Test Problems in Stochastic Multistage Programming

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Abstract

This paper provides a set of stochastic multistage programs where the evolvement of uncertain factors is given by stochastic processes. We treat a practical problem statement within the field of managing fixed-income securities. Detailed information on the used parameter values in various interest rate models is given. Barycentric approximation is applied to obtain computational results; different measures of the achieved goodness of approximation are indicated.

Keywords: Stochastic multistage programming; Barycentric approximation; Interest rate models; Test problems

1 Introduction

Since the beginning of *stochastic programming* in the mid-fifties the progress in the field dealing with the important model type of a *stochastic multistage program* has made great strides. The intention of stochastic multistage programming can be shortly described as optimizing a decision process under uncertainty, i.e. today's and future decisions shall be made with respect to given constraints taking into account the stochastic evolvement of some uncertain factors. In case that the multidimensional distribution of these random variables is approximated by a scenario tree, then the resulting problem may be treated as a large-scale sparse mathematical program, the socalled *deterministic equivalent program*. One of the most important challenges has been to overcome the curse of dimensionality. Therefore, much attention is paid to two major issues: First, to develop adequate mathematical programming algorithms which exploit the structural properties of deterministic equivalent programs, and second, to work out approximation schemes leading to small scenario trees which are still good representations of the uncertain future. Concerning the first aspect, the L-shaped method of Van Slyke/Wets [18] can be mentioned for the 2-stage case which is an extension of the well known Benders decomposition (see Benders [1]) in the framework of stochastic programming. The extension to the multistage case has been done by Birge [3] [2] and Gassmann [13] leading to the socalled Nested Benders Decomposition. Further decomposition methods have been developed, additionally taking into account the advantages of parallelization (see e.g. Ruszczynski [15]

[16]). Often, these methods are combined with different techniques which lead to smaller scenario trees like importance sampling (see e.g. Dantzig/Infanger [5]) or EVPI-based methods (Dempster [6], Dempster/Thompson [7]). Another method providing smaller approximating scenario trees is the barycentric approximation developed in Frauendorfer [9] [10] [11]. Of course, the efficiency of the above mentioned algorithms that exploit the structure of deterministic equivalent programs can be compared without any difficulties using a set of classical test problems in MPS-format.

The aim of this paper is to provide a set of stochastic multistage programs where the stochastic evolvements of uncertain factors are given as stochastic processes. These test problems represent real problem statements within the field of managing fixed income securities. A practical problem formulation and the specification of the stochastic processes are presented in section 2 and 3, respectively. Section 4 gives a brief informal overview of the *barycentric approximation* technique which is applied to the test problems. The obtained computational results for the test problems characterized by different interest rate models, initial interest rate curves and correlation structures are summarized in section 5. In the last section a conclusion is given based on the empirical computational results to get an insight to the following questions: What is the impact of the mean-reverting property of interest rate models and of correlation structures to the goodness of approximation? How do time horizon, the dimension of stochastic factors and the refinement technique affect the solvability of the underlying real problems?

2 Problem Formulation

The model considered in this paper represents the funding non-fixed rate mortgages. An analogous model for the investment of savings account deposits can be easily derived from this funding model and has been worked out in Forrest et al. [8]. Consider a bank which has to refinance the loans by borrowing bonds of different maturities in the money market. This results in a given structure of maturing funds that is actual today, and every time (from now on up to some time horizon T) when an amount of this liability structure matures, this amount again has to be refinanced. The funding volume and the interest rates change stochastically over time. The aim is to evaluate a refinancing strategy for today which minimizes the expected present value of the total refinancing costs from today up to period T. Decisions have to be made at time points $t = 0, \ldots, T$. At $t = 0$ only the actual interest rate curve and the structure of maturing funds v_{-1}^{d} are known. ξ_t denotes the uncertain change in the funding volume at t, and η_t is a vector of risk factors which affect the interest rates at t. Dependent on η_t, the discounted accrued interest payments for refinancing \$ 1 in maturity d at time t is given by $\varphi_t^d(\eta_t)$. Let $\mathcal{D} := \{1, \ldots, 12\}$ denote the set of maturities in months; $\mathcal{D}^{\mathcal{S}} \subset \mathcal{D}$ is the set of standard maturities for which the market is sufficiently liquid. x_t^d is the amount of borrowing at time t with a maturity of d months ($d \in \mathcal{D}^S$); $v_t^d (d \in \mathcal{D})$ indicates the total volume of borrowings at time t with a d-month maturity and is determined by

$$\begin{aligned} v_t^d - v_{t-1}^{d+1} - x_t^d &= 0 \qquad t = 0, \ldots, T, \ d \in \mathcal{D}^S, \\ v_t^d - v_{t-1}^{d+1} &= 0 \qquad t = 0, \ldots, T, \ d \in \mathcal{D} \setminus \mathcal{D}^S . \end{aligned}$$

At time t, the funding volume v_t must equal the total volume of borrowing, i.e.

$$v_t = \sum_{d \in \mathcal{D}} v_t^d \qquad t = 0, \ldots, T.$$

The funding volume at t is given by the volume at the previous period $t-1$ increased by the stochastic volume change ξ_t:

$$v_t = v_{t-1} + \xi_t \qquad t = 1, \ldots, T$$

Finally, with P denoting the joint multidimensional probability distribution of the underlying stochastic process (η_t, ξ_t) the stochastic multistage problem reads as

$$\begin{array}{lll} \min \displaystyle\int \sum_{t=0}^{T} \sum_{d \in \mathcal{D}^S} \varphi_t^d(\eta_t) \cdot x_t^d \, dP(\eta, \xi) & & \\ v_t^d - v_{t-1}^{d+1} - x_t^d \;=\; 0 & & t = 0, \ldots, T, \ \forall d \in \mathcal{D}^S \\ v_t^d - v_{t-1}^{d+1} \;=\; 0 & & t = 0, \ldots, T, \ \forall d \in \mathcal{D} \setminus \mathcal{D}^S \\ v_t - \sum_{d \in \mathcal{D}} v_t^d \;=\; 0 & & t = 0, \ldots, T \\ v_t - v_{t-1} \;=\; \xi_t & & t = 1, \ldots, T \\ v_t, v_t^d \in I\!R \ \text{ nonanticipative} & & t = 0, \ldots, T, \ \forall d \in \mathcal{D} \\ x_t^d \geq 0 \ \text{ nonanticipative} & & t = 0, \ldots, T, \ \forall d \in \mathcal{D}^S \end{array}$$

3 Specification of Input Data

In this section, the stochastic and deterministic input are introduced. The problem presented in section 2 will be solved for two different interest rate models: a simple key rate model and the well known Vasicek model. Both models as well as the specific parameters used in the calculations are stated in subsection 3.1. The dynamics of the volume change are treated in 3.2, the deterministic input is presented in 3.3.

3.1 Interest Rate Models

It is assumed that the set of traded bonds contains the maturities 1 month, 2 months, 3 months, 6 months and 12 months. Hence, the sets $\mathcal{D}$ and $\mathcal{D}^S$ introduced in section 2 are given by $\mathcal{D} := \{1, \ldots, 12\}$ and $\mathcal{D}^S := \{1, 2, 3, 6, 12\}$, respectively.

$\rho = (\rho^1, \rho^2, \rho^3, \rho^6, \rho^{12})$ denotes the interest rates associated with maturities $d \in \mathcal{D}^{\mathcal{S}}$. The term structure $\rho = \rho(\eta)$ itself is determined by the risk factor $\eta \in I\!R^K$ with $K \in 2, 3$ (key rate model) and $K \in 1, 2$ (Vasicek model), respectively.

Key rate model:
The dynamics of the risk factor η are represented by a generalized Wiener process

$$d\eta_t = \alpha dt + \sigma dz_t, \quad t \in [0, \infty)$$

where α is a constant drift vector in $I\!R^K$ and σ a constant dispersion matrix in $I\!R^{K \times K}$; $dz_t = (dz_{1,t}, \ldots, dz_{K,t})'$ with $z = \{z_t : 0 \leq t < \infty\}$ denoting a K-dimensional standard Wiener process. $\eta = (\eta_1, \ldots, \eta_K)$ represents some pre-chosen interest rates, which are called key rates. In case $K = 2$, η_1 and η_2 correspond to ρ^1 and ρ^{12}, respectively. When $K = 3$, η_1, η_2 and η_3 are associated with ρ^1, ρ^3 and ρ^{12}. For those maturities $d \in \mathcal{D}^{\mathcal{S}}$ which are no key rates, ρ^d is obtained by interpolation. For $t = 0$ the observable market interest rate curve is taken (see table 1 in section 3). In the following, three different key rate models are considered which are denoted W2U, W2C, and W3C with W standing for <u>W</u>iener process, 2 or 3 stating the dimension of η, and U or C referring to <u>u</u>ncorrelated or <u>c</u>orrelated factors, respectively:

W2U: $\eta \in I\!R^2,\ \Delta\rho^1 := \eta_1,\ \Delta\rho^{12} := \eta_2$

$$\alpha := 0 \in I\!R^2,\ \sigma := \begin{pmatrix} 0.423084 & 0.0 \\ 0.0 & 0.3535533 \end{pmatrix}$$

W2C: $\eta \in I\!R^2,\ \Delta\rho^1 := \eta_1,\ \Delta\rho^{12} := \eta_2$

$$\alpha := 0 \in I\!R^2,\ \sigma := \begin{pmatrix} 0.423084 & 0.0 \\ 0.276541 & 0.220284 \end{pmatrix}$$

W3C: $\eta \in I\!R^3,\ \Delta\rho^1 := \eta_1,\ \Delta\rho^3 := \eta_2,\ \Delta\rho^{12} := \eta_3$

$$\alpha := 0 \in I\!R^3,\ \sigma := \begin{pmatrix} 0.423084 & 0.0 & 0.0 \\ 0.356903 & 0.116705 & 0.0 \\ 0.276541 & 0.191094 & 0.109582 \end{pmatrix}$$

Vasicek model:
Vasicek [19] assumes that the evolvement of the short rate r follows an Ornstein-Uhlenbeck process

$$dr = \kappa(\Theta - r)dt + \sigma dz \quad (\kappa, \sigma > 0) \tag{1}$$

where the deterministic drift term $\kappa(\Theta - r)$ causes the process to move towards its long term mean Θ at a mean-reversion rate $\kappa > 0$. Using a no-arbitrage argument and assuming a constant market price of risk λ leads to an affine one-factor model, as the zero bond rates are affine functions of the instantaneous rate r. Vasicek's one-factor model is easily extendable to a K-factor model when the increment dr of the instantaneous rate r is expressed as a sum of changes of orthogonal state variables $\eta_1, \ldots, \eta_K$, i.e.

$$dr = d\eta_1, + \cdots + d\eta_K \ .$$

Analogously to (1), the dynamics of the stochastic quantities are given by Ito processes

$$d\eta_i = \kappa_i(\Theta_i - \eta_i)dt + \sigma_i dz_i \ , \quad i = 1, \ldots, K,$$

with $\kappa_i, \sigma_i > 0$. Like in the one-factor model, the interest rates for all maturities are affine functions of the state variables, but now leading to a more complex evolvement of the interest rate structure because of $K > 1$ influencing factors. A detailed treatment of these topics can be found in Schürle [17]. The computational results in section 5 are listed for a one-factor model V1U and a two-factor model V2U with the following parameter values of the Ornstein-Uhlenbeck processes and the market prices of risk:

V1U: $\kappa := 0.4613$, $\Theta := 0.04267$, $\sigma := 0.01590$, $\lambda := 0.5218$

V2U: $\kappa := \begin{pmatrix} 2.3174 \\ 0.1990 \end{pmatrix}$, $\Theta := \begin{pmatrix} 0.02705 \\ 0.01705 \end{pmatrix}$,

$\sigma := \begin{pmatrix} 0.02406 \\ 0.01398 \end{pmatrix}$, $\lambda := \begin{pmatrix} -0.1288 \\ 0.3458 \end{pmatrix}$.

The underlying data set for estimating the above parameters is given by the CHF-Euromarket interest rates from 01/1983 to 11/1996. In particular, a Quasi-Newton method with BFGS-update was used to evaluate Maximum-Likelihood estimators (see Schürle [17]).

Discounting of Cash Flows:

Since cash flows beyond the planning horizon are not relevant for the optimization within the test problems, only interest payments within the interval $[0, T+1]$ are taken into account. Therefore, payments after $t = T+1$ are ignored even if they are induced by a decision in $t = 0, \ldots, T$. Let $a_t(\rho)$ denote the discount factor for \$ 1 where ρ is the capitalization rate p.a. and t the payment date in month:

$$a_t(\rho) = \begin{cases} (1 + \frac{\rho t}{12})^{-1} & t < 12 \\ (1+\rho)^{-n_o}(1 + \rho \frac{t - 12 n_o}{12})^{-1} \ , \ n_o := \lfloor t/12 \rfloor & t \geq 12 \end{cases}$$

Then, the present value of a \$ 1 investment at t with maturity d is given by

$$\varphi_t^d = \sum_{k=1}^{g} a_{t+12k}(\rho_o^{t+12k})\rho_t^d \ + \ a_\Delta(\rho_o^{t+\Delta})\rho_t^d \quad , g := \lfloor \Delta/12 \rfloor$$

where ρ_t^d is the yield of a zero bond with maturity d at time t (depending on η_t) and

$$\Delta := \begin{cases} T + 1 - t & t + d > T \\ d & otherwise \end{cases}$$

3.2 Dynamics of Volume

The stochastic RHS of the problem presented in section 2 is given by $\xi \in I\!R$ which indicates the change of the total funding volume. Its corresponding dynamics are expressed by the SDE

$$d\xi_t = \alpha(\xi_t, t)dt + \sigma(\xi_t, t)dz_t \ , \quad t \in [0, \infty)$$

with $z = \{z_t : 0 \leq t < \infty\}$ denoting a standard Wiener process. For the uncorrelated case W2U, the drift term and the volatility are set to

$$\alpha(\xi_t, t) \equiv 0, \ \ \sigma(\xi_t, t) \equiv 584 \, .$$

In the correlated cases W2C and W3C, the drift term is set to 0; the dispersion matrices corresponding to $\eta_1, \dots, \eta_K$, and ξ are given by

$$\text{W2C: } \sigma(\eta_t, \xi_t, t) \equiv \begin{pmatrix} 0.423084 & 0.0 & 0.0 \\ 0.276541 & 0.220284 & 0.0 \\ 104.582 & 39.7703 & 573.133 \end{pmatrix}$$

$$\text{W3C: } \sigma(\eta_t, \xi_t, t) := \begin{pmatrix} 0.423084 & 0.0 & 0.0 & 0.0 \\ 0.356903 & 0.116705 & 0.0 & 0.0 \\ 0.276541 & 0.191094 & 0.109582 & 0.0 \\ 104.582 & 50.6603 & -8.39645 & 572.211 \end{pmatrix}.$$

Simple calculation show that the last row of the corresponding diffusion matrices $\sigma(\eta_t, \xi_t, t)\sigma(\eta_t, \xi_t, t)'$ contains only positive entries. By this, W2C and W3C model the well-known dependency of funding volume and interest rates in case of non-fixed rate loans which can be observed in reality. In particular, positive correlations reflect the simultaneous occurrence of decreasing interest rates and of a decreasing funding volume. The reason for this behaviour is based on a special aspect of non-fixed rate loans: Customers are entitled to switch from a fixed rate contract to a non-fixed rate one and vice versa. Of course, the latter only happens in phases with a low level of interest rates (so that the customer get the benefit of a lower customer rate for a long period); the opposite is observable in phases with a high level of interest rates.
Apart from the correlated models W2C and W3C, contrary to reality an uncorrelated model W2U is considered assuming the change of the funding volume to be stochastic independent of the change of all key rates. This rather serves for investigational purpose with regard to the value of the applied method against the background of various correlation structures than for an adequate modeling of the funding volume behaviour observable in practice: An insight to the impact of uncorrelation (i.e. stochastic independence in case of normal distributed variables) on the value of the applied method is achieved when solving the multistage funding problem stated in section 2.

Concerning the one-factor and two-factor Vasicek model, the change of the funding volume is supposed to be uncorrelated with the change of the risk factors

$\eta_1, \ldots, \eta_K$; hence, in the model V1U and V2U the dynamics of the volume change are set accordingly $\alpha(\xi_t, t) \equiv 0$, $\sigma(\xi_t, t) \equiv 584$. The reason for this simplification is as follows: Assuming a process with stationary trend (in a normal distributed 2-factor model) for the evolvement of the Swiss savings account deposits from January 1984 to November 1996 reveals the evolvement of the savings volume being influenced to a great extent by the interest rate level (see Schürle [17], p. 235). Transferring this knowledge to the funding situation with a Vasicek model thus would lead – in order to model the volume evolvement in a realistic manner – to the need of modeling stochastic factors driving both the Vasicek state variable (short rate r) and the funding volume itself. This would prevent a strict separation between the risk factors η and the volume ξ and increase the dimension of the joint probability space, but would have no impact on the applicability of the barycentric approximation scheme (see Frauendorfer [11], p. 279). Despite of this theoretical result, the above stated increase of the dimension is avoided to ensure the numerical tractability of the resulting deterministic equivalent programs. Since an estimation of the correlations between the stochastic factors in a Vasicek interest rate model and the funding volume is still an open issue in our framework, these correlations are set to zero for ease of simplicity.

3.3 Deterministic Input

The upcoming calculations are carried out for two representative interest rate curves, a normal one and an inverse one. Table 1 shows the initial values at $t = 0$ for all maturities $d \in \mathcal{D}^S$. These are the starting values ρ_0^d $(d \in \mathcal{D}^S)$ of

Interest Rates					
maturity	1 m	2 m	3 m	6 m	12 m
normal	5.00	5.10	5.20	5.55	6.00
inverse	6.00	5.91	5.82	5.55	5.00

Table 1: Initial interest rates at $t = 0$

the interest rate evolvement in the key rate models W2U, W2C and W3C. Using the Vasicek models V2U and V3U, there is the need of having additional starting values for the evolvement of the risk factors $\eta_1, \ldots, \eta_K$. These can be derived from the initial interest rate curves and the Vasicek parameters κ, Θ, σ and λ by some simple calculations (see e.g. Schürle [17]): Table 2 reveals the starting values corresponding to both Vasicek models and two initial interest rate curves.

The structure of maturing funds v_{-1}^d is given in table 3; it is assumed that a volume of 200 units matures at all time points $t = 0, \ldots, T$.

Starting Values for the Vasicek Models			
Model	interest rate curve	$\eta_{1,0}$	$\eta_{2,0}$
V1U	normal	0.05	–
	inverse	0.06	–
V2U	normal	0.02931011	0.02093309
	inverse	0.02646704	0.02661233

Table 2: Starting values $\eta_{i,0}$ $(i = 1, \ldots, K)$ of the risk factor evolvement

Maturing Funds				
0	1	2	$\cdots$	T
200	200	200	$\cdots$	200

Table 3: Structure of maturing funds v^d_{-1}

4 Barycentric Approximation

This section sketches only briefly the ideas and the method of the barycentric approximation developed in Frauendorfer [9][10][11]. The cost function of a particular funding strategy taken with respect to market dynamics is denoted *value function*. It has to be stressed that the value function is not given analytically, even not explicitely, but implicitely by the solutions of a stochastic multistage optimization problem. Applying duality theory, the value function is a saddle function in case the underlying stochastic multistage optimization problem is convex. More precisely, the value function is convex with respect to the change ξ_t of the funding volume and is concave with respect to the change η_t of the key rates (risk factors, respectively). For details about this topic see e.g. Frauendorfer [11], p.279ff. The way the barycentric approximation technique exploits the inherent saddle structure of the value function can be motivated in a rather easy way:

Let $\Theta_t \subset I\!R^K$ and $\Xi_t \subset I\!R$ be regular simplices which cover the support of η_t and ξ_t; let the vertices of these simplices be denoted with $a_{0,t}, \ldots, a_{K,t}$ and $b_{0,t}, b_{1,t}$, respectively. For $K + 1$ points $u_{i,t} \in a_{i,t} \times \Xi_t$ $(i = 0, \ldots, K)$ take the corresponding first-order approximation of the value function (corresponding to the period $[t - 1, t]$) with respect to the change of funding volume and form a bilinear upper approximation of the value function. Analogously, for 2 points $l_{i,t} \in b_{i,t} \times \Theta_t$ $(i = 0, 1)$ take the corresponding first-order approximation of the value function with respect to the change of the key rates (risk factors, respectively) and form a bilinear lower approximation of the value function. These upper und lower bilinear approximations are easy to integrate and allow for stating and localizing the current inaccuracies, so that potential improvements can be achieved via successive partitionings of the effective domains

$\Theta_t \times \Xi_t\,(t = 1, \ldots, T)$. Having in mind to minimize the inaccuracy, it remains to be answered which sets of points $\{u_{0,t}, \ldots, u_{K,t}\}$ and $\{l_{0,t}, l_{1,t}\}$ are optimal in the sense that they yield the best approximation of the expected costs. It is proven that the optimal points where the value function should be supported for minimizing the expected error solve a corresponding semi-infinite optimization problem and represent *generalized barycenters* of the $\times$-simplex $\Theta_t \times \Xi_t$ (see e.g. Frauendorfer [10]).
The successive partitioning of the effective domains can be done in various ways. Herein, we focus mainly on the partitioning of the root node $\Theta_1 \times \Xi_1$, i.e. the scenario tree is partitioned in the root node. For a thorough discussion of refinement techniques the reader is referred to Frauendorfer/Marohn [12] and Marohn [14].

At any point of time there is no possibility of a „free lunch“, due to the fact that the funding problem stated in section 2 only includes buy-and-hold strategies of bonds in the money market, i.e. all borrowed bonds are held till maturity; selling formerly borrowed bonds and short sellings are not allowable. Hence, the barycentric approximation scheme provides arbitrage-free scenarios within this specific model environment. However, it has to be stressed that the use of the barycentric approximation technique doesn't result necessarily in arbitrage-free scenarios in the classical sense, e.g. when pricing of instruments in complete markets is of interest.

5 Computational Results

This section lists the computational results obtained when the underlying problems are solved with barycentric approximation for different time horizons $T = 4, \ldots, 7$. The barycentric approximation provides lower bounds (LB) and upper bounds (UB) which define the achieved *accuracy* := (UB - LB)/LB as a measure for the goodness of approximation. An additional measure is given by the *local expected value of perfect information* (EVPI) taken at the root node. It can be interpreted as the amount one would pay now to get perfect information about the uncertain future. Using traditional terms, the EVPI is given by the difference between the values of the *wait-and-see* solution and the *here-and-now* solution.[1] Often, the EVPI is taken relative to the optimal value of the objective function of the deterministic equivalent program: The quotient of the EVPI and the optimal objective value is called *relative EVPI* which is also indicated in the tabulated results. Both the EVPI and the relative EVPI refer to the lower approximations which provide more reliable results than upper approximations (see e.g. Marohn [14], p.114). The tables 4, 5, 6 and 7 list the results after 4 refinements in the root node with a time horizon $T = 4, 5, 6$ and 7, respectively. The length of every time period is one month. 'Policy' denotes the maturity which has been

[1] For a detailed mathematical treatment of this topic see e.g. Birge/Louveaux [4] .

evaluated as optimal today's decision for refinancing the volume maturing today; 'Value' indicates the optimal value of the objective function of the deterministic equivalent program. A summary concerning the achieved accuracies (in tables 4 –7) is given in table 8. To give an insight into the improvement of the approximation by partitioning the scenario tree in the root node, table 9 describes the ratio of the accuracy without any refinements and the accuracy after 4 refinements. The expression '(accuracy $< 1\%$)' indicates that the ratio hasn't been evaluated because the inaccuracy itself is too small at the very beginning.

Comprehensive information about the different values of the relative EVPI after 4 refinements in the root node is summarized in table 10. To describe the improvement of the relative EVPI after 4 refinements, table 11 contains the ratio of the relative EVPI without refinement and the relative EVPI after 4 refinements in the root node. An alternative refinement technique is given by partitioning along the scenario tree. The results of this refinement technique are listed in tables 12 and 13 which refer to the ratio of accuracy and the ratio of the relative EVPI, respectively. The size of the deterministic equivalent programs are summarized with respect to the lower approximations after 4 refinements in the root node in table 14.

6 Empirical Observations and Conclusions

An analysis of the computational results for the funding problem (tables 8 – 13) leads to the following empirical observations.

Key rate models: As one could expect, for all three key rate models W2U, W2C and W3C and for both shapes of initial interest rate curves the inaccuracy is monotone increasing in the time horizon T. But it is worth emphasizing that the inaccuracy increases with a constant rate. When applying refinements in the root, the correlated one W2C provides a considerably better accuracy than the uncorrelated case W2U. This is observed for all underlying planning horizons and term structures (table 8). To the contrary, W2C has a larger relative EVPI than W2U (table 10). This leads to the following – a little bit surprising – conclusion: The larger the correlations the larger the relative EVPI and therefore the larger the degree of stochasticity, but the easier to approximate the stochastic multistage program by the barycentric deterministic equivalent programs. Using three instead of two key rates results in a poorer accuracy and a larger relative EVPI. A comparison of tables 9 and 12 shows that partitioning in the root node provides better rates of improvement than partitioning along the scenario tree. Considering the ratios of the relative EVPI (tables 11 and 13) no preferable partitioning technique is observable; at least, one may observe that in many cases partitioning along the scenario tree reduces to a larger extent the degree of stochasticity. Partitioning in the root node can even lead to ratios of the relative EVPI which are < 1.0 and thus reflect an increased stochasticity after 4 refinements.

Vasicek models: In all cases the inaccuracy of the 1-factor model V1U is less than 1 % (table 8). This high accuracy indicates a high degree of bilinearity of the value functions corresponding to two subsequent periods. A possible reason for this characteristic lies in the nature of the 1-factor Vasicek model: The assumption of one single factor implies perfectly correlated yields of all considered maturities. Applying the 2-factor model leads to significantly increased inaccuracies. However, this inaccuracy is relatively stable with respect to the time horizon T.

Conclusions

From a probabilistic viewpoint the barycentric scenario trees are rough approximations of the interest rate process. But the saddle structure of the value functions inherent in funding problems makes the optimization process manageable. The approximate solution values turned out to be sensitive with respect to the parameters of the interest rate process. These data mainly determine the model risk. Hence, the user has to specify the proper interest rate model in a thorough way and has to perform a careful *profit & loss analysis* with respect to these input data to keep the model risk sufficiently small.

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A Appendix

Normal T=4	LB Value	LB Policy	UB Value	UB Policy	Accur. in%	rel. EVPI	EVPI
W2U	10626	1	12432	1	17.00	8.76	931
W2C	11945	1	12441	1	4.15	12.46	1488
W3C	11453	1	12430	1	8.53	13.22	1514
V1U	12468	1	12570	1	0.82	10.51	1211
V2U	11204	1	12427	1	10.91	19.97	2237
Inverse T=4	**LB Value**	**LB Policy**	**UB Value**	**UB Policy**	**Accur. in%**	**rel. EVPI**	**EVPI**
W2U	10991	2	12322	12	12.11	9.03	993
W2C	12029	12	12330	12	2.50	10.08	1213
W3C	11822	12	12317	12	4.19	12.27	1451
V1U	13829	12	13928	12	0.71	6.57	909
V2U	12120	12	13027	12	7.49	17.85	2163

Table 4: Results after 4 refinements in the root, $T = 4$

Normal T=5	LB Value	LB Policy	UB Value	UB Policy	Accur. in %	rel. EVPI	EVPI
W2U	14317	1	17363	1	21.28	10.68	1529
W2C	16449	1	17377	1	5.64	14.54	2392
W3C	15679	1	17359	1	10.71	15.65	2465
V1U	17433	1	17454	1	0.12	12.69	2213
V2U	15604	1	17283	1	10.76	24.16	3770
Inverse T=5	**LB Value**	**LB Policy**	**UB Value**	**UB Policy**	**Accur. in%**	**rel. EVPI**	**EVPI**
W2U	14706	2	17183	12	16.84	10.58	1556
W2C	16613	12	17198	12	3.52	12.80	2126
W3C	16249	12	17174	12	5.69	15.17	2465
V1U	19387	12	19541	12	0.79	8.74	1695
V2U	16940	12	18251	12	7.73	21.85	3701

Table 5: Results after 4 refinements in the root, $T = 5$

Normal T=6	LB Value	LB Policy	UB Value	UB Policy	Accur. in%	rel. EVPI	EVPI
W2U	18370	1	23093	1	25.71	12.16	2235
W2C	21557	1	23115	1	7.22	16.16	3484
W3C	20458	1	23086	1	12.84	17.74	3629
V1U	23014	1	23219	1	0.89	14.69	3380
V2U	20656	1	22910	1	10.91	28.33	5853
Inverse T=6	**LB Value**	**LB Policy**	**UB Value**	**UB Policy**	**Accur. in%**	**rel. EVPI**	**EVPI**
W2U	18862	2	22824	12	21.01	12.44	2346
W2C	21854	12	22848	12	4.55	15.08	3296
W3C	21275	12	22810	12	7.22	17.87	3801
V1U	25811	12	26034	12	0.86	10.67	2755
V2U	22464	12	24293	12	8.14	25.72	5779

Table 6: Results after 4 refinements in the root, $T = 6$

Normal T=7	LB Value	LB Policy	UB Value	UB Policy	Accur. in %	rel. EVPI	EVPI
W2U	22782	1	29615	1	29.99	13.62	3103
W2C	27284	1	29647	1	8.66	17.67	4821
W3C	(not available)						
V1U	29417	1	29561	1	0.49	16.46	4841
V2U	26510	1	29248	1	10.33	31.88	8451
Inverse T=7	**LB Value**	**LB Policy**	**UB Value**	**UB Policy**	**Accur. in %**	**rel. EVPI**	**EVPI**
W2U	23370	2	29240	12	25.12	14.20	3319
W2C	27715	12	29275	12	5.63	17.27	4787
W3C	(not available)						
V1U	33014	12	33299	12	0.86	12.52	4133
V2U	28818	12	31149	12	8.09	29.16	8403

Table 7: Results after 4 refinements in the root, $T = 7$

Normal	T=4	T=5	T=6	T=7
W2U	17.00	21.28	25.71	29.99
W2C	4.15	5.64	7.22	8.66
W3C	8.53	10.71	12.84	n.a.
V1U	< 1.0	< 1.0	< 1.0	< 1.0
V2U	10.91	10.76	10.91	10.33

Inverse	T=4	T=5	T=6	T=7
W2U	12.11	16.84	21.01	25.12
W2C	2.50	3.52	4.55	5.63
W3C	4.19	5.69	7.22	n.a.
V1U	< 1.0	< 1.0	< 1.0	< 1.0
V2U	7.49	7.73	8.14	8.09

Table 8: Accuracy in % after 4 refinements in the root node

Normal	T=4	T=5	T=6	T=7
W2U	1.27	1.25	1.19	1.17
W2C	1.47	1.34	1.26	1.21
W3C	1.22	1.17	1.14	n.a.
V1U	(accuracy $< 1\%$)			
V2U	1.07	1.13	1.01	0.98

Inverse	T=4	T=5	T=6	T=7
W2U	1.73	1.43	1.34	1.28
W2C	1.80	1.59	1.50	1.43
W3C	1.68	1.58	1.49	n.a.
V1U	(accuracy $< 1\%$)			
V2U	1.06	1.13	1.02	0.97

Table 9: Ratio of accuracy with partitioning in the root node

Normal	T=4	T=5	T=6	T=7
W2U	8.76	10.68	12.16	13.62
W2C	12.46	14.54	16.16	17.67
W3C	13.22	15.65	17.74	n.a.
V1U	10.51	12.69	14.69	16.46
V2U	19.97	24.16	28.33	31.88

Inverse	T=4	T=5	T=6	T=7
W2U	9.03	10.58	12.44	14.20
W2C	10.08	12.80	15.08	17.27
W3C	12.27	15.17	17.87	n.a.
V1U	6.57	8.74	10.67	12.52
V2U	17.85	21.85	25.72	29.16

Table 10: Relative EVPI after 4 refinements in the root node (lower approximation)

Normal	T=4	T=5	T=6	T=7
W2U	1.14	1.06	1.06	1.06
W2C	1.02	1.01	1.01	1.01
W3C	1.01	1.01	1.01	n.a.
V1U	1.10	1.08	1.04	1.02
V2U	1.08	1.07	1.05	1.03

Inverse	T=4	T=5	T=6	T=7
W2U	0.99	1.06	1.04	1.03
W2C	1.06	1.01	1.00	0.99
W3C	0.96	0.94	0.94	n.a.
V1U	1.13	1.09	1.07	1.04
V2U	1.13	1.10	1.07	1.04

Table 11: Ratio of relative EVPI with partitioning in the root node

Normal	T=4	T=5	T=6	T=7
W2U	1.19	1.04	1.03	1.06
W2C	1.07	1.07	1.06	1.06
W3C	1.03	1.02	1.02	n.a.
V1U	(accuracy $< 1\%$)			
V2U	1.06	1.13	1.08	0.98

Inverse	T=4	T=5	T=6	T=7
W2U	1.59	1.38	1.32	1.25
W2C	1.06	1.06	1.04	1.03
W3C	1.54	1.01	1.00	n.a.
V1U	(accuracy $< 1\%$)			
V2U	1.04	1.10	1.10	0.98

Table 12: Ratio of accuracy with partitioning along the scenario tree

Normal	T=4	T=5	T=6	T=7
W2U	1.16	1.03	1.03	1.02
W2C	1.04	1.02	1.01	1.01
W3C	1.07	1.05	1.04	1.03
V1U	1.14	1.05	1.03	1.01
V2U	1.08	1.07	1.05	1.03

Inverse	T=4	T=5	T=6	T=7
W2U	1.06	1.03	1.02	1.03
W2C	1.10	1.06	1.05	1.04
W3C	1.01	1.03	1.01	1.02
V1U	1.26	1.22	1.13	1.08
V2U	1.13	1.10	1.07	1.04

Table 13: Ratio of relative EVPI with partitioning along the scenario tree

Normal/Inverse		T=4	T=5	T=6	T=7
W2U	rows	9'015	27'240	81'915	245'940
	columns	10'818	32'688	98'298	295'128
	nonzeros	30'050	90'800	273'050	819'800
W2C	rows	9'015	27'240	81'915	245'940
	columns	10'818	32'688	98'298	295'128
	nonzeros	30'050	90'800	273'050	819'800
W3C	rows	25'515	102'315	409'515	1'638'315
	columns	30'618	122'778	491'418	1'965'978
	nonzeros	85'050	341'050	1'365'050	5'461'050
V1U	rows	2'265	4'665	9'465	19'065
	columns	2'718	5'598	11'358	22'878
	nonzeros	7'550	15'550	31'550	63'550
V2U	rows	9'015	27'240	81'915	245'940
	columns	10'818	32'688	98'298	295'128
	nonzeros	30'050	90'800	273'050	819'800

Table 14: Problem sizes after 4 refinements in the root (lower approximations)

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