Asset Allocation, Longevity Risk, Annuitisation and Bequests

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- Motivation
- Model Assumptions
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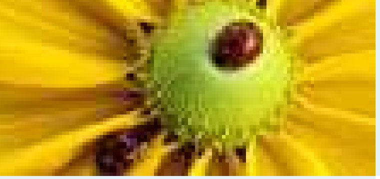

Relevance

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- Extensions

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- Importance of the End of the Life-Cycle:
  - Rising Conditional Life Expectancies
  - Growing Number of DC Plans
  - Continuing Wealth Concentration Among Pensioners
  - Input for Labour Models

Technical Problem

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- Technical View of the Pensioner's Problem:
  - Consumption/Portfolio Optimisation (c, π)
    $\longrightarrow$ Financial Market Risk
  - Optimal Annuitisation Decision (τ)
    $\longrightarrow$ Longevity Risk
- $\Rightarrow$ Combined Optimal Stopping and Optimal Control Problem (COSOCP)

Literature

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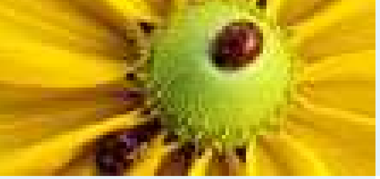
- Literature Overview:
  - Merton (1969) $\longrightarrow$ Stochastic Control
  - Vast Literature Imposing a Fixed or Infinite Planning Horizon
  - Yaari (1965) $\longrightarrow$ Uncertain Lifetime
  - Richard (1975) $\longrightarrow$ Reversible Annuities
- Few Normative Models with Irreversible Annuities and Uncertain Lifetime, i.e.
  - Milevsky and Young (2007):
    Commitment to Predetermined Annuitisation Time
  - Stabile (2006):
    Annuitisation Rule as a Controlled Stopping Time

Extensions

- Our Extensions to the Model of Stabile (2006):
  - Inclusion of a Bequest Motive
  - Prior Life Insurance and Subsistence Level of Bequest
  - Economically Relevant Risk Aversion ($\gamma > 1$)
  - New Solution Method with Duality Arguments



Model Assumptions

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- Utility Maximisation (Consumption, Annuity, Bequest; Identical Relative Risk Aversion)
- No Stochastic Income
  $\longrightarrow$ No Labour Income
- Prior Decision on Annuitisation and Life Insurance Taken as Given
- Annuitisation of Entire Wealth and Consumption of Entire Annuity
- One Riskless Asset, One Risky Asset (Geometric Brownian Motion)
- Exponential Mortality Law

Indirect Utility

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- COSOCP
- Verification Theorem
- Variational Inequality

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Total Expected Discounted Utility $J_{c,\pi,\tau}(w)$:

$$E\left[\int_0^{T_x} e^{-\delta^S t}\left\{U_1(c(t))\,1_{\{t\leq\tau\}}+U_2\left(\frac{W(\tau)}{\bar{a}_{x+\tau}}\right)1_{\{t>\tau\}}\right\}dt\right.$$

$$\left.+\eta e^{-\delta^S T_x}\left\{U_3\left(W(T_x)+Z^s\right)1_{\{T_x\leq\tau\}}+U_3\left(Z^s\right)1_{\{T_x>\tau\}}\right\}\right]$$

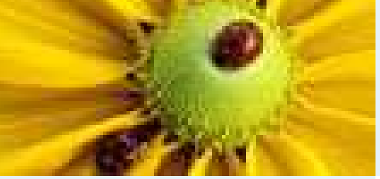
COSOCP

General COSOCP with Exponential Mortality:

$$V(w) = \sup_{(c,\pi,\tau)\in\mathcal{G}(w)} J_{c,\pi,\tau}(w) \qquad \text{for all } w > 0$$

$$J_{c,\pi,\tau}(w) = E^{w}\left[\int_0^{\tau} e^{-\beta^S t} f(c(t), W(t))\, dt + e^{-\beta^S \tau} g(W(\tau))\right]$$

$$dW(t) = W(t)\left[r + \pi(t)(\mu - r)\right] dt - c(t)\, dt + \sigma\pi(t) W(t)\, dB(t)$$



Verification Theorem

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- Optimal Strategies:
  - Annuitisation rule

$$\tau^* = \inf\{t \geq 0 | W^*(t) \notin D\}$$

with

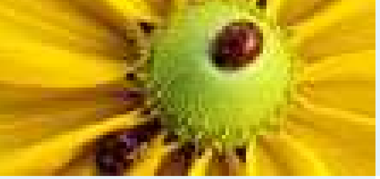
$$D = \{W(t) \in G \,|\, v(W(t)) > g(W(t))\}$$

  - Consumption rule

$$c^* = I\left(v_W\left(W^*(t)\right)\right) 1_{\{t \leq \tau^*\}}$$

  - Investment rule

$$\pi^* = -\frac{\mu - r}{\sigma^2} \frac{v_W\left(W^*(t)\right)}{W^*(t)\, v_{WW}\left(W^*(t)\right)} 1_{\{t \leq \tau^*\}}$$



Variational Inequality

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- The Verification Theorem Reduces the COSOCP to the Variational Inequality:

$$\max\left\{L^{\text{com}}v\left(W\left(t\right)\right), g\left(W\left(t\right)\right)-v\left(W\left(t\right)\right)\right\}=0 \quad \text{for } W\left(t\right)>0$$

with

$$L^{\text{com}}v\left(W\left(t\right)\right)=\sup_{(c,\pi)\in\mathcal{G}^{\tau}(W(t))}\left\{f\left(c\left(t\right),W\left(t\right)\right)-\beta^{S}v\left(W\left(t\right)\right)+Lv\left(W\left(t\right)\right)\right\}$$

- subject to

$$v\left(W\left(t\right)\right)=g\left(W\left(t\right)\right) \quad \text{for all } W\left(t\right)\in\partial D$$

- and

$$v_{W}\left(W\left(t\right)\right)=g_{W}\left(W\left(t\right)\right) \quad \text{for all } W\left(t\right)\in\partial D.$$

No-Bequest Case

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- Bequest Case $\gamma < 1$
- Bequest Case $\gamma > 1$

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- Now-or-Never Annuitisation: M^{nb}
- Natural Parameter Effects
  - Risk Aversion (A$+$)
  - Subjective Life Expectancy (A$+$)
  - Objective Life Expectancy (A$-$)
  - Identical Life Expectancy (A$-$)
  - Sharpe Ratio (A$-$)
- Annuitisation in Most Parameter Settings
  $\longrightarrow$ Important Inclusion of a Bequest Motive

Bequest Case $\gamma < 1$

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- Bequest Case $\gamma < 1$ and $Z^s = 0$:
  - Now-or-Never Annuitisation: $M^b = M^{nb} + \lambda^S \eta$
  - Slight Tendency for the Financial Market
    $\longrightarrow$ Important Inclusion of Bequest Motive
  - Natural Parameter Effects
  - Natural Comparison to No-Bequest Case
    - $\frac{c^b}{W^b} < \frac{c^{nb}}{W^{nb}}$
    - $W^b > W^{nb}$
    - $\pi^b = \pi^{nb}$
    - $\frac{c^b}{W^b}$ decreases in η

Bequest Case $\gamma > 1$

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- Bequest Case $\gamma > 1$ and $Z^s > 0$:
  - Never Annuitisation or Wealth-Dependent Annuitisation with $D = (\underline{W}, \infty)$
  - Natural Comparison to No-Bequest Case
  - Real COSOCP with $D = (\underline{W}, \infty)$:
    - Simplification via Duality Arguments
    - Free Boundary Value Problem
    - Numerical Solution Algorithm
      $\longrightarrow$ Boundaries
      $\longrightarrow$ Value Function
    - Natural Parameter Effects:
      $\longrightarrow$ Life Insurance ($A+$)
      $\longrightarrow$ Bequest Motive ($A-$)
    - Heavy Consumption Smoothing
    - More Aggressive Investment Rule Compared to Merton
      $\longrightarrow$ Additional Option of Annuitisation

Conclusions

- Conclusions:
  - COSOCP: New Solution Method
  - Economically Important Risk Aversion $\gamma > 1$
  - Longevity Risk Is Absolutely Relevant
    $\longrightarrow$ Modelling of Lifetime
    $\longrightarrow$ Role of Pension Funds
  - Essential Inclusion of a Bequest Motive
    $\longrightarrow$ Consumption-Wealth Trade-off
    $\longrightarrow$ Absurd Strong Tendency for the Annuity Market Vanishes

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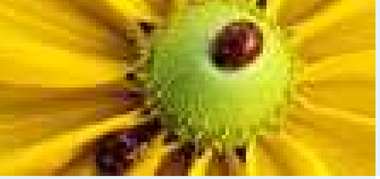
Thank you very much for your attention!



Indirect Utility (2)

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Indirect Utility Function with Exponential Mortality:

$$E^w \left[\int_0^\tau e^{-\beta_x^S t} \left\{ U_1\left(c\left(t\right)\right) + \lambda_x^S \eta U_3\left(W\left(t\right) + Z^s\right) \right\} dt \right.$$

$$\left. + e^{-\beta_x^S \tau} \frac{1}{\beta_x^S} \left\{ U_2\left(W\left(\tau\right)\left(r + \lambda_x^O\right)\right) + \lambda_x^S \eta U_3\left(Z^s\right) \right\} \right]$$

Continuation Region

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- Continuation Region D
  $\rightarrow$ Open and Connected
- $U \subset D$ with

$$U = \left\{ W(t) \in \mathbb{R}^{+} \mid L^{\text{com}} g(W(t)) > 0 \right\}$$

- The set U can be used to infer information about the form of the important continuation region D.

Investment in the Bequest Case $\gamma > 1$

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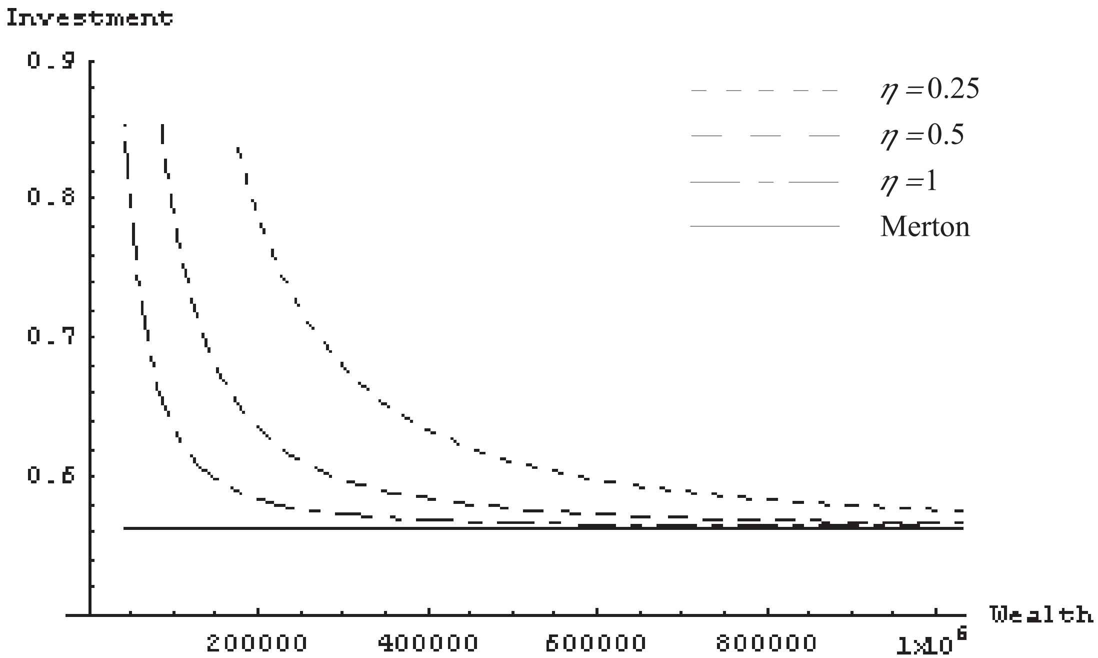

Figure 1: The investment rule for different values of the bequest parameter assuming a subjective and objective life expectancy of 20 years, a subjective and objective discounting parameter of 0.035, $\mu = 0.08$, $\sigma = 0.2$, life insurance net of subsistence of 500 and $\gamma = 2$.

Consumption in the Bequest Case $\gamma > 1$

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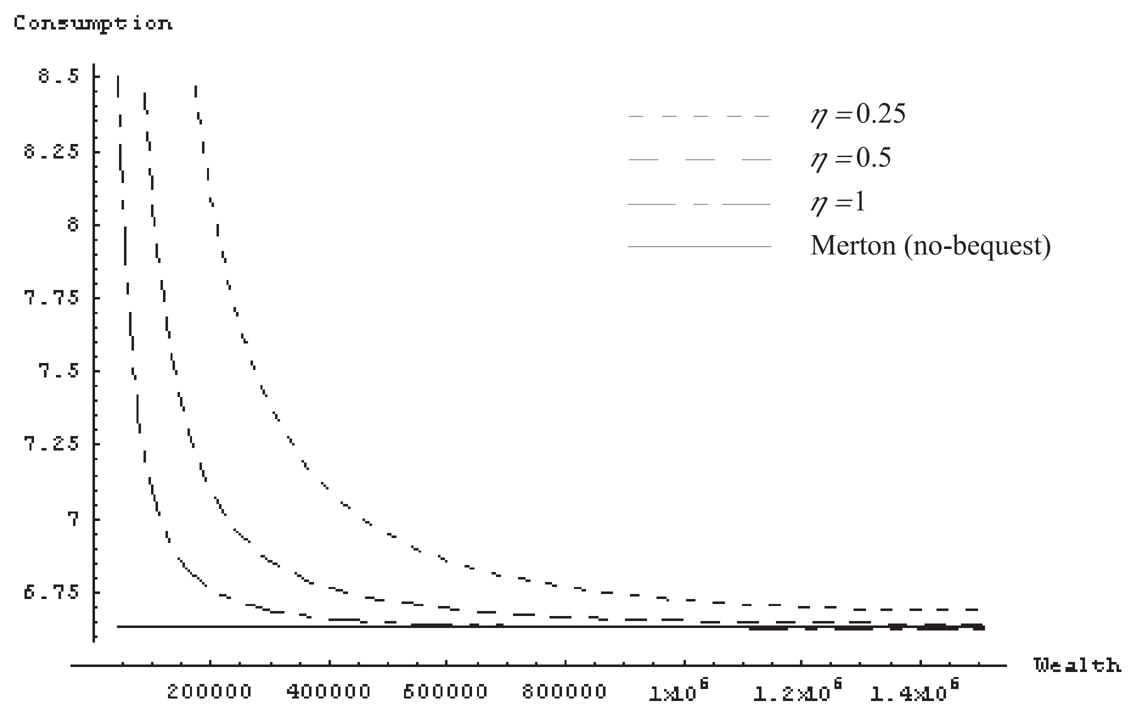
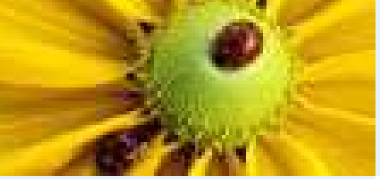


Figure 2: The consumption fraction for different values of the bequest parameter assuming a subjective and objective life expectancy of 20 years, a subjective and objective discounting parameter of 0.035, $\mu = 0.08$, $\sigma = 0.2$, life insurance net of subsistence of 500 and $\gamma = 2$.



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- References (4)

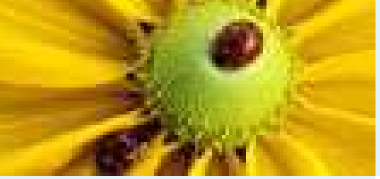
[1] G. A. Akerlof, The Market for Lemons: Quality Uncertainty and the Market Mechanism, Quarterly Journal of Economics **84** (1970) 488-500

[2] R. T. Baumann and H. H. Müller, Pension Funds as Institutions for Intertemporal Risk Transfer, Insurance: Mathematics and Economics (2008)

[3] N. Charupat and M. A. Milevsky, Optimal Asset Allocation in Life Annuities: A Note, Insurance: Mathematics and Economics **30** (2002) 199-209

[4] M. D. Hurd and K. McGarry, Evaluation of the Subjective Probabilities of Survival in the Health and Retirement Study, Journal of Human Resources **30** (1995) 268-291

[5] M. D. Hurd and K. McGarry, The Predictive Validity of Subjective Probabilities of Survival, NBER Working Paper 6193 (1997)



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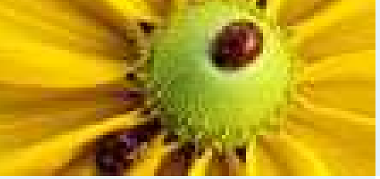
- References (3)
- References (4)

[6] N. Hakansson, Optimal Investment and Consumption Strategies under Risk, an Uncertain Lifetime, and Insurance, International Economic Review (1969)

[7] R. C. Merton, Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case, Review of Economics and Statistics **51** (1969) 247-257

[8] R. C. Merton, Optimum Consumption and Portfolio Rules in a Continuous-Time Model, Journal of Economic Theory **3** (1971) 373-413

[9] M. A. Milevsky, K. S. Moore and V. R. Young, Asset Allocation and Annuity-Purchase Strategies to Minimize the Probability of Financial Ruin, Mathematical Finance **16(4)** (2006) 647-671



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- References (2)
- References (3)
- References (4)

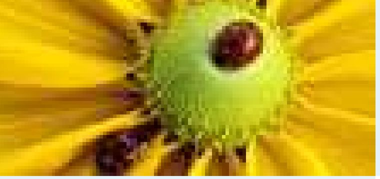
[10] M. A. Milevsky and V. R. Young, Annuitisation and Asset Allocation, Journal of Economic Dynamics and Control **31(9)** (2007) 3138-3177

[11] B. ∅ksendal, Stochastic Differential Equations, Springer, Berlin (2003)

[12] S. F. Richard, Optimal Consumption, Portfolio and Life Insurance Rules for an Uncertain Lived Individual in Continuous Time Model, Journal of Financial Economics **2** (1975) 187-203

[13] D. Schiess, Consumption and Portfolio Optimisation at the End of the Life-Cycle, Doctoral Thesis, University of St. Gallen, Switzerland

[14] D. Schiess, Optimal Strategies During Retirement, Center for Finance Working Paper No. 70, University of St. Gallen, Switzerland



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- References (4)

[15] G. Stabile, Optimal Timing of the Annuity Purchase: A Combined Stochastic Control and Optimal Stopping Problem, International Journal of Theoretical and Applied Finance **9(2)** (2006)

[16] M. Yaari, Uncertain Lifetime, Life Insurance, and the Theory of the Consumer, Review of Economic Studies **2** (1965) 137-150