

A Life Cycle Model with Pension Benefits and Taxes

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Abstract

A life cycle model with pension benefits and taxes is analyzed by means of stochastic control. In the phase of employment an individual earns a stochastic income, contributes to a pension plan and chooses an optimal consumption and investment strategy under a tax system. At the end of the phase of employment the individual decides to fully or partially withdraw capital from the pension plan or to retire with no reduced pension benefits. During retirement an optimal consumption and investment strategy is chosen. It is shown that the individual profits from the financial protection against the uncertainty of her life span. Further, the decision on partial or full capital withdrawal from the pension fund depends crucially on the specification of the tax scheme. Under a uniform linear tax scheme and a fair pension benefit there will be no capital withdrawal. Under a more sophisticated tax scheme no, partial or full withdrawal may occur.

Key Words: Pension Finance, Life Cycle Models, Continuous Time Portfolio Theory, Defined Contribution Pension Plans

1 Introduction

For an individual with stochastic labor income and a pension plan optimal investment and consumption over the life cycle is a central issue. For such an individual our paper analyses these decisions under a tax scheme and examines the option of capital withdrawal from the pension fund. The life cycle consists of a period of activity with stochastic labor income and contributions to the pension plan and a period of retirement with pension benefits. The focus is on

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an individual with infinite planning horizon but facing an uncertain lifetime. For the sake of closed-form results we restrict the instantaneous force of mortality to be constant over each of the two periods. The pension fund is assumed to be risk neutral with respect to the mortality risk and invests only in the riskless asset. The individual with hyperbolic absolute risk aversion (HARA) over consumption compensates with a correspondingly aggressive investment strategy. In the basic model, we assume a universal linear tax scheme on all types of income. Later on, the basic model is extended. In particular, we assume that the investor is exposed to a progressive tax on non-financial income and to a linear tax on financial returns.

Our main findings are as follows. The existence of a pension plan leads to a substantial increase in welfare. For an individual with an infinite planning horizon, who does not take care of a bequest, financial wealth and future pension benefits are exposed to mortality risk in the same way. For the pension fund however, mortality clearly reduces the present value of liabilities. Due to this fact it can also be shown that under simple tax schemes and a fair pension benefit an individual should never withdraw her capital from the pension fund at retirement. Moreover, it is shown that the protection by a pension plan leads to a higher individual investment in equities.

In the extension of the basic model with a progressive tax scheme on non-financial income a partial withdrawal of capital from the pension fund may occur at retirement.

The life-cycle setting of the paper is somewhat related to Huang and Milevsky (2008) and Huang et al. (2008) but their focus is on optimal life insurance and their object of interest is not an individual investor but a so called family unit with one working breadwinner.

The paper falls within the portfolio choice literature. Optimization in the presence of labor income was pioneered by Merton (1971). Merton finds closed-form solution for the case with a constant wage and HARA utility. The work by Merton was extended in several ways. Duffie et al. (1997) discuss properties of the value function and the optimal policies under stochastic labor income. Koo (1998) and He and Pagès (1993) rule out short selling and/or imposing borrowing constraints under non-insurable labor income risk. Bodie et al. (1992) introduce flexible labor supply. Cocco et al. (2005) examine the effect of different labor income risk on the optimal portfolio policies of a life cycle investor under a realistic calibration of the labor income process. Benzoni et al. (2007) introduce a model where stock market and labor market are cointegrated. More recent work by Munk and Sørensen (2010) and Lynch and Tan (2009) deals with state-dependent labor income and financial markets. Portfolio choice with habit formation is treated by Gomes and Michaelides (2003), Polkovnichenko (2007) and Munk (2008). Hubbard (2008) and Sundaresan and Zapatero (1997) examine the impact of pension wealth on optimal asset allocation. Moreover, Bodie et al. (1994) and Dybvig and Liu (2010) introduce the option of a flexible retirement date.

From a technical perspective, the model contains two state variables - the risky stock and labor income. This results in a Hamilton-Jacobi-Bellmann

(HJB) partial differential equation that is non-linear. In order to get closed-form solutions it is necessary to either impose that labor income and the stock return are perfectly positively (negatively) correlated or that labor income volatility is zero. Although these assumptions do not match reality one-to-one, several papers have shown that the results of the exactly solvable special cases are qualitatively similar to the cases with non-perfect correlation¹. In general, these papers use numerical methods to solve an optimization problem for a specific set of parameters and there is no detailed sensitivity analysis. Hence we can state two points. On the one hand, we expect that our results are qualitatively valid even if correlation is not perfect. On the other hand, another contribution of our paper is to point out the sensitivities of the optimal policies with respect to parameter changes in the presence of labor income.

The rest of the paper is organized as follows. In section two we describe the setup of the basic scenario. The subsequent section shows the solution methodology and discusses the results of the base case. Section four shows the result of the model extension with a more realistic tax scheme and compares them to the base case. Mathematical derivations as the solution of the HJB-equation and other non-trivial derivations are provided in the appendices.

2 Life Cycle Model

2.1 Structure of the Model

The life of an individual consists of a phase of employment and a phase of retirement.

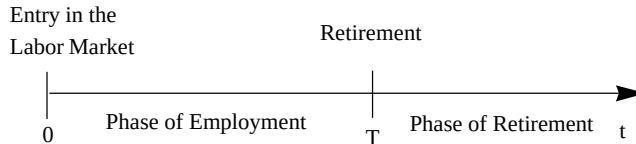


Figure 1: Life Cycle

As illustrated in Figure 1 the individual is active in the labor market from time 0 till time T . Afterwards she retires. During the phase of employment the individual earns a stochastic labor income, contributes to the pension plan, consumes and invests in a riskless and a risky asset. She is subject to a linear tax scheme on all types of income. Through the whole phase of activity the individual faces a constant force of mortality. In the phase of retirement the individual enjoys pension benefits and consumes. The tax scheme and the investment opportunities are the same as in the phase of employment. Again

¹See Cocco et al. (2005), Huang et al. (2008), Huang and Milevsky (2008), Bick et al. (2009) and Dybvig and Liu (2010)

there is a constant force of mortality that may differ from the force of mortality in the active period.

2.2 Phase of Employment

The labor income is given by the stochastic process

$$Y_l(t) = \bar{Y} + Y(t) \quad , \quad t \geq 0$$

with

$$\bar{Y} > 0$$

and

$$\frac{dY(t)}{Y(t)} = \mu_l dt + \sigma_l dZ_l(t)$$

where $Z_l(t)$ denotes a standard Brownian motion. For the sake of closed-form solutions, the dynamics of labor income are kept simple²

Consumption $c(t)$ at time t leads to an utility

$$u_1(c(t)) = \frac{e^{-\delta t}}{1-\gamma} (c(t) - \bar{c}_1)^{1-\gamma} \quad , \quad \gamma > 1$$

where $\bar{c}_1 \geq 0$ is the subsistence level of consumption, γ the coefficient of relative risk aversion and $\delta \geq 0$ is the time discount rate.

The financial market is given by a riskless and a risky investment opportunity with price processes

$$\frac{dS_0(t)}{S_0(t)} = r dt$$

with $r > 0$ and

$$\frac{dS_1(t)}{S_1(t)} = (r + \pi) dt + \sigma dZ(t)$$

with $\pi > 0$, $\sigma > 0$ and where $Z(t)$ denotes a standard Brownian motion and $dZ_l(t) dZ(t) = \rho dt$.

The investment policy is given by choosing the relative weight $\alpha(t)$ of the risky asset return and leads to a rate of return of the financial asset portfolio of

$$(r + \alpha(t)\pi) dt + \alpha(t)\sigma dZ(t)$$

$p \geq 0$ denotes the contribution to the pension plan which is assumed to be constant over time. $\tau \in [0, 1]$ is the tax rate which applies to all types of net

²In this setting, a low risky investment will be mainly due to a strong positive correlation between the risky asset and labor income, which is not consistent with empirical research. More realistic risky investment policies without the assumption of a strongly positive contemporaneous correlation can result with dynamic labor income. See, for example, Benzoni et al. (2007) or Lynch and Tan (2009). It can be shown that under the assumption of complete markets a specification similar to Lynch and Tan (2009) could be integrated and more realistic optimal investment policies would result. On the other hand, mathematical derivations would become more complicated and the impact on our results is of minor importance. As a consequence, this extension is omitted.

income. $\lambda_1 \geq 0$ denotes the force of mortality during the phase of employment. For the sake of closed-form solutions it is assumed to be constant³.

Finally, the dynamics of the financial wealth $M(t)$ during the phase of employment is given by

$$\begin{aligned} dM(t) = & (1 - \tau) M(t) [(r + \alpha(t) \pi) dt + \alpha(t) \sigma dZ(t)] \\ & + [(1 - \tau) (Y_l(t) - p) - c(t)] dt \end{aligned}$$

2.3 Phase of Retirement

Consumption $c(t)$ leads to

$$u_2(c(t)) = \frac{e^{-\delta t}}{1 - \gamma} (c(t) - \bar{c}_2)^{1-\gamma}, \quad \gamma > 1 \quad (1)$$

where δ and γ are the same as in the active period and $\bar{c}_2 \geq 0$ is the subsistence level of consumption that may differ from $\bar{c}_1$ ⁴.

The benefits from the pension plan are denoted by b . Investment opportunities and taxation are as in the phase of employment. λ_2 denotes the constant force of mortality during the phase of retirement and it is assumed that $\lambda_2 \geq \lambda_1$.

Summing up, the dynamics of financial wealth $M(t)$ during the phase of retirement are given by

$$\begin{aligned} dM(t) = & (1 - \tau) M(t) [(r + \alpha(t) \pi) dt + \alpha(t) \sigma dZ(t)] \\ & + [(1 - \tau) b - c(t)] dt \end{aligned} \quad (2)$$

2.4 Pension Plan

The pension fund is assumed to invest in the riskless asset only and is risk neutral with respect to mortality risk⁵. Therefore the accrued retirement benefits for an individual at T are

$$\int_0^T p e^{(r+\lambda_1)t} dt = p \frac{e^{(r+\lambda_1)T} - 1}{r + \lambda_1} \quad (3)$$

The present value of pension benefits for an individual at T is

$$\int_T^\infty b e^{-(r+\lambda_2)(t-T)} dt = \frac{b}{r + \lambda_2} \quad (4)$$

³Nevertheless, we show below that our main results are valid in the presence of more realistic hazard rates.

⁴Since retirement allows an individual to move into an other environment, the assumption of different levels of subsistence consumption seems to be reasonable.

⁵For the sake of analytical tractability, the investment policy of the pension plan is kept simple. The individual is able to offset the conservative investment policy of the pension plan by choosing a correspondingly aggressive investment strategy for financial wealth. For analytical tractability contributions proportional to income are excluded as well. In this case the fair contribution level would have to be calculated numerically and does not add much insights.

Equating (3) with (4) yields

$$b = p \frac{r + \lambda_2}{r + \lambda_1} \left[e^{(r + \lambda_1)T} - 1 \right] \quad (5)$$

As can be directly seen from the integrals, the sensitivity of the pension benefits b with respect to r , p , λ_1 , λ_2 and T are all non-negative.

3 Optimization and Results

The probability law for the lifetime θ of the individual is given by

$$\Pr \{ \theta \geq t \} = \begin{cases} e^{-\lambda_1 t} & , t \leq T \\ e^{-\lambda_1 T - \lambda_2 (t - T)} & , t > T \end{cases}$$

The conditional expected utility over the remaining lifetime for an individual at t is

$$E_t \left[\begin{aligned} & \int_0^{\min(\theta, T)} \frac{e^{-\delta s}}{1 - \gamma} (c(s) - \bar{c}_1)^{1 - \gamma} ds \\ & + \int_T^{\max(\theta, T)} \frac{e^{-\delta s}}{1 - \gamma} (c(s) - \bar{c}_2)^{1 - \gamma} ds \end{aligned} \right] , \quad t \leq \min(\theta, T)$$

$$E_t \left[\int_T^\theta \frac{e^{-\delta s}}{1 - \gamma} (c(s) - \bar{c}_2)^{1 - \gamma} ds \right] , \quad T < t < \theta$$

In the next steps we discuss the control strategy of the individual by using the dynamic programming principle. We will examine the associated Hamilton–Jacobi–Bellman (HJB) equation and focus on closed-form solutions.

3.1 Phase of Retirement

Given the utility function as specified in (1) and the wealth dynamics (2), the HJB-equation for $t \geq T$ is given by

$$\max_{c, \alpha} \left\{ \begin{aligned} & J_t + \frac{e^{-\delta t}}{1 - \gamma} (c(t) - \bar{c}_2)^{1 - \gamma} - \lambda_2 J \\ & + J_M [(1 - \tau) M(t) (r + \alpha(t) \pi) + (1 - \tau) b - c(t)] \\ & + \frac{1}{2} J_{MM} (1 - \tau)^2 M(t)^2 \alpha(t)^2 \sigma^2 \end{aligned} \right\} = 0 \quad (6)$$

It has to be noticed that during retirement consumption $c(t)$ must exceed the subsistence level $\bar{c}_2$. For this reason, feasible consumption plans over the phase of retirement exist under the assumption

$$M(T) + \frac{(1 - \tau) b - \bar{c}_2}{(1 - \tau) r} > 0 \quad (C.1)$$

As will become more clear below, (C.1) ensures that at the beginning of the phase of retirement financial wealth and pension benefits are sufficient to afford the future subsistence consumption. The standard transversality condition

$$\delta + \lambda_2 + (\gamma - 1) \left[(1 - \tau) r + \frac{1}{2\gamma} \frac{\pi^2}{\sigma^2} \right] > 0 \quad (T.1)$$

that is known from Merton (1971) adjusted for taxes and the hazard rate is obviously satisfied since we assume $\gamma > 1$.

As shown in Appendix A.1 the following result holds.

Proposition 1. *Under condition (C.1) one obtains*

$$1) \quad J(M(t), t) = \frac{ae^{-\delta t}}{1-\gamma} (M(t) + k)^{1-\gamma}, t \geq T \quad (7)$$

with

$$\begin{aligned} k &= \frac{(1-\tau)b - \bar{c}_2}{(1-\tau)r} \\ a &= \gamma^\gamma \left\{ \delta + \lambda_2 + (\gamma - 1) \left[(1-\tau)r + \frac{1}{2\gamma} \frac{\pi^2}{\sigma^2} \right] \right\}^{-\gamma} \\ 2) \quad c^*(t) &= \frac{1}{\gamma} \left\{ \delta + \lambda_2 + (\gamma - 1) \left[(1-\tau)r + \frac{1}{2\gamma} \frac{\pi^2}{\sigma^2} \right] \right\} (M(t) + k) + \bar{c}_2 \quad (8) \\ 3) \quad \alpha^*(t) &= \frac{1}{\gamma} \frac{\pi}{(1-\tau)\sigma^2} \frac{M(t)+k}{M(t)} \quad (9) \end{aligned}$$

Comments

ad 1) The constant k consists of the component $\frac{b}{r}$ and $\frac{\bar{c}_2}{(1-\tau)r}$. It can be noticed that the individual mortality risk has the same effect on future pension benefits as on financial wealth. Moreover, taxes on pension benefits and on the riskless return cancel out. Hence $\frac{b}{r}$ is the subjective present value of future pension benefits and represents the social security wealth. Similarly $\frac{\bar{c}_2}{(1-\tau)r}$ may be interpreted as the individual's covering to ensure her subsistence level. In other words, despite the fact that the future pension benefit (subsistence consumption) stream has a positive probability to end abruptly due to the mortality risk, the individual values the stream similar to a perpetual bond. The neglect of the mortality risk in the valuation of the future pension benefits will become important for the decision whether an annuity or a capital withdrawal is preferred.

This separation allows the following interpretation

$$W^{tot}(t) = M(t) + k = M(t) + W^s - R$$

with $W^{tot}(t)$ denoting the total wealth, $M(t)$ is the financial wealth, $W^s = \frac{b}{r}$ is the subjective social security wealth and $R = \frac{\bar{c}_2}{(1-\tau)r}$ denotes the reserves covering the subsistence level.

ad 2) Not surprisingly, optimal consumption depends linearly on total wealth $W^{tot}(t)$.

ad 3) In our model the pension fund chooses a riskless investment policy. As a compensation the ratio of total wealth to financial wealth occurs as a leverage factor in the investment policy of the individual. In fact, $\alpha^*(t) M(t) = \frac{1}{\gamma} \frac{\pi}{(1-\tau)\sigma^2} W^{tot}(t)$ may be interpreted as the Merton solution for the investment policy of total wealth with an adjustment for taxes.

3.2 Phase of Employment

For $t \leq T$ the HJB-equation for the J -function is given by

$$\max_{c, \alpha} \left\{ \begin{aligned} & \hat{J}_t + \frac{e^{-\delta t}}{1-\gamma} (c(t) - \bar{c}_1)^{1-\gamma} + \hat{J}_Y \mu_l Y(t) - \lambda_1 \hat{J} \\ & + \hat{J}_M [(1-\tau) M(t) (r + \alpha(t) \pi) + (1-\tau) (\bar{Y} + Y(t) - p) - c(t)] \\ & + \frac{1}{2} \left[\hat{J}_{MM} (1-\tau)^2 M(t)^2 \alpha(t)^2 \sigma^2 + \hat{J}_{YY} Y(t)^2 \sigma_l^2 \right. \\ & \quad \left. + 2 \hat{J}_{MY} (1-\tau) M(t) Y(t) \alpha(t) \rho \sigma \sigma_l \right] \end{aligned} \right\} = 0 \quad (10)$$

We shall need the two following assumptions.

$$\left(\begin{aligned} & M(0) + \frac{1-\tau}{\hat{g}} [1 - e^{-\hat{g}T}] Y(0) \\ & + \frac{(1-\tau)(\bar{Y}-p) - \bar{c}_1}{(1-\tau)r} + \frac{(1-\tau)(b+p-\bar{Y}) + \bar{c}_1 - \bar{c}_2}{(1-\tau)r} e^{-(1-\tau)rT} \end{aligned} \right) \geq 0 \quad (C.2)$$

where $\hat{g} = (1-\tau)r + \frac{\rho\sigma_l}{\sigma}\pi - \mu_l$

$$(1 - \rho^2) \sigma_l^2 = 0 \quad (C.3)$$

Under assumption (C.3) human capital can be completely hedged and there exists an analytical solution of the HJB-equation. Assumption (C.2) guarantees the existence of feasible consumption plans. Although (C.2) looks rather complicated at first sight, an intuitive interpretation will be provided in the first comment on the subsequent proposition. In fact, (C.2) states that the combination of initial wealth and income has to be sufficiently large to afford the subsistence level of consumption⁶.

As shown in Appendix A.2 the following result holds.

Proposition 2. *Under conditions (C.2) and (C.3) one obtains*

$$1) \quad \hat{J}(M(t), Y(t), t) = \frac{h(t)}{1-\gamma} (M(t) + j(t) Y(t) + k(t))^{1-\gamma}$$

with

$$h(t) = \left\{ \frac{\gamma}{\delta + g\gamma} e^{-\frac{\delta}{\gamma}t} - \left[\frac{\gamma}{\delta + g\gamma} - \frac{\gamma}{\delta + g\gamma + \lambda_2 - \lambda_1} \right] e^{gt - (g + \frac{\delta}{\gamma})T} \right\}^\gamma \quad (11)$$

⁶Moreover, it is easy to show that given $(r, T, \bar{c}_1, \bar{c}_2)$ for a sufficiently high $\bar{Y}$, every combination of non-negative $(M(0), Y(0))$ is admissible. If $\bar{Y}$ is not sufficiently large, then the $(M(0), Y(0))$ set is restricted but not empty.

where $g = \frac{\gamma-1}{\gamma} \left[(1-\tau)r + \frac{1}{2\gamma} \frac{\pi^2}{\sigma^2} \right] + \frac{\lambda_1}{\gamma}$

$$j(t) = \begin{cases} \frac{1-\tau}{\hat{g}} [1 - e^{\hat{g}(t-T)}] & , \hat{g} \neq 0 \\ (1-\tau)(T-t) & , \hat{g} = 0 \end{cases}$$

$$k(t) = \frac{(1-\tau)(\bar{Y}-p) - \bar{c}_1}{(1-\tau)r} + \frac{(1-\tau)(b+p-\bar{Y}) + \bar{c}_1 - \bar{c}_2}{(1-\tau)r} e^{(1-\tau)r(t-T)}$$

$$2) \quad c^*(t) = e^{-\frac{\delta}{\gamma}t} h(t)^{-\frac{1}{\gamma}} (M(t) + j(t)Y(t) + k(t)) + \bar{c}_1$$

$$3) \quad \alpha^*(t) = \frac{1}{\gamma} \frac{\pi}{(1-\tau)\sigma^2} \frac{M(t)+j(t)Y(t)+k(t)}{M(t)} - \frac{\rho\sigma_1}{(1-\tau)\sigma} \frac{j(t)Y(t)}{M(t)}$$

Comments

ad 1) As above, rearranging terms leads to

$$W^{tot}(t) = M(t) + j(t)Y(t) + k(t) = M(t) + H(t) + W^s(t) - R(t)$$

with $W^{tot}(t)$ denoting total wealth, $M(t)$ is the financial wealth,

$$H(t) = j(t)Y(t) + \frac{\bar{Y}}{r} [1 - e^{(1-\tau)r(t-T)}]$$

corresponds to human capital,

$$W^s(t) = -\frac{p}{r} [1 - e^{(1-\tau)r(t-T)}] + \frac{b}{r} e^{(1-\tau)r(t-T)} \quad (12)$$

is the subjective social security wealth and

$$R(t) = \frac{\bar{c}_1}{(1-\tau)r} [1 - e^{(1-\tau)r(t-T)}] + \frac{\bar{c}_2}{(1-\tau)r} e^{(1-\tau)r(t-T)}$$

denotes the reserves covering the subsistence level.

Obviously, human capital becomes zero at $t = T$, i.e. $H(T) = 0$. Furthermore, as expected, subjective social security wealth is strictly increasing in t and attains its maximum at $\frac{b}{r}$ at $t = T$. More surprising is the fact that subjective social security wealth is already strictly positive at $t = 0$, i.e. $W^s(0) > 0$. In fact, inserting (5) into (12) leads to

$$\begin{aligned} W^s(0) &= -\frac{p}{r} [1 - e^{-(1-\tau)rT}] + \frac{p}{r} \frac{r + \lambda_2}{r + \lambda_1} [e^{(r+\lambda_1)T} - 1] e^{-(1-\tau)r(t-T)} \\ &\geq -\frac{p}{r} [1 - e^{-(1-\tau)rT}] + \frac{p}{r} [e^{(\tau r + \lambda_1)T} - e^{-(1-\tau)rT}] \\ &= \frac{p}{r} [e^{(\tau r + \lambda_1)T} - 1] > 0 \end{aligned}$$

Tax advantages and the different role of mortality risk for the individual and the pension plan lead to this surprising result.

ad 2) Appendix A.2 contains the following result

$$\frac{c^*(t) - \bar{c}_1}{W^{tot}(t)} = \frac{1}{\gamma} \frac{(\delta + g\gamma)(\delta + g\gamma + \lambda_2 - \lambda_1)}{\delta + g\gamma + (\lambda_2 - \lambda_1) \left(1 - e^{(g + \frac{\delta}{\gamma})(t-T)}\right)}$$

Hence, $\frac{c^*(t) - \bar{c}_1}{W^{tot}(t)}$ is increasing in t for $\lambda_2 > \lambda_1$, and constant for $\lambda_2 = \lambda_1$. Moreover, from $H(T) = 0$ and $j(T) = 0$ and a comparison with Proposition 1 one finds no jump in the consumption exceeding the subsistence level $c^*(T^-) - \bar{c}_1 = c^*(T^+) - \bar{c}_2$.

ad 3) As in the phase of retirement the investment policy contains myopic demand $\frac{1}{\gamma} \frac{\pi}{(1-\tau)\sigma^2} \frac{W^{tot}(t)}{M(t)}$. In addition, the second term of the optimal investment policy is a hedging component⁷ for the stochastic part of human capital. Now one obtains,

$$\alpha^*(t) M(t) = \frac{1}{\gamma} \frac{\pi}{(1-\tau)\sigma^2} W^{tot}(t) - \frac{\rho\sigma_l}{(1-\tau)\sigma} j(t) Y(t)$$

In analogy to the preceding comment, one concludes no jump in investment strategy $\alpha^*(T^-) = \alpha^*(T^+)$.

3.3 Annuity or Capital Withdrawal at Retirement

According to Section 3.1 and the comments on Proposition 1, subjective social security wealth at retirement is

$$W^s(T) = \frac{b}{r} = \frac{p}{r} \frac{r + \lambda_2}{r + \lambda_1} \left[e^{(r+\lambda_1)T} - 1 \right]$$

For the pension fund, which is risk neutral with respect to uncertainty, the present value of the annuity is

$$V(T) = \frac{b}{r + \lambda_2} = \frac{p}{r + \lambda_1} \left[e^{(r+\lambda_1)T} - 1 \right]$$

The rate of substitution is

$$\frac{dV(T)}{dW^s(T)} = \frac{r}{r + \lambda_2} < 1 \quad (13)$$

Hence there will be no capital withdrawal.

Comments

- In our model the pension plan offers an insurance against the uncertainty about the individual life span without charging a risk premium. In the absence of a bequest motive this is highly beneficial for the individual. In

⁷This part is well known from the portfolio choice literature with labor income. See for example Koo (1998) or Munk (2000).

fact, the subjective valuation of the pension benefit stream does not involve the force of mortality. This is intuitive as the stream can be replicated by an investment of $\frac{b}{r}$ in the riskless asset. Since the pension fund is not willing to pay this amount, the individual is clearly better off without withdrawal. Finally, the force of mortality has an impact only on the subjective discount factor⁸ of the individual and pushes her to consume at a higher rate.

- A more realistic increasing hazard rate would not alter this result. For example, Pliska and Ye (2007) assume a linearly increasing hazard rate of the form $\lambda_2(t) = \bar{\lambda}_2 + \hat{\lambda}_2 t$. From (4) it becomes evident that $V(T)$ becomes even smaller. In particular, $\tilde{V}(T) = \int_T^\infty b e^{-(r+\bar{\lambda}_2+\hat{\lambda}_2 t)(t-T)} dt \leq \int_T^\infty b e^{-(r+\bar{\lambda}_2)(t-T)} dt = \frac{b}{r+\bar{\lambda}_2}$, since $\hat{\lambda}_2 > 0$.
- Under a universal linear tax scheme, taxes on pension benefits are offset by taxes on the riskfree rate that is used for discounting. Therefore, subjective social security wealth $W^s(T)$ does not depend on the tax rate and capital withdrawal has no tax advantage⁹. In an extended model (see Section 4) with a non-linear tax scheme a partial withdrawal of capital will be optimal in some cases.

Further, the results rely on the assumptions that the pension fund works efficiently and charges no risk premium. In order to shed light on this issue, we assume that the pension fund does not pay out an annuity of b but of $\hat{b} = \hat{s}b$ with $\hat{s} \in [0, 1]$ and we assume that the withdrawal amount remains¹⁰ $\frac{b}{r+\lambda_2}$. It is straightforward that (13) changes to

$$\frac{d\hat{V}(T)}{dW^s(T)} = \frac{r}{\hat{s}(r+\lambda_2)}$$

Hence withdrawal occurs if

$$\hat{s} \leq \frac{r}{(r+\lambda_2)} \quad (14)$$

As can be seen from (14) for realistic interest rates and hazard rates the individual is better off with the annuity even if the pension fund absorbs a considerable part of the fair benefit.

3.4 Protection by a Pension Plan and Investment Policy

Subjective Social Security Wealth and Valuation by the Pension Fund

⁸In Proposition 1, the subjective discount factor may be defined as $\delta_s = \delta + \lambda_2$ which explains the statement. A similar statement was already mentioned by Merton (1971).

⁹In this section it is implicitly assumed that there is no tax on the withdrawal amount. Considering a tax would even make the result more clear in favor of the annuity.

¹⁰Of course we could assume that at withdrawal the pension fund pays out $\tilde{s}\frac{b}{r+\lambda_2}$ with $\tilde{s} > \hat{s}$ but the argument remains the same.

According to Section 3.1 and Propositions 1 and 2 subjective social security wealth is given by

$$\begin{aligned}
W^s(t) &= \frac{b}{r} = \frac{p}{r} \frac{r + \lambda_2}{r + \lambda_1} \left[e^{(r+\lambda_1)T} - 1 \right] \quad , \quad t \geq T \\
W^s(t) &= -\frac{p}{r} \left[1 - e^{(1-\tau)r(t-T)} \right] + \frac{b}{r} e^{(1-\tau)r(t-T)} \\
&= -\frac{p}{r} + \frac{p}{r} \frac{1}{r + \lambda_1} \left[\begin{array}{c} (r + \lambda_2) e^{(r+\lambda_1)T} \\ -(\lambda_2 - \lambda_1) \end{array} \right] e^{(1-\tau)r(t-T)} \quad , \quad 0 \leq t \leq T
\end{aligned}$$

The corresponding valuations by the pension fund are

$$\begin{aligned}
V(t) &= \frac{b}{r + \lambda_2} = \frac{p}{r + \lambda_1} \left[e^{(r+\lambda_1)T} - 1 \right] \quad , \quad t \geq T \\
V(t) &= \frac{p}{r + \lambda_1} \left[e^{(r+\lambda_1)t} - 1 \right] \quad , \quad 0 \leq t \leq T
\end{aligned}$$

Proposition 3. *For $\lambda_2 \geq \lambda_1$ one obtains*

$$W^s(t) \geq V(t), t \geq 0$$

For the proof see Appendix A.3.

Comment

- According to Proposition 3 the individual is always better off with the pension plan than without. Hence, the existence of a pension plan leads to an increase in total wealth.

Investment over the Life Cycle

According to Propositions 1 and 2 the amount invested in equity is given by

$$\begin{aligned}
\alpha^*(t) M(t) &= \frac{1}{\gamma} \frac{\pi}{(1-\tau)\sigma^2} W^{tot}(t) \quad , \quad t \geq T \\
\alpha^*(t) M(t) &= \frac{1}{\gamma} \frac{\pi}{(1-\tau)\sigma^2} W^{tot}(t) - \frac{\rho\sigma_l}{(1-\tau)\sigma} j(t) Y(t) \quad , \quad 0 \leq t \leq T
\end{aligned}$$

Hence it can be concluded:

1. A withdrawal of capital from the pension fund would lead to a decrease of the amount invested in equity at any point in life cycle. This generalizes the statement from Section 3.2.
2. From $H(T) = 0$, $j(T) = 0$ it follows immediately that there is no jump in the investment strategy at retirement.

Comment

- These results clearly show that the conservative investment strategy by the pension fund is compensated by the individual's investment. Even more, according to Proposition 3 protection by a pension plan leads to higher total wealth and therefore to a higher amount invested in equities.

3.5 Sensitivity Analysis

3.5.1 Phase of Retirement

First of all, it should be noted that under the uniform linear tax scheme the withdrawal decision is not sensitive to parameter changes. Hence, in this section only the sensitivities of welfare, consumption and investment must be analyzed.

Proposition 4. *Under (C.1) for fixed $M(t), t > T$, the following results hold*

- 1) $\frac{\partial}{\partial \pi} J(M(t), t) > 0, \frac{\partial}{\partial \sigma} J(M(t), t) < 0, \frac{\partial}{\partial \tau} J(M(t), t) < 0$
- 2) $\frac{\partial}{\partial \pi} c^*(t) > 0, \frac{\partial}{\partial \sigma} c^*(t) < 0, \frac{\partial}{\partial \tau} c^*(t) < 0$
- 3) $\frac{\partial}{\partial \pi} \alpha^*(t) > 0, \frac{\partial}{\partial \sigma} \alpha^*(t) < 0$

Under the additional assumptions $\lambda_2 = \lambda_1$ and $\left(1 + \frac{\bar{c}_2}{(1-\tau)^p}\right) e^{-(r+\lambda_1)T} + Tr > 1$, for fixed $M(t)$ the following result holds for $t > T$,

- 4) $\frac{\partial}{\partial r} J(M(t), t) > 0, \frac{\partial}{\partial r} c^*(t) > 0, \frac{\partial}{\partial r} \alpha^*(t) > 0$

The proof can be found in Appendix A.4.

Comments

The results are as one would expect. Nevertheless, a few comments seem appropriate.

- For fixed pension benefits subjective social security wealth declines with an increasing interest rate r . However, an increasing interest rate leads also to a higher level of pension benefits and the reserves for the subsistence level of future consumption decreases as well. Therefore, under the additional assumptions of Proposition 4 one can show that the first effect dominates and an increase in r leads to a higher non-financial wealth and hence higher total wealth. As a consequence, the sensitivities with respect to the interest rate follow¹¹.

¹¹From the proof of Proposition 4 it should be noticed that the assumption $\lambda_2 = \lambda_1$ has no impact on the qualitative results here. There exists always a critical $\bar{r}$ where for $r > \bar{r}$ the sensitivities with respect to r are unambiguously positive. Our assumption ensures only that the critical region can be formulated by a simple formula.

- While the sensitivities of the value function and of consumption with respect to the tax rate are unambiguous, the tax rate has an ambiguous effect on the investment policy. On the one hand taxes reduce financial risk and the investment strategy becomes more aggressive. On the other hand non-financial wealth and therefore the leverage ratio is reduced.

3.5.2 Phase of Employment

The sensitivity analysis for the phase of employment is less straightforward. In particular, most sensitivities are ambiguous. In Proposition 5 we state results that rely on either no or only simple assumptions.

Proposition 5. *Given assumption (C.2), for fixed $M(t), Y(t)$ the following results hold for $0 \leq t \leq T$*

- 1) $\frac{\partial}{\partial \mu_l} \hat{J}(M(t), Y(t), t) > 0, \frac{\partial}{\partial \mu_l} c^*(t) > 0, \frac{\partial}{\partial \mu_l} \alpha^*(t) > 0$
- 2) $\frac{\partial}{\partial \sigma_l} \hat{J}(M(t), Y(t), t) > 0, \frac{\partial}{\partial \sigma_l} c^*(t) > 0, \frac{\partial}{\partial \sigma_l} \alpha^*(t) > 0$ for $\rho = -1$
- 3) $\frac{\partial}{\partial \sigma_l} \hat{J}(M(t), Y(t), t) < 0, \frac{\partial}{\partial \sigma_l} c^*(t) < 0$ for $\rho = 1$

Under the additional assumptions¹² that $\lambda_1 = \lambda_2$ and either $\rho = -1$ or $\sigma_l = 0$ one obtains

- 4) $\frac{\partial}{\partial \pi} \hat{J}(M(t), Y(t), t) > 0, \frac{\partial}{\partial \sigma} \hat{J}(M(t), Y(t), t) < 0$
- 5) $\frac{\partial}{\partial \pi} c^*(t) > 0, \frac{\partial}{\partial \sigma} c^*(t) < 0$
- 6) $\frac{\partial}{\partial \pi} \alpha^*(t) > 0, \frac{\partial}{\partial \sigma} \alpha^*(t) < 0$

Under the preceding assumptions and $\mu_l = 0, \sigma_l = 0$,

$$\begin{aligned} \bar{Y} - p - \frac{\bar{c}_1}{1 - \tau} &> 0, \\ b - \frac{\bar{c}_1}{1 - \tau} &< 0 \end{aligned}$$

one obtains

- 7) $\frac{\partial}{\partial \tau} \hat{J}(M(t), Y(t), t) < 0, \frac{\partial}{\partial \tau} c^*(t) < 0$

The proof can be found in Appendix A.5.

Comments

- The sensitivities with respect to labor income growth μ_l and risk σ_l are rather clear, since they have an impact on human capital only.

¹²It should be noticed that the results in 4), 5) and 6) are valid without the assumption $\lambda_1 = \lambda_2$. Nevertheless, this assumption simplifies the proof considerably. A proof without the assumption is available from the authors upon request.

- Since discounting depends on the tax rate τ one needs a strong assumption to prove that human capital is decreasing in τ . In particular, in order to guarantee negative sensitivities of welfare and consumption with respect to the tax rate we have to assume $\mu_l = \sigma_l = 0$. It should be noticed that the ambiguity arises due to the assumption of a uniform tax for all types of income¹³.
- Furthermore, a linear tax scheme on all types of income reduces financial risk. Hence, one expects that a higher tax rate leads to a more aggressive investment strategy. However, a higher tax rate reduces as well the ratio of total wealth to financial wealth. Therefore, the overall impact on the investment strategy is ambiguous.
- There are no clear results for the interest rate. The ambiguity with respect to the interest rate r stems from the fact that an increase in r leads to a decrease of human capital.

4 Modification: Progressive Taxes

In the standard model there was an universal tax scheme for all types of income. Under the assumption of a fair premium neither a full nor partial withdrawal of capital from the pension fund was attractive for the individual. In this section we show that a more sophisticated tax scheme may lead to a partial withdrawal of capital at retirement.

4.1 Model with a Progressive Income Tax

In order to keep the model tractable we assume¹⁴

$$Y_l(t) = \bar{Y} \quad (\text{C'.0})$$

The tax scheme is presented in Table 1. There is still a linear tax on capital income. However a progressive tax is applied on non-capital income. Due to the specification of the progressive tax as a quadratic function it is assumed that the parameters are chosen that

$$2\tau_2 \max(b, \bar{Y} - p) < 1 - \tau_1 \quad (15)$$

This assumption ensures that the marginal tax rate on non-financial income is less than one. Further, a progressive tax implies $\tau_2 > 0$.

¹³In the extended case we distinguish between tax rate on financial and non-financial income. In this case it is straightforward to show that human capital is decreasing in the tax rate on non-financial income.

¹⁴Including stochastic income with the assumptions from above is still possible but does not yield more insights with respect to the results below.

	Phase of employment	Phase of retirement
Taxes on capital income	$\tau_r M(t) \left[\begin{array}{c} (r + \alpha(t)\pi) dt \\ + \alpha(t)\sigma dZ(t) \end{array} \right]$	$\tau_r M(t) \left[\begin{array}{c} (r + \alpha(t)\pi) dt \\ + \alpha(t)\sigma dZ(t) \end{array} \right]$
Taxes on non-capital income	$\tau_0 + \tau_1 (\bar{Y} - p) + \tau_2 (\bar{Y} - p)^2$	$\tau_0 + \tau_1 b + \tau_2 b^2$
Withdrawal tax		τ_w

Table 1: Tax Scheme with a Progressive Tax on Non-Financial Income

4.1.1 Phase of Retirement

For $t \geq T$ the HJB-equation for the J -function $J(M(t), t)$ is given by

$$\max_{c, \alpha} \left\{ \begin{array}{c} J_t + \frac{e^{-\delta t}}{1-\gamma} (c(t) - \bar{c}_2)^{1-\gamma} - \lambda_2 J \\ + J_M [(1 - \tau_r) M(t) (r + \alpha(t)\pi) + b - (\tau_0 + \tau_1 b + \tau_2 b^2) - c(t)] \\ + \frac{1}{2} J_{MM} (1 - \tau_r)^2 M(t)^2 \alpha(t)^2 \sigma^2 \end{array} \right\} = 0$$

Assumption (C.1) has to be replaced by

$$M(T) + \frac{b - \bar{c}_2 - [\tau_0 + \tau_1 b + \tau_2 b^2]}{(1 - \tau_r) r} > 0 \quad (C'.1)$$

Modifying Proposition 1 of Section 3.1 leads to

Proposition 6. *Under condition (C'.1) one obtains*

$$1) \quad J(M(t), t) = \frac{ae^{-\delta t}}{1-\gamma} (M(t) + k)^{1-\gamma}, t \geq T$$

with

$$\begin{aligned} k &= \frac{b - \bar{c}_2 - (\tau_0 + \tau_1 b + \tau_2 b^2)}{(1 - \tau_r) r} \\ a &= \gamma^\gamma \left\{ \delta + \lambda_2 + (\gamma - 1) \left[(1 - \tau_r) r + \frac{1}{2\gamma} \frac{\pi^2}{\sigma^2} \right] \right\}^{-\gamma} \\ 2) \quad c^*(t) &= \frac{1}{\gamma} \left\{ \delta + \lambda_2 + (\gamma - 1) \left[(1 - \tau_r) r + \frac{1}{2\gamma} \frac{\pi^2}{\sigma^2} \right] \right\} (M(t) + k) + \bar{c}_2 \\ 3) \quad \alpha^*(t) &= \frac{1}{\gamma} \frac{\pi}{(1 - \tau_r) \sigma^2} \frac{M(t) + k}{M(t)} \end{aligned}$$

The proof is analogous to Proposition 1 and therefore omitted.

4.1.2 Phase of Employment

For $t \leq T$ the HJB-equation for the J -function $\hat{J}(M(t), t)$ is given by

$$\max_{c, \alpha} \left\{ \begin{aligned} & \hat{J}_t + \frac{e^{-\delta t}}{1-\gamma} (c(t) - \bar{c}_1)^{1-\gamma} - \lambda_1 \hat{J} \\ & + \hat{J}_M \left[\begin{aligned} & (1 - \tau_r) M(t) (r + \alpha(t) \pi) + \bar{Y} - p - c(t) \\ & - \left(\tau_0 + \tau_1 (\bar{Y} - p) + \tau_2 (\bar{Y} - p)^2 \right) \end{aligned} \right] \\ & + \frac{1}{2} \left[\hat{J}_{MM} (1 - \tau_r)^2 M(t)^2 \alpha(t)^2 \sigma^2 \right] \end{aligned} \right\} = 0$$

Assumption (C.2) has to be replaced by

$$M(0) + \frac{\bar{Y} - p - \bar{c}_1 - \left(\tau_0 + \tau_1 (\bar{Y} - p) + \tau_2 (\bar{Y} - p)^2 \right)}{(1 - \tau_r) r} (1 - e^{-(1 - \tau_r) r T}) + \frac{b - \bar{c}_2 - \left(\tau_0 + \tau_1 b + \tau_2 b^2 \right)}{(1 - \tau_r) r} e^{-(1 - \tau_r) r T} > 0 \quad (C'.2)$$

Modifying Proposition 2 of Section 3.2 leads to

Proposition 7. *Under conditions (C'.0) and (C'.2) one obtains*

$$1) \quad \hat{J}(M(t), t) = \frac{h(t)}{1-\gamma} (M(t) + k(t))^{1-\gamma}$$

with

$$h(t) = \left\{ \frac{\gamma}{\delta + g\gamma} e^{-\frac{\delta}{\gamma} t} - \left[\frac{\gamma}{\delta + g\gamma} - \frac{\gamma}{\delta + g\gamma + \lambda_2 - \lambda_1} \right] e^{gt - (g + \frac{\delta}{\gamma}) T} \right\}^\gamma$$

$$\text{where } g = \frac{\gamma-1}{\gamma} \left[(1 - \tau_r) r + \frac{1}{2\gamma} \frac{\pi^2}{\sigma^2} \right] + \frac{\lambda_1}{\gamma}$$

$$k(t) = \frac{\bar{Y} - p - \bar{c}_1 - \left(\tau_0 + \tau_1 (\bar{Y} - p) + \tau_2 (\bar{Y} - p)^2 \right)}{(1 - \tau_r) r} (1 - e^{(1 - \tau_r) r (t - T)}) + \frac{b - \bar{c}_2 - \left(\tau_0 + \tau_1 b + \tau_2 b^2 \right)}{(1 - \tau_r) r} e^{(1 - \tau_r) r (t - T)}$$

$$2) \quad c^*(t) = e^{-\frac{\delta}{\gamma} t} h(t)^{-\frac{1}{\gamma}} (M(t) + k(t)) + \bar{c}_1$$

$$3) \quad \alpha^*(t) = \frac{1}{\gamma} \frac{\pi}{(1 - \tau_r) \sigma^2} \frac{M(t) + k(t)}{M(t)}$$

The proof is analogous to Proposition 2 and therefore omitted.

4.2 Capital Withdrawal at Retirement

According to Proposition 7 subjective social security wealth at retirement is

$$W^s(T) = \frac{b - \tau_1 b - \tau_2 b^2}{(1 - \tau_r) r}$$

For the pension fund which is risk neutral with respect to mortality, the present value of the annuity is as above

$$V(T) = \frac{b}{r + \lambda_2}$$

Under the assumption that withdrawals are linearly taxed at a rate τ_w , the rate of substitution becomes

$$(1 - \tau_w) \frac{dV(T)}{dW^s(T)} = \frac{(1 - \tau_w)(1 - \tau_r)r}{r + \lambda_2} \frac{1}{1 - \tau_1 - 2\tau_2 b}$$

Obviously, the rate of substitution is increasing in b .

If the initial rate of substitution exceeds one a partial or full withdrawal of capital is optimal. The exact results are as follows:

Proposition 8. *Define*

$$b^* = \max \left(0, \frac{1}{2\tau_2} \left[1 - \tau_1 - \frac{(1 - \tau_w)(1 - \tau_r)r}{r + \lambda_2} \right] \right) \quad (16)$$

Then

1. *If $b^* = 0 \Leftrightarrow (1 - \tau_1)\lambda_2 \leq (\tau_1 - \tau_w - \tau_r + \tau_r\tau_w)r = (\tau_1 - \tau_w - (1 - \tau_w)\tau_r)r$ a full withdrawal of capital is optimal.*
2. *For $b > b^*$ benefits should be reduced to b^* which leads to a capital withdrawal of $\frac{b-b^*}{r+\lambda_2}$.*
3. *For $b \leq b^*$ capital should not be withdrawn.*

The proof is given in Appendix A.6.

Comments

- Due to the force of mortality λ_2 the pension plan can provide an additional marginal after tax rate $(1 - \tau_1)\lambda_2$. If the marginal tax advantage $(\tau_1 - \tau_w - \tau_r + \tau_r\tau_w)r$ offsets this ratio a full capital withdrawal becomes optimal¹⁵. Moreover, it is obvious that if there are no tax advantages $\tau_w \rightarrow \tau_1$ a full withdrawal will not occur, since $-(1 - \tau_w)\tau_r r < 0$.
- Under progressive taxation the rate of substitution $\frac{dV(T)}{dW^s(T)}$ depends on the level of pension benefits. Therefore, a partial withdrawal may be optimal.
- The progressive part of the tax rate τ_2 is not involved in the decision whether a full withdrawal is optimal or not. However, τ_2 determines whether a partial or no withdrawal takes place.

¹⁵Marginal tax rates at zero non-capital income.

- In the special case of $\tau_2 = 0$, it can be seen that the ratio $\frac{(1-\tau_w)(1-\tau_r)}{1-\tau_1}$ becomes important. In fact, a high tax rate on non-financial income compared to the tax rate of financial income is in favor of withdrawal. Moreover, the effect is the same as in the case of pension benefits which are lower as its fair value. Hence, the two effects go in the same direction and a withdrawal becomes more likely. It can be summarized that Proposition 8 illustrates that the decision on capital withdrawal depends crucially on the details of the tax scheme.
- As in the basic model, it can be shown that if the pension fund charges a risk premium on the paid out pension benefits, there exists a critical level $\hat{s}$ for s such that above this level no withdrawal occurs and below this level there is a partial/full withdrawal.

5 Conclusions

A life cycle model with a phase of employment, a phase of retirement, stochastic mortality and financial protection by a pension plan was analyzed by means of stochastic control. In order to get closed form solutions we assumed constant forces of mortality during the phase of employment and the phase of retirement. Since we analyzed an individual with no bequest motive the pension plan's financial protection against the uncertainty of the individual life span turned out to be beneficial. In fact, it can be stated that under an uniform linear tax scheme the individual should not withdraw capital from the fund. Under a more sophisticated tax system with progressive taxes on non-financial income and a linear tax on financial income partial or full withdrawal of capital may occur. It could also be shown that there is no jump in the investment policy at retirement. Since human capital tends to zero as retirement approaches, the hedging component in the investment portfolio tends to zero as well. Further, it is shown that a conservative investment strategy is fully compensated by a more aggressive investment strategy of the individual for her financial wealth.

It should be noticed that our model is clear in favor of no withdrawal because the force of mortality does not appear in the subjective valuation of the pension benefit stream. In this sense, our model takes into account that pension plans and life insurance institutions offer a financial protection against the uncertainty of the individual life-span. In the presence of market frictions as, for example, borrowing constraints, these results might change. In fact, Koo (1998) and Munk (2000) show that the value of future labor income lowers for an individual facing borrowing constraints, especially for individuals with low financial wealth. Hence, it can be expected that borrowing constraints would lower the value of the future benefit stream and make withdrawal more attractive since the present value of the benefits from the pension plan's point of view remain unchanged. A similar statement might be true under the assumption of a bequest motive. However, such models can not be solved by analytical methods and this task is left for future research.

Finally, the sensitivities of the investment strategy, consumption and welfare with respect to the riskfree rate, risk premium, the volatility of the risky investment and to the tax rate have an impact on income and discounting. Therefore, during the phase of employment the sensitivities with respect to the riskfree rate are ambiguous and results for the tax rate could only be obtained under restrictive assumptions.

A Appendix

A.1 Proof of Proposition 1

Condition (C.1) guarantees the existence of a feasible consumption/investment plan. Given (C.1), from the HJB-equation (6) one gets the optimal policies

$$c^*(t) = e^{-\frac{\delta}{\gamma}t} J_M^{-\frac{1}{\gamma}} + \bar{c}_2, \quad \alpha^*(t) = -\frac{J_M}{J_{MM}M(t)} \frac{\pi}{(1-\tau)\sigma^2}$$

Inserting $c^*(t)$ and $\alpha^*(t)$ into the HJB-equation leads to

$$J_t + \frac{\gamma}{1-\gamma} e^{-\frac{\delta}{\gamma}t} J_M^{1-\frac{1}{\gamma}} + J_M [(1-\tau)(M(t)r + b) - \bar{c}_2] - \frac{1}{2} \frac{J_M^2}{J_{MM}} \frac{\pi^2}{\sigma^2} - \lambda_2 J = 0 \quad (\text{A.1.1})$$

One tries

$$J(M(t), t) = \frac{ae^{-\delta t}}{1-\gamma} (M(t) + k)^{1-\gamma}, t \geq T$$

Plugging in the relevant partial derivatives into (A.1.1) leads to¹⁶

$$\begin{aligned} -\frac{\delta}{1-\gamma} (M+k) + \frac{\gamma}{1-\gamma} a^{-\frac{1}{\gamma}} (M+k) + (1-\tau)r(M+k) - (1-\tau)rk \\ + (1-\tau)b - \bar{c}_2 + \frac{1}{2} \frac{1}{\gamma} \frac{\pi^2}{\sigma^2} (M+k) - \frac{\lambda_2}{1-\gamma} (M+k) = 0 \end{aligned} \quad (\text{A.1.2})$$

Setting the coefficient of M equal to zero implies

$$a^{-\frac{1}{\gamma}} = \frac{1}{\gamma} \left\{ \delta + \lambda_2 + (\gamma-1) \left[(1-\tau)r + \frac{1}{2\gamma} \frac{\pi^2}{\sigma^2} \right] \right\} \quad (\text{A.1.3})$$

Setting the constant terms to zero yields

$$(1-\tau)rk = [(1-\tau)b - \bar{c}_2]$$

The results stated in Proposition 1 follow now immediately. ■

¹⁶For the sake of readability, the time subscripts are omitted.

A.2 Proof of Proposition 2

Condition (C.2) guarantees the existence of a feasible consumption/investment plan. Given conditions (C.2) and (C.3), from the HJB-equation (10) one gets the optimal policies

$$c^*(t) = e^{-\frac{\delta}{\gamma}t} \hat{J}_M^{-\frac{1}{\gamma}} + \bar{c}_1, \quad \alpha^*(t) = -\frac{\hat{J}_M}{\hat{J}_{MM}M(t)} \frac{\pi}{(1-\tau)\sigma^2} - \frac{\hat{J}_{MY}}{(1-\tau)\hat{J}_{MM}M(t)} \frac{\rho\sigma_l}{\sigma} Y(t)$$

Inserting $c^*(t)$ and $\alpha^*(t)$ into the HJB-equation and arranging properly yields

$$\begin{aligned} \hat{J}_t + \frac{\gamma}{1-\gamma} e^{-\frac{\delta}{\gamma}t} \hat{J}_M^{1-\frac{1}{\gamma}} + \hat{J}_M [(1-\tau)(M(t)r + Y(t) + \bar{Y} - p) - \bar{c}_1] \\ - \frac{1}{2} \frac{1}{\hat{J}_{MM}} \left(\hat{J}_M \frac{\pi}{\sigma} + \hat{J}_{MY} \rho\sigma_l \right)^2 + \hat{J}_Y \mu_l Y(t) + \frac{1}{2} \hat{J}_{YY} \sigma_l^2 Y(t)^2 - \lambda_1 \hat{J} = 0 \end{aligned} \quad (\text{A.2.1})$$

One tries

$$\hat{J}(M(t), Y(t), t) = \frac{h(t)}{1-\gamma} (M(t) + j(t)Y(t) + k(t))^{1-\gamma}$$

Plugging in the relevant partial derivatives into (A.2.1) leads to¹⁷

$$\begin{aligned} 0 = & \frac{1}{1-\gamma} \frac{h'(t)}{h(t)} (M + j(t)Y + k(t)) + j'(t)Y + k'(t) \\ & + \frac{\gamma}{1-\gamma} e^{-\frac{\delta}{\gamma}t} h(t)^{-\frac{1}{\gamma}} (M + j(t)Y + k(t)) \\ & + (1-\tau)(rM + Y + \bar{Y} - p) - \bar{c}_1 + j(t)\mu_l Y - \frac{\lambda_1}{1-\gamma} (M + j(t)Y + k(t)) \\ & + \frac{1}{2\gamma} (M + j(t)Y + k(t)) \left(\frac{\pi}{\sigma} - \gamma j(t)(M + j(t)Y + k(t))^{-1} Y \rho\sigma_l \right)^2 \\ & - \frac{1}{2} \gamma j(t)^2 (M + j(t)Y + k(t))^{-1} Y^2 \sigma_l^2 \end{aligned} \quad (\text{A.2.2})$$

Under assumption (C.3) (A.2.2) simplifies to¹⁸

$$\begin{aligned} 0 = & \frac{1}{1-\gamma} \frac{h'(t)}{h(t)} (M + j(t)Y + k(t)) + j'(t)Y + k'(t) \\ & + \frac{\gamma}{1-\gamma} e^{-\frac{\delta}{\gamma}t} h(t)^{-\frac{1}{\gamma}} (M + j(t)Y + k(t)) + (1-\tau)r(M + j(t)Y + k(t)) \\ & - (1-\tau)r(j(t)Y + k(t)) + (1-\tau)(Y + \bar{Y} - p) - \bar{c}_1 + j(t)\mu_l Y \\ & - \frac{\lambda_1}{1-\gamma} (M + j(t)Y + k(t)) + \frac{1}{2\gamma} \frac{\pi^2}{\sigma^2} (M + j(t)Y + k(t)) - j(t)Y \frac{\rho\sigma_l}{\sigma} \pi \end{aligned} \quad (\text{A.2.3})$$

¹⁷For the sake of readability, the time subscripts of $M(t)$ and $Y(t)$ are omitted.

¹⁸Without (C.3) there are no closed-form solutions available. See Duffie et al. (1997) or Koo (1998). However, it can be noticed that the critical terms $j(t)^2 Y^2 (M + j(t)Y + k(t))^{-1}$ vanish as the ratio of $M + k(t)$ and Y becomes large. In that case, the solution converges to the Merton solution as noted by Munk (2000).

Matching coefficients on M and setting to zero implies

$$\frac{1}{1-\gamma} \frac{h'(t)}{h(t)} + \frac{\gamma}{1-\gamma} e^{-\frac{\delta}{\gamma}t} h(t)^{-\frac{1}{\gamma}} + (1-\tau)r - \frac{\lambda_1}{1-\gamma} + \frac{1}{2\gamma} \frac{\pi^2}{\sigma^2} = 0 \quad (\text{A.2.4})$$

Thus,

$$h'(t) = -\gamma e^{-\frac{\delta}{\gamma}t} h(t)^{1-\frac{1}{\gamma}} + \left\{ (\gamma-1) \left[(1-\tau)r + \frac{1}{2\gamma} \frac{\pi^2}{\sigma^2} \right] + \lambda_1 \right\} h(t) \quad (\text{A.2.5})$$

with terminal condition $h(T) = ae^{-\delta T}$. (A.2.5) is a Bernoulli differential equation and the solution approach is well known¹⁹. Hence, a detailed derivation is omitted and the solution of $h(t)$ is given as in Proposition 2.

Matching coefficients of Y yields²⁰

$$j'(t) = \hat{g}j(t) - (1-\tau) \quad (\text{A.2.6})$$

where $\hat{g} = (1-\tau)r + \frac{\rho\sigma_1}{\sigma}\pi - \mu_l$ and with terminal condition $j(T) = 0$. (A.2.6) is a linear differential equation with constant coefficients. It can be verified that the solution in Proposition 2 is in accordance with (A.2.6) and the terminal condition²¹.

Finally, matching constant terms leads to

$$k'(t) - (1-\tau)rk(t) = \bar{c}_1 - (1-\tau)(\bar{Y} - p) \quad (\text{A.2.7})$$

Proposition 1 provides the terminal condition $k(T) = \frac{(1-\tau)b - \bar{c}_2}{(1-\tau)r}$. As before, (A.2.7) is a linear differential equation with constant coefficients and it can be verified that the solution is as stated in Proposition 2. The optimal policies follow directly from these results. ■

Analysis of $\frac{c^*(t) - \bar{c}_1}{W^{tot}(t)}$

$\frac{c^*(t) - \bar{c}_1}{W^{tot}(t)}$ is the net consumption wealth ratio where net consumption is defined as the consumption in excess of the subsistence level. From Proposition 2 we know that

$$\begin{aligned} \frac{c^*(t) - \bar{c}_1}{W^{tot}(t)} &= e^{-\frac{\delta}{\gamma}t} h(t)^{-\frac{1}{\gamma}} \\ &= \left\{ \frac{\gamma}{\delta + g\gamma} - \left[\frac{\gamma}{\delta + g\gamma} - \frac{\gamma}{\delta + g\gamma + \lambda_2 - \lambda_1} \right] e^{(g + \frac{\delta}{\gamma})(t-T)} \right\}^{-1} \\ &= \frac{1}{\gamma} \frac{(\delta + g\gamma)(\delta + g\gamma + \lambda_2 - \lambda_1)}{\delta + g\gamma + (\lambda_2 - \lambda_1) \left(1 - e^{(g + \frac{\delta}{\gamma})(t-T)} \right)} \end{aligned}$$

¹⁹See Zwillinger (1998, pp. 235-236).

²⁰Terms similar to (A.2.4) are directly set to zero.

²¹In the case of $\hat{g} = 0$ (A.2.6) can be solved by integration.

It should be noted that the time-variation stems exclusively from the difference in the hazard rates. Moreover, at $t = T$

$$\frac{c^*(T) - \bar{c}_1}{W^{tot}(T)} = \frac{1}{\gamma} (\delta + g\gamma + \lambda_2 - \lambda_1)$$

which is equal to the (constant) net consumption wealth ratio during the retirement phase.

A.3 Proof of Proposition 3

i) For $t \geq T$ one gets immediately

$$W^s(t) - V(t) = b \left[\frac{1}{r} - \frac{1}{r + \lambda_2} \right] \geq 0$$

ii) For $0 \leq t \leq T$ one obtains

$$\begin{aligned} W^s(t) - V(t) &= -\frac{p}{r} + \frac{p}{r} \frac{1}{r + \lambda_1} \left[(r + \lambda_2) e^{(r + \lambda_1)T} - (\lambda_2 - \lambda_1) \right] e^{(1 - \tau)r(t - T)} \\ &\quad - \frac{p}{r + \lambda_1} (e^{(r + \lambda_1)t} - 1) \\ &= -\frac{p}{r} + \frac{p}{r} \frac{1}{r + \lambda_1} \left[(r + \lambda_1) e^{(r + \lambda_1)T} + \lambda_1 (e^{(r + \lambda_1)T} - 1) \right. \\ &\quad \left. + \lambda_2 (e^{(r + \lambda_1)T} - 1) - \frac{p}{r + \lambda_1} (e^{(r + \lambda_1)t} - 1) \right] e^{(1 - \tau)r(t - T)} \\ &\geq \frac{p}{r} \left(e^{(r + \lambda_1)T} e^{(1 - \tau)r(t - T)} - 1 \right) - \frac{p}{r + \lambda_1} (e^{(r + \lambda_1)t} - 1) \\ &\geq \frac{p}{r} (e^{(r + \lambda_1)t} - 1) - \frac{p}{r + \lambda_1} (e^{(r + \lambda_1)t} - 1) \\ &\geq p (e^{(r + \lambda_1)t} - 1) \left[\frac{1}{r} - \frac{1}{r + \lambda_1} \right] \geq 0 \end{aligned}$$

■

A.4 Proof of Proposition 4

The sensitivities of k with respect to π, τ and σ and the sensitivities of a are straightforward and given by

$$\begin{aligned} \frac{\partial k}{\partial \pi} &= 0, \frac{\partial k}{\partial \tau} < 0, \frac{\partial k}{\partial \sigma} = 0 \\ \frac{\partial a}{\partial r} &< 0, \frac{\partial a}{\partial \pi} < 0, \frac{\partial a}{\partial \tau} > 0, \frac{\partial a}{\partial \sigma} > 0 \end{aligned} \tag{A.3.1}$$

The first part of Proposition 4 follows immediately from (A.3.1) and from Proposition 1.

Under the additional assumptions of Proposition 4

$$k = \frac{(1 - \tau)b - \bar{c}_2}{(1 - \tau)r}, \quad b = p [e^{(r + \lambda_1)T} - 1]$$

one obtains

$$k = \frac{p}{r} \left[e^{(r+\lambda_1)T} - 1 \right] - \frac{\bar{c}_2}{(1-\tau)r}$$

Hence,

$$\begin{aligned} \frac{\partial k}{\partial r} &= -\frac{p}{r^2} \left[e^{(r+\lambda_1)T} - 1 \right] + \frac{pT}{r} e^{(r+\lambda_1)T} + \frac{\bar{c}_2}{(1-\tau)r^2} \\ &= \frac{p}{r^2} e^{(r+\lambda_1)T} \left[\left(1 + \frac{\bar{c}_2}{(1-\tau)p} \right) e^{-(r+\lambda_1)T} + Tr - 1 \right] > 0 \end{aligned} \quad (\text{A.3.2})$$

The second part of Proposition A.4 follows immediately from (A.3.2) and Proposition 1. $\blacksquare$

A.5 Proof of Proposition 5

According to Proposition 2 it becomes evident that μ_l and σ_l have an impact on human capital only. The first derivatives of human capital with respect to μ_l and σ_l are straightforward and therefore omitted.

From Proposition 2 and $\lambda_1 = \lambda_2$ one obtains

$$h(t) = e^{-\delta t} \left\{ \frac{\gamma}{\delta + g\gamma} \right\}^\gamma \quad (\text{A.4.1})$$

Therefore

$$\frac{\partial h(t)}{\partial g} < 0 \quad (\text{A.4.2})$$

Moreover, Proposition 2 leads to

$$\frac{\partial g}{\partial \pi} > 0, \frac{\partial g}{\partial \sigma} < 0, \frac{\partial g}{\partial \tau} < 0 \quad (\text{A.4.3})$$

Furthermore²²

$$\begin{aligned} \frac{\partial j(t)}{\partial \hat{g}} &= -\frac{1-\tau}{\hat{g}^2} \left[1 - e^{\hat{g}(t-T)} \right] + \frac{(1-\tau)(T-t)}{\hat{g}} e^{\hat{g}(t-T)} \\ &= \frac{1-\tau}{\hat{g}^2} e^{-\hat{g}(T-t)} \left[1 + \hat{g}(T-t) - e^{\hat{g}(T-t)} \right] < 0 \end{aligned} \quad (\text{A.4.4})$$

The condition $\rho = -1$ or $\sigma_l = 0$ yields

$$\frac{\partial \hat{g}}{\partial \pi} \leq 0, \frac{\partial \hat{g}}{\partial \sigma} \geq 0 \quad (\text{A.4.5})$$

Obviously

$$\frac{\partial k(t)}{\partial \pi} = 0, \frac{\partial k(t)}{\partial \sigma} = 0 \quad (\text{A.4.6})$$

²²Although not explicitly shown, the case $\hat{g} = 0$ is well-defined.

hold.

From Proposition 2 and (A.4.1) - (A.4.6) parts 4-6 of Proposition 5 follow immediately.

Assuming $\mu_l = 0$ and $\sigma_l = 0$ leaves no stochastic income and hence, the part of total wealth containing $j(t)$ disappears. Under the assumptions

$$\begin{aligned}\bar{Y} - p - \frac{\bar{c}_1}{1 - \tau} &> 0 \\ b - \frac{\bar{c}_1}{1 - \tau} &< 0\end{aligned}$$

and rewriting $k(t)$ yields

$$k(t) = \underbrace{\left(\frac{\bar{Y} - p}{r} - \frac{\bar{c}_1}{(1 - \tau)r}\right)}_{>0} \underbrace{\left(1 - e^{-(1-\tau)r(T-t)}\right)}_{>0} + \underbrace{\left(\frac{b}{r} - \frac{\bar{c}_2}{(1 - \tau)r}\right)}_{<0} e^{-(1-\tau)r(T-t)} \quad (\text{A.4.7})$$

Inspecting (A.4.7) leads to

$$\frac{\partial k(t)}{\partial \tau} < 0 \quad (\text{A.4.8})$$

Now, part 7 of Proposition 5 follows from Proposition 2 and (A.4.1) - (A.4.3), (A.4.7) and (A.4.8). $\blacksquare$

A.6 Proof of Proposition 8

Given the opportunity of full or partial withdrawals the optimal level of pension benefits b^* is attained at

$$(1 - \tau_w) \frac{dV(T)}{dW^s(T)} = 1, \text{ for } b^* > 0$$

or

$$\frac{(1 - \tau_w)(1 - \tau_r)r}{r + \lambda_2} \frac{1}{1 - \tau_1 - 2\tau_2 b^*} = 1, \text{ for } b^* > 0$$

This leads to

$$b^* = \max\left(0, \frac{1}{2\tau_2} \left[1 - \tau_1 - \frac{(1 - \tau_w)(1 - \tau_r)r}{r + \lambda_2}\right]\right)$$

A full withdrawal is optimal for

$$b^* = 0$$

or

$$(r + \lambda_2)(1 - \tau_1) - (1 - \tau_w)(1 - \tau_r)r \leq 0$$

$$\Leftrightarrow$$

$$(1 - \tau_1)\lambda_2 \leq (\tau_1 - \tau_r - \tau_w + \tau_r\tau_w)r$$

$\blacksquare$

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