Optimal Learning via Moderate Deviations Theory

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joint work with A. Ganguly

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Learning from data: setting

Function to learn	Family of prob. measures	Data-gen. process
$c(\theta)$	$\{\mathbb{P}_\theta : \theta \in \Theta\}$	$\{X_t\}_{t \in \mathbb{N}}$

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Cost $c : \Theta \to \mathbb{R}$ uniformly continuous on bounded subsets of Θ

Examples:

- Expected loss $c(\theta) = \mathbb{E}_\theta[\ell(X)]$
- Risk of loss $c(\theta) = \rho_\theta[\ell(X)]$
- Long-run average loss $c(\theta) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_\theta[\ell(\pi(s_t), s_t)]$
- Stochastic optimization $c(\theta) = \min_z \mathbb{E}_\theta[\ell(z, X)]$
- ...

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Assumptions:

- All random variables defined on $(\Omega, \mathcal{F})$
- $\Theta \subseteq \mathcal{V}$ open, $\mathcal{V}$ normed vector space
  - e.g., non-parametric setting $\Theta = \mathcal{P}(\mathcal{S})$, $\mathcal{S} \subset \mathbb{R}^n$

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$c(\theta)$	$\{\mathbb{P}_\theta : \theta \in \Theta\}$	$\{X_t\}_{t\in\mathbb{N}}$

- $\{X_t\}_t$ is an $\mathcal{F}_t$-adapted stochastic process with state space $\mathcal{S} \subset \mathbb{R}^n$
- Estimator $\{\widehat{\theta}_T\}$ is $\mathcal{F}_t$-adapted and weakly consistent for θ

Examples:

- I.i.d. processes
- Markov chains
- Vector-autoregressive processes
- I.i.d. processes with parametric distribution function

Motivating example: asymptotic variance of OU process

- $\mathrm{d}X_t = -\theta X_t \mathrm{d}t + \mathrm{d}W_t, \quad X_0 = 0$
- $\Theta = \mathbb{R}_{>0}$, $\mathcal{V} = \mathbb{R}$
- Asymptotic variance

$$c(\theta) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \left(\mathbb{E}_\theta[X_t^2] - \mathbb{E}_\theta[X_t]^2\right) = \frac{1}{2\theta}$$

- Maximum likelihood estimator

$$\widehat{\theta}_T = -\frac{X_T^2 - T}{2\int_0^T X_t^2 \mathrm{d}t}$$

Interval estimation

Goal: Estimate $c(\theta)$ via a confidence interval $\widehat{\mathcal{I}}^{\star}_{T,r} = \mathcal{I}^{\star}_{T,r}(\widehat{\theta}_T)$ where $\mathcal{I}^{\star}_{T,r}(\cdot) = \left[\underline{c}^{\star}_{T,r}(\cdot), \bar{c}^{\star}_{T,r}(\cdot)\right]$ such that $(r > 0)$

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1. **Exponential accuracy:**

$$\mathbb{P}_{\theta}(c(\theta) \notin \widehat{\mathcal{I}}^{\star}_{T,r}) \leq e^{-rb_T}, \qquad 1 \ll b_T \ll T$$

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3. **Consistency:** $\widehat{\mathcal{I}}^\star_{T,r} \to \{c(\theta)\}$ as $T \to \infty$
4. **Mischaracterization probability:**

$$\mathbb{P}_\theta(c(\theta') \notin \widehat{\mathcal{I}}^\star_{T,r}) > e^{-rb_T}, \qquad \forall \theta' \ : \ c(\theta') \neq c(\theta)$$

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5. **Uniformly most accurate (UMA):** Any interval $\widehat{\mathcal{I}}_{T,r}$ satisfying 1 is such that

$$\mathbb{P}_\theta(c(\theta') \in \widehat{\mathcal{I}}^\star_{T,r}) \leq \mathbb{P}_\theta(c(\theta') \in \widehat{\mathcal{I}}_{T,r}), \qquad \forall \theta' \ : \ c(\theta') \neq c(\theta)$$

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- Confidence interval family of the form

$$\mathcal{I}_{T,r}(\cdot) = \left[\underline{c}_{T,r}(\cdot), \bar{c}_{T,r}(\cdot)\right]$$

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Definition. An interval family $\{\mathcal{I}_{T,r}\} \in \mathbb{I}$ is called exponentially accurate if

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- $\mathbb{I}_r \subset \mathbb{I}$ denotes the set of exponentially accurate confidence interval families

Idea: Find an interval family in $\mathbb{I}_r$ that satisfies ② - ⑤

① Exponential accuracy

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Heuristic CLT based confidence intervals:

- Given a fixed α, CLT guarantees

$$\lim_{T\to\infty} \mathbb{P}_\theta(c(\theta) \notin \mathcal{I}^{\mathrm{CLT}}_{T,\alpha}(\widehat{\theta}_T)) \leq \alpha$$

- for the CLT-based interval

$$\mathcal{I}^{\mathrm{CLT}}_{T,\alpha}(\widehat{\theta}_T) = \left[c(\widehat{\theta}_T) - \kappa^{\mathrm{CLT}}_T(\alpha), c(\widehat{\theta}_T) + \kappa^{\mathrm{CLT}}_T(\alpha)\right]$$

$$\kappa^{\mathrm{CLT}}_T(\alpha) = \Phi^{-1}(1-\alpha/2)\sqrt{\nabla c(\widehat{\theta}_T)^\top S(\widehat{\theta}_T)\nabla c(\widehat{\theta}_T)}/\sqrt{T}$$

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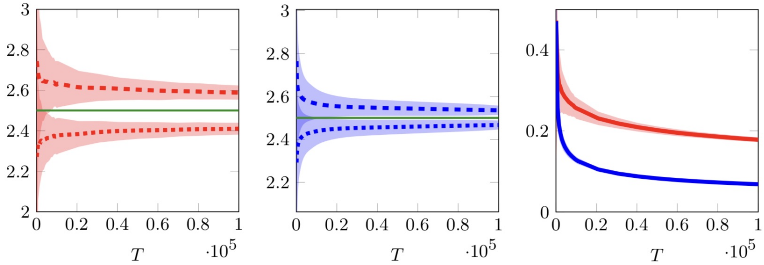
Question: How about minimality of the interval $\mathcal{I}^{\mathrm{CLT}}_{T,\alpha}(\widehat{\theta}_T)$?

Example: Asymptotic variance of OU process



(a) Upper and lower confidence bound (b) Upper and lower confidence bound (c) Interval length

- Optimal value $c(\theta)$
- Optimal interval $\mathcal{I}_T^\star(\widehat{\theta}_T)$
- CLT interval $\mathcal{I}_{T,\alpha}^{\text{CLT}}(\widehat{\theta}_T)$, $\alpha = e^{-rb_T}$

② Minimality

- $\mathcal{W}$ be a metric space
- $\mathcal{M}(\mathcal{W}, \mathbb{R})$ the class of measurable functions from $\mathcal{W}$ to $\mathbb{R}$

Definition: A family of functions $\{\phi_T : T > 0\} \subset \mathcal{M}(\mathcal{W}, \mathbb{R})$ is called *eventually and uniformly smaller* than $\{\psi_T : T > 0\}$ on compact sets if for any compact subset $\mathcal{W}_0 \subset \mathcal{W}$,

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- An interval family $\mathcal{I}_T^\star(\cdot) = [\underline{c}_T^\star(\cdot), \bar{c}_T^\star(\cdot)] \in \mathbb{I}_r$ is called *locally uniform minimal*, if for any other interval $\mathcal{I}_T(\cdot) = [\underline{c}_T(\cdot), \bar{c}_T(\cdot)] \in \mathbb{I}_r$
  - $\underline{c}_T^\star(\cdot)$ is eventually and uniformly larger than $\underline{c}_T(\cdot)$, and
  - $\bar{c}_T^\star(\cdot)$ is eventually and uniformly smaller than $\bar{c}_T(\cdot)$

③ Consistency

- Confidence interval of the form

$$\mathcal{I}_{T,r}(\widehat{\theta}_T) = [\underline{c}_{T,r}(\widehat{\theta}_T), \bar{c}_{T,r}(\widehat{\theta}_T)]$$

- **Consistency:** for every $\theta \in \Theta$, both

$$\underline{c}_{T,r}\left(\widehat{\theta}_T\right), \bar{c}_{T,r}\left(\widehat{\theta}_T\right) \xrightarrow{\mathbb{P}_\theta} \theta, \quad \text{as } T \to \infty$$

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- Asymptotic version of what is often referred to as an unbiased confidence interval

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Definition: We call a family $\{\mathcal{I}^{\star}_{T,r}\} \in \mathbb{I}_r$ (eventually) uniformly most accurate (UMA) if for any other family $\{\mathcal{I}_{T,r}\} \in \mathbb{I}_r$ and as $T \to \infty$

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- UMA-CIs are typically constructed via uniformly most powerful (UMP) tests
- in "simple" settings the Neyman-Pearson lemma ensures that the likelihood ratio defines an UMP test
- Often existence and construction of UMA-CIs is nontrivial

Assumptions

- $1 \ll b_T \ll T$, e.g., $b_T = \sqrt{T}$
- weak consistency: $\widehat{\theta}_T \overset{\mathbb{P}_{\theta_0}}{\to} \theta_0$ as $T \to \infty$ for all $\theta_0 \in \Theta$
- $\{\widehat{\theta}_T\}$ satisfies an Moderate Deviations Principle (MDP) in $\mathcal{V}$ with speed b_T and rate function $I^M(\cdot, \theta_0)$
- "Some more" technical assumptions

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Question: Find family $\mathcal{I}_T^\star(\cdot) = [\underline{c}_{T,r}^\star(\cdot), \bar{c}_{T,r}^\star(\cdot)] \in \mathbb{I}_r$ that satisfies 2 - 5?

Optimal confidence interval

- $\underline{c}^{\star}_{T,r}(\theta') = \inf_{\theta\in\Theta}\{c(\theta) \ : \ I^M(a_T(\theta'-\theta),\theta) \le r\}$
- $\bar{c}^{\star}_{T,r}(\theta') = \sup_{\theta\in\Theta}\{c(\theta) \ : \ I^M(a_T(\theta'-\theta),\theta) \le r\}$
- where
  - $I^M(\cdot,\theta)$: Moderate deviation rate function of $\{\widehat{\theta}_T\}_T$
  - $a_T = \sqrt{T/b_T}, \quad 1 \ll b_T \ll T, \quad (\text{ex. } a_T = T^{1/4})$

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- Θ can be infinite dimensional
- mild assumptions on $I^M(\cdot,\theta)$

Example: Asymptotic variance of OU process

- Ornstein-Uhlenbeck process

$$\mathrm{d}X_t = -\theta X_t \mathrm{d}t + \mathrm{d}W_t, \quad X_0 = 0$$

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$$c(\theta) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \left(\mathbb{E}_\theta[X_t^2] - \mathbb{E}_\theta[X_t]^2 \right) = \frac{1}{2\theta}$$

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- Optimal CI $\mathcal{I}^\star_{T,r}(\widehat{\theta}_T) = [c(\widehat{\theta}_T) + \kappa_T^-, c(\widehat{\theta}_T) + \kappa_T^+]$

$$\kappa_T^\pm = \frac{1}{2\widehat{\theta}_T{}^2} \left(r_T \pm \sqrt{r_T^2 + 2\widehat{\theta}_T r_T} \right), \quad r_T = r \cdot b_T / T$$

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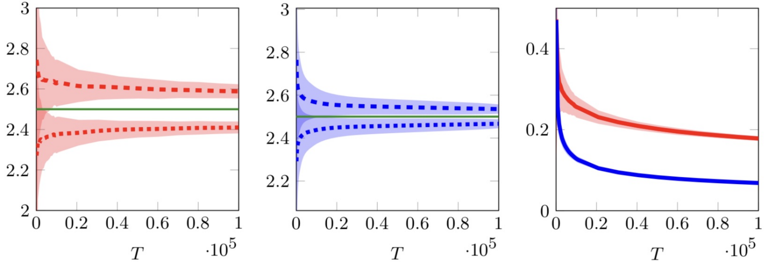

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Example: Cts. state i.i.d. model with unknown mean

- $\mathbb{R}^n$-valued i.i.d. process $\{X_t\}$
- Parametric distribution F_θ of X_t under $\mathbb{P}_\theta$
- F_θ is known except its mean (denoted by $\theta \in \mathbb{R}^n$)
- Parameter $\theta \in \Theta \subset \mathbb{R}^n$, and $\mathcal{V} = \mathbb{R}^n$
- Estimator $\widehat{\theta}_T = \frac{1}{T} \sum_{t=1}^T X_t$ (strongly consistent)
- Moderate deviation principle: Under "mild assumptions" $\{\widehat{\theta}_T\}_T$ satisfies an MDP with rate function

$$I^M(\vartheta, \theta) = \vartheta^\top C(\theta)^{-1} \vartheta,$$

where $C(\theta)$ is the covariance matrix of X_1

Example: Cts. state i.i.d. model with unknown mean

Distribution F_θ	Model θ	Variance $C(\theta)$	Rate fct $I^M(\vartheta, \theta)$
(a) Normal, $\mathcal{N}(\mu, \Sigma)$	$\theta = \mu \in \mathbb{R}^d$	Σ	$\vartheta^\top \Sigma^{-1} \vartheta$
(b) Exponential, $\mathrm{Exp}(\lambda)$	$\theta = \frac{1}{\lambda} \in \mathbb{R}_+$	θ^2	$\frac{\vartheta^2}{\theta^2}$
(c) Gamma, $\mathrm{Gam}(p, \nu)$	$\theta = p\nu \in \mathbb{R}_+$	$\theta\nu$	$\frac{\vartheta^2}{\nu\theta}$
(d) Poisson, $\mathrm{Poi}(\lambda)$	$\theta = \lambda \in \mathbb{R}_+$	θ	$\frac{\vartheta^2}{\theta}$
(e) Bernoulli, $\mathrm{Ber}(p)$	$\theta = p \in (0, 1)$	$\theta(1-\theta)$	$\frac{\vartheta^2}{\theta(1-\theta)}$
(f) Binomial, $\mathrm{Bin}(n, p)$	$\theta = np \in \mathbb{R}_+$	$\theta(1 - \frac{\theta}{n})$	$\frac{\vartheta^2}{\theta(1-\frac{\theta}{n})}$
(g) Geometric, $\mathrm{Geom}(p)$	$\theta = \frac{1}{p} \in \mathbb{R}_+$	$\theta(\theta - 1)$	$\frac{\vartheta^2}{\theta(\theta-1)}$

Example: Non-parametric i.i.d. model

- $\mathcal{S}$-valued i.i.d. process $\{X_t\}$ for $\mathcal{S} \subset \mathbb{R}^n$ compact
- Parameter space $\Theta = \mathcal{P}(\mathcal{S})$ probability measures on $\mathcal{S}$
- Vector space $\mathcal{V} = \mathcal{M}(\mathcal{S})$ finite signed measures on $\mathcal{S}$
- Empirical probability measure $\widehat{\theta}_T = \frac{1}{T}\sum_{t=1}^{T} \delta_{X_t}$ (weakly consistent)
- Moderate deviation principle: Under "mild assumptions" $\{\widehat{\theta}_T\}_T$ satisfies an MDP with rate function

$$I^M(\vartheta, \theta) = \begin{cases} \frac{1}{2}\int_{\mathcal{S}} \left(\frac{\mathrm{d}\vartheta}{\mathrm{d}\theta}\right)^2 \mathrm{d}\theta, & \vartheta \ll \theta, \vartheta(\mathcal{S}) = 0 \\ +\infty, & \text{o.w.} \end{cases}$$

- Consider expected loss, i.e., $c(\theta) = \mathbb{E}_{\theta}[\ell(X)]$

Example: Non-parametric i.i.d. model

Duality of convex programming: Let $\bar{\ell} = \max_{t \in \{1,\dots T\}} \ell\left(X_t\right)$, then

$$\underline{c}_{T,r}\left(\widehat{\theta}_T\right) = -\min_{\alpha \in \mathcal{A}} \left\{ \alpha - \frac{1}{2r_T + 1} \left(\frac{1}{T} \sum_{t=1}^{T} \sqrt{\alpha + \ell\left(X_t\right)} \right)^2 \right\}$$

$$\bar{c}_{T,r}\left(\widehat{\theta}_T\right) = \min_{\alpha \in \mathcal{B}} \left\{ \alpha - \frac{1}{2r_T + 1} \left(\frac{1}{T} \sum_{t=1}^{T} \sqrt{\alpha - \ell\left(X_t\right)} \right)^2 \right\}$$

- $\mathcal{A} = \left\{ \alpha \in \mathbb{R} : \bar{\ell} \le \alpha \le \bar{\ell}\left(2r_T + 1\right) / \left(2r_T\right) + \frac{1}{2r_T} c\left(\widehat{\theta}_T\right) \right\}$
- $\mathcal{B} = \left\{ \alpha \in \mathbb{R} : \bar{\ell} \le \alpha \le \bar{\ell}\left(2r_T + 1\right) / \left(2r_T\right) - \frac{1}{2r_T} c\left(\widehat{\theta}_T\right) \right\}$
- $r_T = r / a_T^2$

Outlook

Can these tools be of any use to solve control and decision-making problems?

- Offline reinforcement learning
  $\Longrightarrow$ M. Li, D. Kuhn and TS, *Towards Optimal Offline Reinforcement Learning*, 2025, arXiv:2503.12283
- Risk sensitive decision making
  $\Longrightarrow$ R. Salac, M. Kupper, and TS, *Asymptotic Optimality in Data-driven Decision Making*, 2025, arXiv:2507.15215

References

This talk:

- A. Ganguly and TS, *Optimal Learning via Moderate Deviations Theory* arXiv:2305.14496, 2025

Related concepts

- TS, B.P. Van Parys, and D. Kuhn, *A Pareto Dominance Principle for Data-Driven Optimization*, Operations Research, 2024
- R. Salac, M. Kupper, and TS, *Asymptotic Optimality in Data-driven Decision Making*, 2025, arXiv:2507.15215
- M. Li, D. Kuhn and TS, *Towards Optimal Offline Reinforcement Learning*, 2025, arXiv:2503.12283