\right] \leq v\left(w\right). \tag{73}$$

We next take the limit as t approaches infinity, which means that $t\wedge\tau$ will tend to τ. The right hand side will trivially remain the same. We therefore discuss the left hand side separately. In accordance with the previously mentioned future needs we discuss two cases: First, we assume that the running and terminal reward function f and g, respectively, are both positive and secondly, we assume that they are both negative. We begin with the easier positive case. We then use the monotone convergence theorem to conclude that

$$\begin{aligned}
&\lim_{t\to\infty} E^w \left[\int_0^{t\wedge\tau} e^{-\beta^S s} f\left(c\left(s\right), W\left(s\right)\right) ds + e^{-\beta^S (t\wedge\tau)} g\left(W\left(t\wedge\tau\right)\right)\right] \\
= \;& E^w \left[\lim_{t\to\infty} \left\{\int_0^{t\wedge\tau} e^{-\beta^S s} f\left(c\left(s\right), W\left(s\right)\right) ds + e^{-\beta^S (t\wedge\tau)} g\left(W\left(t\wedge\tau\right)\right)\right\}\right] \\
= \;& E^w \left[\int_0^{\tau} e^{-\beta^S s} f\left(c\left(s\right), W\left(s\right)\right) ds + e^{-\beta^S \tau} g\left(W\left(\tau\right)\right)\right].
\end{aligned} \tag{74}$$

Now we consider the case where both functions f and g are negative. By definition we then have $f^- = -f$ and $g^- = -g$. Thus, exploiting the monotone convergence theorem, we get [24]

$$\begin{aligned}
&\lim_{t\to\infty} E^w\left[\int_0^{t\wedge\tau} e^{-\beta^S s} f\left(c\left(s\right), W\left(s\right)\right) ds + e^{-\beta^S (t\wedge\tau)} g\left(W\left(t\wedge\tau\right)\right)\right] \\
= \; & -\lim_{t\to\infty} E^w\left[\int_0^{t\wedge\tau} e^{-\beta^S s} f^-\left(c\left(s\right), W\left(s\right)\right) ds + e^{-\beta^S (t\wedge\tau)} g^-\left(W\left(t\wedge\tau\right)\right)\right] \\
= \; & -E^w\left[\int_0^{\tau} e^{-\beta^S s} f^-\left(c\left(s\right), W\left(s\right)\right) ds + e^{-\beta^S \tau} g^-\left(W\left(\tau\right)\right)\right] \\
= \; & E^w\left[\int_0^{\tau} e^{-\beta^S s} f\left(c\left(s\right), W\left(s\right)\right) ds + e^{-\beta^S \tau} g\left(W\left(\tau\right)\right)\right]. \qquad (75)
\end{aligned}$$

In both cases taking the limit of (73) as t approaches infinity therefore yields

$$J_{c,\pi,\tau}\left(w\right) \leq v\left(w\right).$$

The arbitrariness of the considered triple (c, π, τ) and of the initial wealth level implies

$$V\left(w\right) = \sup_{(c,\pi,\tau)\in\mathcal{G}(w)} J_{c,\pi,\tau}\left(w\right) \leq v\left(w\right) \quad \text{for all } w > 0,$$

which completes the first part of the proof.

b) Let us now consider the strategy (c^*, π^*, τ^*) defined in (19), (20) and (18) with corresponding wealth $W^*\left(t\right)$. The same steps as in a) lead to

$$\begin{aligned}
&\int_0^{t\wedge\tau^*} e^{-\beta^S s} f\left(c^*\left(s\right), W^*\left(s\right)\right) ds + e^{-\beta^S (t\wedge\tau^*)} g\left(W^*\left(t\wedge\tau^*\right)\right) \\
= \; & v\left(w\right) + e^{-\beta^S (t\wedge\tau^*)}\left[g\left(W^*\left(t\wedge\tau^*\right)\right) - v\left(W^*\left(t\wedge\tau^*\right)\right)\right] \\
& + \int_0^{t\wedge\tau^*} e^{-\beta^S s}\left[Lv\left(W^*\left(s\right)\right) - \beta^S v\left(W^*\left(s\right)\right) + f\left(c^*\left(s\right), W^*\left(s\right)\right)\right] ds \\
& + \int_0^{t\wedge\tau^*} e^{-\beta^S s} \sigma\pi^*\left(s\right) W^*\left(s\right) v_W\left(W^*\left(s\right)\right) dB\left(s\right).
\end{aligned}$$

Tautological arguments involving $e^{-\beta^S (t\wedge\tau^*)} g\left(W^*\left(t\wedge\tau^*\right)\right)$ and $e^{-\beta^S (t\wedge\tau^*)} v\left(W^*\left(t\wedge\tau^*\right)\right)$

[24] Note that we need to use the negative parts of the running and terminal reward functions, since the monotone convergence theorem assumes a monotonically increasing series of nonnegative functions.

yield

$$\begin{aligned}
&\int_0^{t\wedge\tau^*} e^{-\beta^S s} f\left(c^*\left(s\right), W^*\left(s\right)\right) ds + e^{-\beta^S (t\wedge\tau^*)} g\left(W^*\left(t\wedge\tau^*\right)\right) 1_{\{t\geq\tau^*\}} \\
= \;& v\left(w\right) + e^{-\beta^S (t\wedge\tau^*)} \left[g\left(W^*\left(t\wedge\tau^*\right)\right) - v\left(W^*\left(t\wedge\tau^*\right)\right)\right] 1_{\{t\geq\tau^*\}} \\
& - e^{-\beta^S (t\wedge\tau^*)} v\left(W^*\left(t\wedge\tau^*\right)\right) 1_{\{t<\tau^*\}} \\
& + \int_0^{t\wedge\tau^*} e^{-\beta^S s} \left[Lv\left(W^*\left(s\right)\right) - \beta^S v\left(W^*\left(s\right)\right) + f\left(c^*\left(s\right), W^*\left(s\right)\right)\right] ds \\
& + \int_0^{t\wedge\tau^*} e^{-\beta^S s} \sigma\pi^*\left(s\right) W^*\left(s\right) v_W\left(W^*\left(s\right)\right) dB\left(s\right). \qquad (76)
\end{aligned}$$

By the strict concavity assumption in (vi) of the functions $v\left(.\right)$ and $f\left(.\right)$ we know that the chosen strategy (c^*, π^*) defined in (19) and (20) satisfies

$$Lv\left(W^*\left(s\right)\right) - \beta^S v\left(W^*\left(s\right)\right) + f\left(c^*\left(s\right), W^*\left(s\right)\right) = L^{\text{com}} v\left(W^*\left(s\right)\right) \qquad (77)$$

for all $s < t\wedge\tau^*$.

Note next that because $W^*\left(s\right) \in D$ for all $s < t\wedge\tau^*$, we have

$$\int_0^{t\wedge\tau^*} e^{-\beta^S s} \left[Lv\left(W^*\left(s\right)\right) - \beta^S v\left(W^*\left(s\right)\right) + f\left(c^*\left(s\right), W^*\left(s\right)\right)\right] ds = 0 \qquad (78)$$

with (77) and (17). Furthermore, we know that

$$\begin{aligned}
&\left[g\left(W^*\left(t\wedge\tau^*\right)\right) - v\left(W^*\left(t\wedge\tau^*\right)\right)\right] 1_{\{t\geq\tau^*\}} \\
&= \left[g\left(W^*\left(\tau^*\right)\right) - v\left(W^*\left(\tau^*\right)\right)\right] 1_{\{t\geq\tau^*\}} = 0 \qquad (79)
\end{aligned}$$

because of (14), the definition of the continuation region D in (16) and the definition of the optimal stopping time τ^* in (18). Using (78) and (79), we can simplify (76) considerably to

$$\begin{aligned}
&\int_0^{t\wedge\tau^*} e^{-\beta^S s} f\left(c^*\left(s\right), W^*\left(s\right)\right) ds + e^{-\beta^S \tau^*} g\left(W^*\left(\tau^*\right)\right) 1_{\{t\geq\tau^*\}} \\
= \;& v\left(w\right) - e^{-\beta^S t} v\left(W^*\left(t\right)\right) 1_{\{t<\tau^*\}} + \int_0^{t\wedge\tau^*} e^{-\beta^S s} \sigma\pi^*\left(s\right) W^*\left(s\right) v_W\left(W^*\left(s\right)\right) dB\left(s\right).
\end{aligned}$$

Taking the expectation and using (iv) we obtain

$$\begin{aligned}
&E^w \left[\int_0^{t\wedge\tau^*} e^{-\beta^S s} f\left(c^*\left(s\right), W^*\left(s\right)\right) ds + e^{-\beta^S \tau^*} g\left(W^*\left(\tau^*\right)\right) 1_{\{t\geq\tau^*\}}\right] \\
= \;& v\left(w\right) - e^{-\beta^S t} E^w \left[v\left(W^*\left(t\right)\right) 1_{\{t<\tau^*\}}\right].
\end{aligned}$$

Next, we take the limit as $t \to \infty$, which means that $t \wedge \tau^* \to \tau^*$ and that $1_{\{t \geq \tau^*\}} \to 1_\Omega$. Now we would have to use the monotone convergence theorem and manipulations with negative parts in the case of a negative running and terminal reward function f and g, respectively. For the sake of brevity we omit this here, since it would work analoguously to the derivation of (74) and (75) in part a). Additionally, we then make use of (vii) to obtain

$$E^w \left[\int_0^{\tau^*} e^{-\beta^S s} f\left(c^*(s), W^*(s)\right) ds + e^{-\beta^S \tau^*} g\left(W^*(\tau^*)\right) \right] = v(w)$$

or stated differently,

$$J_{c^*,\pi^*,\tau^*}(w) = v(w).$$

We conclude that the upper bound of the value function $v(w)$[25] is attained by applying the strategy (c^*, π^*, τ^*). This and the arbitrariness of the initial wealth level w complete the proof.

■

B Proof of Lemma 2.8

Proof:
We choose an arbitrary initial wealth level $w \in U$ and let τ_U denote the first exit time from the set U. We impose that the pensioner behaves optimally. An application of Dynkin's formula to $e^{-\beta^S t} g(W^*(t))$ then yields

$$E^w \left[e^{-\beta^S(\tau_U \wedge t)} g\left(W^*(\tau_U \wedge t)\right) \right] = g(w) + E^w \left[\int_0^{\tau_U \wedge t} e^{-\beta^S s} \left(Lg\left(W^*(s)\right) - \beta^S g\left(W^*(s)\right) \right) ds \right]$$

for all $t > 0$. We use the operator L^{com} given in (12) to establish

$$\begin{aligned} & E^w \left[e^{-\beta^S(\tau_U \wedge t)} g\left(W^*(\tau_U \wedge t)\right) \right] \\ = \; & g(w) + E^w \left[\int_0^{\tau_U \wedge t} e^{-\beta^S s} \left(L^{\text{com}} g\left(W^*(s)\right) - f\left(c^*(s), W^*(s)\right) \right) ds \right]. \end{aligned}$$

The definition of the set U now implies

$$\begin{aligned} & E^w \left[\int_0^{\tau_U \wedge t} e^{-\beta^S s} f\left(c^*(s), W^*(s)\right) ds + e^{-\beta^S(\tau_U \wedge t)} g\left(W^*(\tau_U \wedge t)\right) \right] \\ = \; & g(w) + E^w \left[\int_0^{\tau_U \wedge t} e^{-\beta^S s} L^{\text{com}} g\left(W^*(s)\right) ds \right] > g(w). \end{aligned}$$

Thus, it is not optimal to stop if $w \in U$. Consequently, we have $w \in D$ and the arbitrariness of w completes the proof. ■

[25] See part a) of the proof.

C Proof of Theorem 3.1

Proof:

1. Cases 1 and 2 are easy to analyse. According to (34), we have $U = \mathbb{R}^+ \subset D \subset \mathbb{R}^+$ trivially implying that $D = \mathbb{R}^+$.

2. We next turn to cases 3 and 4. Since $U = \emptyset$, we have $L^{\text{com}} g(W(t)) \leq 0$ for all $W(t) > 0$. Hence, we know that $g(W(t))$ is a superharmonic function in the current setting. But since it is trivially a majorant of itself, it is also the least superharmonic majorant and therefore equal to the value function.[26] This implies that D is empty, too.

■

D Proof of Theorem 4.3

Proof:
We will exploit lemma 2.8, which states that U is a subset of D. Hence, we will first determine the set U, which by definition (26) consists of all wealth levels satisfying $L^{\text{com}} g(W(t)) > 0$. We will therefore make extensive use of the expression of $L^{\text{com}} g(W(t))$ in (28). Lastly, we will repeatedly exploit lemma 2.7, which states that D is an open and connected set. For convenience we recall from (28) that

$$L^{\text{com}} g(W(t)) = \frac{W(t)^{1-\gamma}}{1-\gamma} M^{nb} + \frac{\lambda^S \eta}{1-\gamma} \left[(W(t) + Z^s)^{1-\gamma} - (Z^s)^{1-\gamma} \right].$$

Using $1 - \gamma < 0$, we get the limits

$$\lim_{W(t) \downarrow 0} L^{\text{com}} g(W(t)) = -M^{nb} \infty \tag{80}$$

and

$$\lim_{W(t) \to \infty} L^{\text{com}} g(W(t)) = -\frac{\lambda^S \eta}{1-\gamma} (Z^s)^{1-\gamma} > 0. \tag{81}$$

Computing the derivative of $L^{\text{com}} g(W(t))$ with respect to wealth gives

$$\frac{\partial}{\partial W(t)} L^{\text{com}} g(W(t)) = M^{nb} W(t)^{-\gamma} + \lambda^S \eta (W(t) + Z^s)^{-\gamma}.$$

Evaluating the above derivative in the limit as wealth approaches zero yields

$$\lim_{W(t) \downarrow 0} \frac{\partial}{\partial W(t)} L^{\text{com}} g(W(t)) = M^{nb} \infty + \lambda^S \eta (Z^s)^{-\gamma}. \tag{82}$$

Clearly, we have to discuss two cases for the crucial quantity M^{nb}.

[26] The interested reader is referred to [15].

(i) We first study $M^{nb} \leq 0$.

1. If $M^{nb} < 0$, then we get $L^{\text{com}} g\left(W\left(t\right)\right) > 0$ for very small and very large wealth levels with (80) and (81). As U is a subset of the connected set D, we can conclude that $D = (0, \infty)$.
2. We next study $M^{nb} = 0$. We exploit (82) and (80) to establish that $L^{\text{com}} g\left(W\left(t\right)\right)$ is positive for very small wealth values. Using (81), we still have $L^{\text{com}} g\left(W\left(t\right)\right) > 0$ for very large wealth levels. Thus, with the same arguments as before we get $D = (0, \infty)$.

(ii) Lastly, we discuss $M^{nb} > 0$. According to (80) and (81) $L^{\text{com}} g\left(W\left(t\right)\right)$ becomes negative for very small values of wealth, while it is positive for very big wealth levels. With the continuity of $L^{\text{com}} g\left(W\left(t\right)\right)$ on $(0, \infty)$ we can apply the intermediate value theorem, which guarantees at least one root of $L^{\text{com}} g\left(W\left(t\right)\right)$ on $(0, \infty)$. We select the smallest such root in case there are several and denote it by $\underline{\mathrm{W}}^{U}$. The fact that U is a subset of the connected set D then implies that $D = (\underline{\mathrm{W}}, \infty)$ with $\underline{\mathrm{W}} \leq \underline{\mathrm{W}}^{U}$.

■

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