ON THE CONSISTENCY OF CHOICE

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Abstract. Consistency of choice is a fundamental and recurring theme in decision theory, social choice theory, behavioral economics, and psychological sciences. The purpose of this paper is to study consistency of choice independent of the particular decision model at hand. Consistency is viewed as an inherently logical concept that is fundamentally void of connotation and is thus disentangled from traditional rationality or consistency conditions imposed on decision models. The proposed formalization of consistency takes two forms: internal consistency, which refers to the property that a choice model does not generate contradictory statements; and semantic consistency, which refers to the idea that a theory's predictions are valid with respect to some observed data. In addressing semantic consistency, the relationship between theory and data is analyzed in terms of so-called duality mappings, which allow a passage between the two universes in a way that consistency is preserved. The formalization of consistency concepts relies on adapting revealed preference theory to the context-dependent setting. The paper concludes by discussing the implications of the proposed framework and how it relates to classical revealed preference theory and other formalizations of the relationship between the theory and reality of choice.

1. Introduction

Consistency of choice is a fundamental and recurring theme in decision theory, social choice theory, behavioral economics, and psychological sciences. The purpose of this paper is to formalize the notion of consistency of choice at both a theoretical and a behavioral level, independent of the particular economic theory or choice model at hand.

The realm of definitions of consistency has traditionally been restricted to either an identification with some form of rationality or to an imposition of properties that choice functions and preference relations must satisfy. Such restrictions have been categorized as being *rationality* or *consistency* conditions, two terms that are often used synonymously and have become loaded with connotations in choice theoretic writings. Any such conditions, however, strictly relate to properties that a given preference relation or a choice function needs to satisfy in order for it to be considered consistent, and not to the inherently logical meaning of consistency. What one considers to be consistent in models of choice thus essentially amounts to a subjective model-dependent evaluation of properties of choice functions and preference relations. There is, however, an intrinsically objective and unambiguous view of consistency that is linked to the logical quality of deductive reasoning.

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The purpose of this paper is hence to provide a concise and objective formalization of the notion of consistency in choice theoretic models. It is argued that such an objective evaluation of consistency of an arbitrary theory of choice reduces to an analysis of the choice system as a whole as opposed to a subjective analysis of properties of preference relations or choice functions. More specifically, for an economic theorist who comes equipped with a choice model, an evaluation of consistency can be categorized as taking one of two forms. One one hand, one hopes that the given theoretical choice system is in itself consistent in the sense that it does not generate contradictory statements. This is referred to as *internal consistency*. On the other hand, consistency of an abstract choice model is often evaluated with respect to some real world observations. This shall be called *semantic consistency*. It refers to the idea that a given choice theory's preferences are valid with respect to some observed choice data. These two pillars of consistency are arguably used implicitly in any evaluation of a choice theoretic model, and their formalization is the subject of this paper.

The crux of this foundational view of consistency is threefold. First, internal consistency of a choice model is defined in terms of impossibility of deriving a contradiction from the model. Such a contradiction consists of the logical incompatibility of two or more choices made. To be able to objectively formulate what constitutes contradictory choices, a context-dependent viewpoint of choices and preferences takes central stage throughout, and it is argued that context-dependence enables a precise definition of contradictory choice in terms of the negation of context-dependent preference relations. Second, internal consistency is used to define an objective notion of rationality of an economic agent that is linked to his quality of reasoning. More precisely, consistent axiomatizability of a decision maker's choices may fail, and if his behavior contains a logically contradictory choice, then a choice theory representing his observed behavior must be internally inconsistent. Third, semantic consistency represents an inherent analysis of the relationship between theory and observation, and a substantial part of this paper is dedicated to examining this relationship more closely. In particular, the passage between the universe of axiomatic choice theories and the universe of observed choice structures is formalized via transformational mappings, which are referred to as the *duality* mappings. They are essential ingredients in the analysis of the semantic consistency of choice and may in fact already be implicitly used, whether one is studying a proposed axiomatic model of choice, or whether one is trying to model observed data. Beyond being a formalization of this relationship, the duality mappings provide insights into the process of evaluating choice consistency in practice.

The remainder of the paper is structured as follows. The foundational view of internal and semantic consistency as opposed to traditional consistency conditions is motivated and discussed in Section 2. Section 3 sets up the choice theoretic framework and provides the context-dependent definition, which is a fundamental concept in the development of internal consistency. In Section 4, the relationship between theory and observation is analyzed in terms of the second pillar of consistency, namely semantic consistency. The passage between the universe of axiomatic choice theories and the universe of observed choice structures is then formalized via transformational mappings, which are referred to as the duality mappings. In Section 5, the process of evaluating semantic consistency is examined more closely in terms of the duality mappings and the relationship between theory and reality of choice is revisited. In particular, the duality mappings are used to show that (i) an observed choice is semantically consistent with a given theory if and only if it is a consequence of the theory's axioms; (ii) a consistent axiomatization, even though based on observed data, implicitly also axiomatizes what is unobserved; (iii) the universes of theory and reality are not equivalent; (iv) one strand of rationality capturing logical quality of reasoning can be defined in terms of internal consistency. Section 6 concludes by discussing the implications of the proposed framework and how it relates to classical revealed preference theory and other formalizations of the relationship between the theory and reality of choice.

2. Towards a well-defined notion of consistency

The axiomatics of choice has been given a central place in decision theory, social sciences and economics. The traditional approach to modeling choice behavior in terms of preferences involves imposing axioms on preference relations and choice functions. One reason is often the need to work with utility functions, which exist if and only if preference relations are complete preorders, hence imposing completeness and transitivity assumptions. Another reason is the goal of modeling the decision maker's *rationality* or *consistency*, which roughly denote what is required of the decision maker for his preferences and choices to be considered sensible. Examples of such requirements include transitivity of preferences, the weak axiom of revealed preference introduced by Samuelson (1938), Sen's α and β axioms (Sen 1969), and Suzumara's consistency condition, which is a weakening of transitivity (Suzumara 1976). Any such conditions, however, strictly relate to properties that a given preference relation or a choice function needs to satisfy in order for it to be considered consistent, and not to the inherently logical meaning of consistency. With this in mind, the goal of this paper is not to debate the economic or choice theoretic soundness of imposing certain axioms on preferences and choice functions. It is also not arguing for or against the epistemological concept of coherentism, which roughly states that internal consistency of a set of beliefs is the only criterion for truth. The primary purpose of this paper is, however, to disentangle the notion of consistency, an inherently logical concept that is fundamentally void of connotation, from traditional rationality or consistency conditions imposed on choice models.

From the previous discussion, it becomes clear that the presumably well defined concept of choice, when looked at closely from a metamathematical point of view, ramifies into many different notions. The view taken in this paper is that different choice theories are defined for different purposes[1], and each theory can be evaluated in terms of its consistency. A theory of choice is thus taken to be a general abstract meta-model of choice behavior made up of three crucial components: a set of alternatives X, a collection of binary relations on subsets of X enabling the definition of menu-dependent preferences, together with a set of axioms describing properties of the binary relations. For example, a theory whose set of axioms contains completeness and transitivity of menu-independent preferences is the traditional *rational* theory of choice; and a theory whose set of alternatives is given by lotteries and whose set of axioms contains the continuity and independence

[1]For a discussion of a similar view, the reader is referred to Gilboa (2010).

axioms on top of completeness and transitivity can be viewed as a representation of the theory of rational expected utility maximizers of von Neumann and Morgenstern (1953).

A core goal of this paper is hence to disentangle the notion of consistency from traditional imposed requirements by developing a clear formalism for discussing consistency of choice at an axiomatic level. Consider the following scrutiny of the concept of *internal consistency* of choice by Sen (1993):

> *The problems with the idea and use of conditions of internal consistency of choice can be seen at two rather different levels:* foundational *and* practical. *At the foundational level, the basic difficulty arises from the implicit presumption underlying that approach that* acts *of choice are, on their own, like* statements *which can contradict, or be consistent with, each other. That diagnosis is deeply problematic. Statements A and* not-*A are contradictory in a way that choosing x from $\{x, y\}$ and y from $\{x, y, z\}$ cannot be. If the latter pair of choices were to entail respectively the statements (1) x is a better alternative than y, and (2) y is a better alternative than x, then there would indeed be a contradiction here (assuming that the content of "being better than" requires asymmetry). But those choices do not,* in themselves, *entail any such statements.* Given *some ideas as to what the person is trying to do (this is an external correspondence), we might be able to "interpret" these actions as implied statements. But we cannot do that without invoking such external reference. There is no such thing as* purely *internal consistency of choice.*

I shall not dispute the idea that internal consistency conditions do not suffice to describe choice behavior as external criteria need to be invoked. However, whether one argues that external factors or motivations can be imposed internally, or whether one agrees that internal consistency conditions are inconclusive if they are not linked to external criteria, consistency of choice can be accurately and unambiguously formalized at the foundational level and its investigation can yield valuable insights into theoretical choice models and their relationship with observed choice data.[2]

2.1. **Non-contradictory consistency.** To define an unambiguous notion of internal consistency of choice, the intrinsically logical definition of consistency in studies of mathematical formalisms is taken as a heuristic guideline for developing its choice-theoretic counterpart. According to Tarski (1946), "*a [...] theory is called consistent or non-contradictory if no two asserted statements of this theory contradict each other*", where "*two sentences, of which the first is a negation of the second, are called contradictory sentences.*" Emulating this idea, this paper takes the view that internal consistency, as opposed to rationality, is a property of an axiomatic choice system and not of the preference relation or choice function representing a decision maker. More specifically, an internally consistent choice theory is one which does not contain contradictory acts of choice. Such contradictions consist of the logical incompatibility of two or more choices made.

[2]Because this assertion is foundational in nature and its topic is, at least tangentially, tied to diverse philosophical and methodological questions regarding choice, the boundaries of the proposed definition are stressed throughout this paper. The purpose of this is not to repudiate a viewpoint, but to clarify the extent and, more importantly, the limitations of the proposed formalization of consistency.

As Sen points out, a statement A (say, "the earth is round") and its negation *not-A* ("the earth is not round") clearly contradict one another, whereas the two acts of choosing x from $\{x, y\}$ and y from $\{x, y, z\}$ do not present a logical incompatibility per se. To be able to define a logically precise and intuitively sensible notion of a contradictory choice, classical revealed preference theory is adopted and a context- or menu-dependent choice setting is imposed. The epistemic value of the menu offered for choice is thus a crucial ingredient in this setting. An imperative assumption underlying the definition of consistency is that the choice context must be the same when determining whether two or more acts of choice are contradictory. As Sen (1993) writes: "*what is offered for choice can give us information about the underlying situation, and can thus influence our preferences over the alternatives, as we see them.*" In this regard, and to continue using an example provided by Sen (1993), a person who chooses not to have anything (x) at a dinner table when the only other choice is one apple (y) may also decide to choose an apple (y) when an additional option, say choosing another apple (z), is added to the menu, without violating any real consistency conditions. This idea of menu- or context-dependence takes central stage in this paper's examination of consistency.

More formally, consider the following two acts of choice: the decision maker chooses x from $\{x, y, z\}$ and the decision maker does *not* choose x from the same choice menu $\{x, y, z\}$. These are not only contradictory statements in the classical logical sense, but they also bring about a contradiction when one adopts the interpretation of revealed preference theory. In particular, if the first act of choice implies that x is at least as preferred to y given the choice set $\{x, y, z\}$, and if the second act implies that x is *not* as least as preferred to y under the same choice context $\{x, y, z\}$, then the two do contradict one another. Hence, by introducing context-dependent preferences, an axiomatic theory of choice can be seen as consistent or inconsistent on purely internal grounds. The context-dependent viewpoint of choice enables a definition of contradiction and consistency in a manner that applies to acts of choice rather than to statements under an arbitrary theory of choice. Inspired by Tarski (1946) and his laws of deductive logic, this identification between consistency and non-contradictory choice is adapted throughout.[3]

Given this view of contradictory choices, it is necessary to point out that the temporal dimension of the act of choice is not considered. In other words, the main interest in this paper lies in the consistency of an instantaneous snapshot of a decision maker's preferences and not in how his preferences evolve over time. In particular, this implies that contradictory choices in the form of

[3] Note that the type of deductive logic considered is formal, free of judgement, experience or perception. This coincides with Ramsey's notion of the logic of consistency or formal logic. In his seminal work, Ramsey (1926) writes: "*it seems to me, therefore, that we can divide arguments into two radically different kinds, which we can distinguish in the words of Peirce as (1) 'explicative, analytic, or deductive' and (2) 'amplifiative, synthetic, or (loosely speaking) inductive'. Arguments of the second type are from an important point of view much closer to memories and perceptions than to deductive arguments. We can regard perception, memory and induction as the three fundamental ways of acquiring knowledge; deduction on the other hand is merely a method of arranging our knowledge and eliminating inconsistencies or contradictions. Logic must then fall very definitely into two parts: (excluding analytic logic, the theory of terms and propositions) we have the lesser logic, which is the logic of consistency, or formal logic; and the larger logic, which is the logic of discovery, or inductive logic.*"

preference reversals, whether or not they are a result of new information, lie beyond the scope of this formalism.

2.2. **Semantic consistency.** Testing the validity of a theoretical model of choice is an integral part of the economic theorist's profession. Consistency of choice hence not only addresses the question of whether a contradiction may arise from within the rules of a given theory, but it also seeks to understand what a given theory says about data. It thus inherently seeks to formalize the relationship between theory and reality. Economic professionals typically want to judge whether or not observed data are consistent with a given theory, for example by comparing the theory's predictions to observations. If observed economic phenomena are consistent with one's theory, one may gain a better understanding of data by analyzing the properties of the theory rather than concluding that the theory is in fact correct or true. This second pillar of consistency shall be called *semantic* consistency, as it analyzes the semantics or meaning given to an abstract theory in terms of the observed reality.[4]

Suppose that an economic theorist comes equipped with an axiomatic theory, denoted by $\mathcal{A}$, and some observed data, represented by a choice structure $\mathcal{C}$ (Arrow 1959, Richter 1966). The typical evaluation of the relationship between theory and data involves a twofold comparison, aiming to understand on one hand whether the theory is consistent with observed choices, and on the other hand how the observed choices compare to the theory's predictions. More specifically, an axiomatic theory $\mathcal{A}$ has many implications that can be tested. These implications are the theory's theorems. They are mappings from postulated statements into derived choices made. Conventionally, these derived conclusions of a theory are what the empirical or experimental economist uses to formulate specific hypotheses that motivate the comparison of data with the theory's predictions. If one prediction of the theory is deduced from another using its axiomatic rules, then one may not claim that the theory is consistent with some observables if whenever the consequent is verified but the antecedent is refuted. And so, a given structure $\mathcal{C}$ should be consistent with some axiomatic choice theory $\mathcal{A}$ if none of the observables in the structure contradict the theorems of $\mathcal{A}$. This relationship is formalized via the concept of *semantic consistency*. For semantic consistency to hold, the idea is roughly that none of the observations in $\mathcal{C}$ and the predictions of $\mathcal{A}$ contradict each other. Semantic consistency is thereby an approach closely related to that of revealed preference theory, which also essentially seeks to understand consistency between theory and data. In its most classical form, revealed preference theory captures a notion of semantic consistency via *rationalizability*, which only holds when certain properties of preference relations and choice functions are satisfied. The extent of the relationship between rationalizability and semantic consistency is discussed more thoroughly in the concluding section of this paper.

[4]The terminology of *semantic consistency* is borrowed from formal studies of logic and model theory. Indeed, in traditional Aristotelian logic, consistency is a semantic notion: two or more statements are called consistent if they are simultaneously true in a model under some interpretation. In modern logic, internal consistency is also called *syntactic* consistency, and a set of statements is called syntactically consistent with respect to a certain logical calculus, if no formula and its negation are derivable from those statements by the rules of the calculus, that is the theory is free from contradictions. If these definitions are equivalent for a logic, the equivalence amounts to the completeness of its system of rules, a notion that is far beyond the scope of this paper.

3. Theoretical setup

Axiomatic theories of choice are abstract models describing observed behavior. An axiomatic theory represents the formal language and rules for reasoning about choice and thus provides a systematic way for forming hypotheses concerning choice independent of the particular economic model under consideration. For the purpose of studying an intrinsic notion of choice consistency, a general model-independent view of theories of choice is taken in this paper. It consists of three essential components: a set of alternatives; a collection of context-dependent binary relations expressing paired preference comparisons with respect to a fixed choice menu; and a set of axioms for making deductions on preferences.

The first ingredient is a set of possible, mutually exclusive alternatives on top of which preferences are formed. The elements of this set of hypothetical alternatives, denoted by X, represent real choice alternatives that one aims to model. Depending on the choice problem at hand, one can impose a topological or algebraic structure on top of X to describe properties of the real-world alternatives more precisely.[5]

3.1. **Preferences-in-context.** Preferences are formal representations of choice that are meant to express paired comparisons with the interpretation of an alternative being preferred to or at least as good as another. No further properties are imposed on preferences. In particular, preferences are not assumed to be partial or total orders as is often the case in choice-theoretic settings. Traditionally, a preference is defined as a binary relation on the set of alternatives X. Motivated by Sen's observation that the two acts of choosing x from $\{x, y\}$ and y from $\{x, y, z\}$ do not really present a logical incompatibility, the focus in this paper is on preferences that are *menu-* or *context-dependent*.

Traditional theories of choice assume that a decision maker has a complete preference order over all alternatives in X and is not affected by the presence or absence of alternatives, a principle also known as the independence of irrelevant alternatives. Indeed, a condition of context-independence is typically implicitly imposed on preference relations in mainstream choice theory, and variations of the context or menu would appear to be a change of preference. Context-dependence of preferences is precisely what is ruled out by standard "consistency" assumptions, such as the weak axiom of revealed preference (WARP) proposed by Samuelson (1938), the strong axiom of revealed preference (SARP) by Houthakker (1950), and the basic contraction and expansion consistency conditions (properties α and β) of Sen (1971). Even though these fundamental conditions have powerful uses across decision, consumer and equilibrium theory, consistency in the strict internal, non-contradictory, coherent sense is not inherent in these choice heuristics, and hence logically consistent reasoning about choice can coexist with context-dependence. This ambiguity between internal inconsistency and preference change has been previously diagnosed in the literature. The

[5]For example, in the traditional setup of mathematical finance, decision makers choose from the vector space $\mathbb{L}^\infty$ of essentially bounded real-valued random variables on a probability space. In expected utility theory, alternatives are considered over the space of lotteries, which are formalized via probability distributions. The agent's choice set is hence the set $\mathcal{M}$ of all probability measures on a separable metric space. The formalization of the algebraic structure of a set relies on work by Birkhoff (1935) and is beyond the scope of this article.

reader is referred to Becker (1976), whose analysis uses menu-dependent preference relations and who points out that many cases of apparent preference change in fact arise from inadequate characterizations of preferences and not from internal contradictions.[6]

In reality, a large body of studies uses empirical evidence to indicate that such context-independence is often violated. The need for context-dependent variations to choice models has hence long been recognized and models incorporating context-dependent preferences have been developed over the past few decades, see for example Tversky and Simonson (1993), Sen (1994), Sen (1997) and more recently Saran (2011). The primary motivation for incorporating context-dependent preferences when studying consistency of choice goes beyond modeling empirical anomalies, however. Context-dependence enables a precise definition of contradictory choice in terms of the negation of preferences-in-context. Moreover, it allows an axiomatic choice theory to be as general as possible, where traditional context-independent preferences can also be modeled.

Let X_Γ be the set of all nonempty subsets of X. An element $\Gamma \subseteq X$ of X_Γ contains a collection of alternatives and is referred to as a *choice context* or simply a *context*. For each context Γ, define $\succsim^\Gamma$ to be a binary relation on Γ. Two alternatives $x, y \in \Gamma$ related through $\succsim^\Gamma$ are denoted by $\Gamma \vdash x \succsim y$ instead of $x \succsim^\Gamma y$. The notation $\Gamma \vdash x \succsim y$ denotes a *preference-in-context* P. It is interpreted as x being at least as good as y when confronted with the menu of choice alternatives Γ.

Towards the end of defining internal consistency in terms of contradictory choices, define the negation $\neg P$ of a preference-in-context $P = \Gamma \vdash x \succsim y$ to be the negation of the binary relation $x \succsim^\Gamma y$, meaning that x is not related through $\succsim^\Gamma$ to y. It is interpreted as the statement that x is not at least as good as y when confronted with the context Γ. In that case, $\Gamma \vdash x \not\succsim y$ is used as a shorthand for the negation $\neg(\Gamma \vdash x \succsim y)$.

Note that $\succsim^\Gamma$ expresses weak preference. For $\Gamma \in X_\Gamma$, a preference relation $\succsim^\Gamma$ induces a strict preference relation $\succ^\Gamma$ on Γ defined by $\Gamma \vdash x \succ y$ if and only if $\Gamma \vdash x \succsim y$ and $\Gamma \vdash y \not\succsim x$. This is intrepreted as x being strictly preferred to y when confronted with the choice context Γ. Each relation $\succsim^\Gamma$ also induces an indifference relation $\sim^\Gamma$ defined by $\Gamma \vdash x \sim y$ if and only if $\Gamma \vdash x \succsim y$ and $\Gamma \vdash y \succsim x$. The relation $\sim^\Gamma$ expresses an absence of strict preference in either direction rather than active indifference, the latter notion being closer to that of equivalence. Indifference relations are symmetric by definition, but not necessarily reflexive or transitive.

When the context of choice is unspecified, the statement "$x \succsim y$" is meaningless in this setup. In particular, the preference relation defined on top of the whole set X is denoted by $\succsim^X$, and a corresponding preference-in-context is denoted by $X \vdash x \succsim y$. Traditional context-independent preferences can be expressed using this formalism. They are represented by binary relations defined over the universal choice set X in such a way that their restriction to contexts coincides with the

[6] Even with this distinction in mind, one may then question whether in the study of choice behavior the idea of context-dependence can be used to form aggregate prediction over a collection of different menus. This is an especially relevant question in social choice theory and aggregation questions lie far beyond the scope of this paper. Note that context-dependence in our framework would for example imply that weak Pareto optimality or the independence of irrelevant alternatives in social choice theory are not necessarily satisfied.

original relation. More formally, a preference relation $\succsim$ on X is *context-independent* if for all contexts $\Gamma \subseteq X$ with $x, y \in \Gamma$, $X \vdash x \succsim y$ if and only if $\Gamma \vdash x \succsim y$.[7]

3.2. **Axiomatics.** Reasoning about preference relations $\succsim^\Gamma$ and their corresponding preferences-in-context is formalized in terms of axioms. These are deduction rules that describe valid logical consequences of given premises. An *axiom*, is denoted by

$$\frac{\text{COND}}{\Gamma \vdash x \succsim y} .$$

It states that, assuming the (set of) conditions COND, the preference-in-context $\Gamma \vdash x \succsim y$ is valid. Conclusions of these rules are always preferences-in-context and are referred to as *derivable* preferences-in-context. There is no restriction on what the set COND may include, and hence conditions can, for example, be imposed on the properties of contexts, the properties of preference relations, underlying behavioral traits, moral or ethical motivations, etc. What constitutes a good set of axioms largely depends on the application one has in mind. In general, axioms are meant to be basic, intuitive and qualitative descriptions of preference relations with a particular setting or application in mind.

Most properties that are traditionally imposed on preference relations can be formulated as axiomatic deduction rules. For example, irreflexivity of strict preferences is the axiom

$$\frac{\Gamma \subseteq X}{\Gamma \vdash x \not\succ x} ,$$

and context-independent transitivity of preferences is the axiom

$$\frac{\Gamma_1 \vdash x \succsim y \wedge \Gamma_2 \vdash y \succsim z}{\Gamma \vdash x \succsim z} \quad \text{for all } \Gamma \subseteq X \text{ with } x, z \in \Gamma .$$

Other standard axioms that are assumed in choice theoretic frameworks because of their economic soundness can be expressed as logical deduction rules. For example, assuming that X comes equipped with an ordering $\geq$, the monotonicity axiom can be stated as

$$\frac{x \geq y}{\Gamma \vdash x \succsim y} \quad \text{for all } \Gamma \subseteq X \text{ with } x, y \in \Gamma .$$

Moreover, classical menu-dependent behavior, which often involves more complex considerations of motivations, ethics and epistemology, can be naturally accommodated in the framework of context-dependent axiomatics. Consider again the case introduced by Sen (1993) that an apple is always chosen from a basket offered to the decision maker, unless the set of alternatives in the basket is reduced to make it the last apple in the basket. Let X represent the set of all possible alternatives that can be added to the basket, and let $\Gamma_a \subset X$ denote the subset of all (and only) apples. Then this type of menu-dependent choice motivated by not wanting to take the last apple offered can be formalized as the rule:

[7]See Sen (1997) for a formalization of menu-independent preferences, menu-independent choice functions, and their relationship.

$$\frac{x \in \Gamma_a \wedge \Gamma \cap (\Gamma_a \setminus \{x\}) \neq \emptyset}{\Gamma \vdash x \succsim y} \quad \text{for all } \Gamma \subseteq X \text{ with } x, y \in \Gamma \ .$$

3.3. **Axiomatic theories and internal consistency of choice.** An *axiomatic theory of choice* or simply an *axiomatization* $\mathcal{A} = (X, \mathbb{A})$ is given by a set of alternatives X and a set of axioms $\mathbb{A}$ specifying the axiomatic deduction rules for preferences-in-context. For an axiomatic theory $\mathcal{A}$, the set $\mathcal{P}_{\mathcal{A}}$ denotes all preferences-in-context P that are derivable from the axioms in $\mathbb{A}$. Such derivable preferences-in-context $P = \Gamma \vdash x \succsim y$ will be denoted by $\Gamma \vdash_{\mathcal{A}} x \succsim y$ and referred to as *theorems* of the theory $\mathcal{A}$.

Definition 3.1 (Internal consistency)**.** An axiomatization $\mathcal{A} = (X, \mathbb{A})$ is *internally consistent* if and only if

$$\Gamma \vdash_{\mathcal{A}} x \succsim y \iff \Gamma \vdash_{\mathcal{A}} x \not\precsim y \ .$$

An axiomatic choice theory is thus internally consistent whenever a theorem and its negation can never be derived from its axioms. Note that absence of contradiction is defined in syntactic terms via a restricted context-dependent form of asymmetry, as opposed to traditionally assumed context-independent asymmetry of preferences, which is a stronger condition.

Clearly, an infinite number of axiomatic theories can be built on top of a given set of alternatives X. Let **Ax** be the set of all possible choice axiomatizations of the form $\mathcal{A} = (X, \mathbb{A})$. It comes equipped with a partial order given by inclusion $\subseteq$ on the sets $\mathbb{A}$ of axioms. For $\mathcal{A}' = (X, \mathbb{A}')$, $\mathcal{A} \subseteq \mathcal{A}'$ means that the set of axioms $\mathbb{A}$ is a subset of $\mathbb{A}'$ and hence $\mathcal{A}$ is to be thought of as a subtheory of $\mathcal{A}'$. Axiomatic theories in **Ax** are thus partially ordered from less specific (requiring less axioms) to more restrictive (including more axioms). The minimal element of **Ax** is the empty theory with no axioms, and the maximal element is the necessarily inconsistent theory containing all possible axioms.

3.4. **Observed choice structures.** Axiomatic theories of choice are abstract models describing how an individual makes decisions. They represent an attempt to describe and understand observed behavior. The semantic universe for interpreting formal theories of choice is given by empirical choice data in the real world. Formally, the semantic universe of choice structures, consisting of budget spaces and choice functions as defined by Arrow (Arrow 1959) and Richter (Richter 1966), is adopted here. A choice structure is an abstract representation of an individual's behavior based on hypothetical choice experiments, actual observed choices, or empirical data (as opposed to being derived from introspection, as discussed in Savage (1954)).

For the set of alternatives X, the set X_Γ of all nonempty subsets of X is referred to as the *budget space* in this setting, and its elements $\Gamma \subseteq X$ are called *budget sets* rather than contexts. A *choice function* $c : X_\Gamma \to \mathcal{P}(X)$ is a mapping from the budget space to the powerset of X satisfying $c(\Gamma) \subseteq \Gamma$. It assigns to each budget set $\Gamma \in X_\Gamma$ a subset $c(\Gamma)$ of Γ. Note that this definition of choice function does not require that the choice set be nonempty. A *choice structure* $\mathcal{C}$ is then given by the tuple (X, c).

For a fixed set of alternatives X, one can now think of all possible collections of choices that can be made on top of X. Denote this universe of observations by **Obs**, which is formally the powerset of the set of all possible semantic choice structures $\mathcal{C}$. An element of **Obs** is hence a set containing a collection of choice structures, which will be denoted by $\mathcal{C}^*$. Suppose for example that an economic theorist observes some choice regularity by running a behavioral experiment over a set of individuals, each with their own choice function and structure. Such regularities can be thought of as elements of a collection of choice structures $\mathcal{C}^*$. A member of **Obs** can also be thought of as a set of alternative semantic choice 'universes' with which an axiomatic theory can be consistent, or, more concretely, the set of all economic agents whose choices are consistent with a given axiomatization. Analogous to the set **Ax** of axiomatic choice theories, the set **Obs** also comes equipped with a partial order given by subset inclusion. Its elements, which are collections of choice structures, are then partially ordered from less specific (including more alternatives) to more specific (including a narrower range of alternatives).

4. Semantic consistency and the duality of axiomatic and observed choice

The relationship between axiomatizations of choice and observed structures of choice amounts to the relationship between what is hypothetical and what is real. I shall refer to this relationship between choice theory and observed choice as a *duality*, a concept which aims to capture a recurring theme within the study of abstract, often times axiomatic, models of observed economic phenomena. Economists repeatedly concentrate in one moment on the deduction of results from theoretical models, and in the next consider the observed universes to which these apply. They essentially deal with two worlds, the abstract and the real, and two transforming functions back and forth between these worlds. In the context of individual choice, this arises for example when reasoning about risk aversion (observed behavior) using concave utility functions (its formal representation), or when reasoning about diversification using convexity of preference functionals. More generally, one can say that economic observed phenomena are fundamentally the subject matter of formal economic theories and models, and that axiomatic theories of choice are formulated for the purpose of reasoning about and predicting observed behavioral choice data.

In this section, the relationship between theory and observation is analyzed in terms of the second pillar of consistency, namely *semantic consistency*. The passage between the universe of axiomatic choice theories and the universe of observed choice structures is then formalized via transformational mappings, which are referred to as the *duality mappings*. The duality mappings are essential ingredients in the analysis of the semantic consistency of choice. Indeed, they are used implicitly by the economic theorist, whether he is studying a proposed axiomatic model of choice, or whether he is trying to make sense of, generally limited, observed data.

4.1. **Semantic consistency.** An axiomatic theory $\mathcal{A}$ has many implications that can be tested. These implications are the theory's theorems, which are the preferences-in-context in $\mathcal{P}_{\mathcal{A}}$ and can be thought of as mappings from postulated statements into derived choices made. Conventionally, these derived conclusions of a theory are what the empirical or experimental economist uses to formulate specific hypotheses that motivate the comparison of data with the theory's predictions. If one prediction $P \in \mathcal{P}_{\mathcal{A}}$ of the theory is deduced from another using its axiomatic rules, then one may not claim that the theory is consistent with some observables if whenever the consequent P is verified but the antecedent is refuted. And so, a given structure $\mathcal{C}$ should be consistent with some axiomatic choice theory $\mathcal{A}$ if none of the observables in the structure contradict the theorems of $\mathcal{A}$. This relationship is formalized via the concept of *semantic consistency*.

Suppose that an economic theorist comes equipped with a theory $\mathcal{A} = (X, \mathbb{A})$ and some observed choice data $\mathcal{C} = (X, c)$. The typical consistency evaluation scenario involves a twofold comparison between the theory and the observed data:

(1) Does the theory contradict the observed choices? In order to analyze how choices of $\mathcal{C}$ compare to the theorems of $\mathcal{A}$, revealed preference theory (Samuelson 1938) is adapted to the context-dependent setting. A choice structure $\mathcal{C} = (X, c)$ induces preference relations $\succsim_{\mathcal{C}}^{\Gamma}$ on every context $\Gamma \subseteq X$ with *revealed preferences-in-context* defined by $\Gamma \vdash x \succsim_{\mathcal{C}} y$ if and only if $x \in c(\Gamma)$. Denote by $\mathcal{P}_{\mathcal{C}}$ the set of all revealed preferences-in-context of $\mathcal{C}$. A preference-in-context $P := \Gamma \vdash_{\mathcal{A}} x \succsim y \in \mathcal{P}_{\mathcal{A}}$ is *consistently satisfied in* or *consistent with* $\mathcal{C}$ if $P \in \mathcal{P}_{\mathcal{C}}$, that is if $\Gamma \vdash x \succsim_{\mathcal{C}} y$ is a revealed preference-in-context. In that case, the notation $\mathcal{C} \models P$ is used to denote that the structure $\mathcal{C}$ satisfies the thorem $P \in \mathcal{P}_{\mathcal{A}}$.

(2) How do the observed choices compare to the theory's predictions? Formally, the predictions that the theory $\mathcal{A} = (X, \mathbb{A})$ produces is a choice structure $\hat{C}_{\mathcal{A}} = (X, \hat{c}_{\mathcal{A}})$ whose predicted choice function $\hat{c}_{\mathcal{A}} : X_{\Gamma} \to \mathcal{P}(X)$ is defined by $x \in \hat{c}_{\mathcal{A}}(\Gamma)$ if and only if $\Gamma \vdash_{\mathcal{A}} x \succsim y$ is a theorem of $\mathcal{A}$ for all $y \in \Gamma$.

For consistency between a theory $\mathcal{A}$ and observed data $\mathcal{C}$ to hold, one requires on one hand that revealed preferences-in-context of $\mathcal{C}$ do not contradict the theorems of $\mathcal{A}$, and on the other hand, that the predictions of $\mathcal{A}$, which are not necessarily observable, do not contradict the limited observations given by the structure C. If this holds, theory and observables are called semantically consistent.

Definition 4.1 (Semantic Consistency)**.** Let $\mathcal{A} = (X, \mathbb{A})$ be an axiomatic theory of choice with theorems $\mathcal{P}_{\mathcal{A}}$ and predictions $\hat{C}_{\mathcal{A}} = (X, \hat{c}_{\mathcal{A}})$, and let $\mathcal{C} = (X, c)$ be an observed choice structure. $\mathcal{A}$ and $\mathcal{C}$ are *semantically consistent* if

(i) observed revealed preferences-in-context coincide with the theory's preferences-in-context: for all $\Gamma \subseteq X$, if $\Gamma \vdash x \succsim_{\mathcal{C}} y \in P_{\mathcal{C}}$, then $\Gamma \vdash x \succsim_{\mathcal{A}} y \in P_{\mathcal{A}}$; and
(ii) actual observed choice coincides with predicted choice: for all $\Gamma \in X_{\Gamma}$, if $x \in c(\Gamma)$, then $x \in \hat{c}_{\mathcal{A}}(\Gamma)$.

Semantic consistency between $\mathcal{A}$ and $\mathcal{C}$ is denoted by $\mathcal{C} \models \mathcal{A}$.

Conditions (i) and (ii) of Definition 4.1 are by construction equivalent, and hence one only needs to verify one of them in practice.

Note that Definition 4.1 does not impose internal consistency conditions on the axiomatic theory $\mathcal{A}$. Hence, semantic consistency between theory and observation may be analyzed even for theories that are internally inconsistent. The main requirement for theory and observation to be semantically consistent is that revealed preferences-in-context coincide with the theory's theorems even if some of these preferences-in-context are contradictory. If a theory $\mathcal{A}$ is internally consistent, then any structure $\mathcal{C}$ that is semantically consistent with $\mathcal{A}$ cannot produce contradictory revealed preferences-in-context.[8]

4.2. **The duality mappings.** As formally described above, the approach to theorizing about empirical and experimental choice phenomena is twofold. From a methodological viewpoint, it has been classified as being either deductive or inductive. The traditional deductive method starts with a general theory of choice whose axioms are assumed to hold and accepted as true. Empirical and experimental data is then used to test the theory by comparing the evidence to the theory's predictions, evaluating the consistency between the two and recording how successfully the theory has been able to withstand or succumb to testing. The second, inductive, approach proceeds from observed regularities in the real world and attempts to distill semantically consistent theories from the observations to help organize and explain the findings. These two approaches are dual in the sense that they are reverse, and not necessarily inverse, processes between two universes. With this duality in mind, two natural mappings representing the processes of moving between axiomatic theories and observed choice structures arise. Fix X to be the set of alternatives on top of which theories in **Ax** and choice structures in **Obs** are built. In one direction, define the function

$$\sigma : \mathbf{Ax} \to \mathbf{Obs}$$

by

$$\sigma(\mathcal{A}) := \{\mathcal{C} \mid \mathcal{C} \models \mathcal{A}\} \ .$$

This is referred to as the *semantics mapping*. It assigns to a theory $\mathcal{A}$ all possible choice structures that are semantically consistent with $\mathcal{A}$. In the other direction, define the function

$$\tau : \mathbf{Obs} \to \mathbf{Ax}$$

by mapping a collection of choice structure $\mathcal{C}^*$ to the theory $\tau(\mathcal{C}^*)$[9] whose axioms are given by

$$\tau(\mathcal{C}^*) := \{P \mid \forall \mathcal{C} \in \mathcal{C}^*, \mathcal{C} \models P\} \ .$$

This is referred to as the *axiomatics* or *theory* mapping. For a collection of choice structures $\mathcal{C}^*$ representing some regularity, it finds the set of all axioms describing this regularity in the sense that they are semantically consistent with each choice structure in $\mathcal{C}^*$. Note that, by construction,

[8]The definition of semantic consistency along with the symbol $\models$ are inspired by the view taken in classical deductive logic and model theory, where lack of contradiction can be defined in either semantic or syntactic terms. The semantic definition states that a theory is consistent if and only if it has a model, i.e. there exists an interpretation under which all theorems in the theory are true. The syntactic definition requires that there is no theorem in the theory such that both the theorem and its negation can be derived from the theory, which corresponds to internal consistency in the choice theoretic setting.

[9]For notational simplicity, the set $\tau(\mathcal{C}^*)$ is overloaded to denote both the theory itself and its set of axioms.

axioms of the theory $\tau(\mathcal{C}^*)$ corresponding to a collection of choice structures $\mathcal{C}^*$ are essentially revealed preferences-in-context (under no assumptions, i.e. under an empty condition set) that are common to all elements of $\mathcal{C}^*$.

The two mappings σ and τ are referred to as the *duality* mappings. They are well-behaved in the sense that more restrictive theories are mapped to smaller collections of structures and, vice versa, larger collections of structures are mapped to less restrictive axiomatic theories. Technically, this means that the semantics and axiomatics mappings are order reversing: starting with a theory $\mathcal{A}$ with its fixed set of axioms, $\sigma(\mathcal{A})$ contains all the structures satisfying the axioms of $\mathcal{A}$. Expanding the axioms of $\mathcal{A}$ to include new ones should yield a smaller number of agents or data consistent with $\mathcal{A}$, and hence one should expect a decrease in the number of choice structures that satisfy the expansion of $\mathcal{A}$. In the other direction, for a given set of choice structures $\mathcal{C}^*$, a subset of $\mathcal{C}^*$ can be thought of as economic agents which have all the properties of those in $\mathcal{C}^*$ plus some additional more restrictive attributes. Therefore, the collection of axioms $\tau(\mathcal{C}^*)$ satisfied by all choice structures in $\mathcal{C}^*$ is contained in the set of axioms satisfied by all the structures in its subset only.

Proposition 4.2. *The axiomatics and semantics mappings are monotonically order-reversing:*

(i) $\mathcal{A} \subseteq \mathcal{B} \implies \sigma(\mathcal{B}) \subseteq \sigma(\mathcal{A})$*; and*

(ii) $\mathcal{C}^* \subseteq \mathcal{D}^* \implies \tau(\mathcal{D}^*) \subseteq \tau(\mathcal{C}^*)$*.*

Proof. (i) Let $\mathcal{C} \in \sigma(\mathcal{B})$. This means that $\mathcal{C} \models P'$ for all preferences-in-context $P' \in \mathcal{B}$. Because $\mathcal{A} \subseteq \mathcal{B}$ by assumption, this in particular implies that $\mathcal{C} \models P$ for all preferences-in-context $P \in \mathcal{A}$. Because by definition of the semantics mapping, $\sigma(\mathcal{A}) = \{\mathcal{C} \mid \mathcal{C} \models \mathcal{A}\} = \{\mathcal{C} \mid \forall P \in \mathcal{A}, \mathcal{C} \models P\}$, we have $\mathcal{C} \in \sigma(\mathcal{A})$. (ii) Let $P \in \tau(\mathcal{D}^*)$. This means that $\mathcal{D} \models P$ for all choice structures $\mathcal{D} \in \mathcal{D}^*$. Because $\mathcal{C}^* \subseteq \mathcal{D}^*$ by assumption, this in particular means that $\mathcal{C} \models P$ for all $\mathcal{C} \in \mathcal{C}^*$. Hence, $P \in \tau(\mathcal{C}^*)$. $\square$

In general, the duality mappings are not inverses of each other. However, the functions σ and τ do interrelate via a duality property, an instance of a general abstract mathematical relationship knows as the *Galois connection*, which are ubiquitous in a number of research areas, ranging from the most theoretical to the most applied.[10] They essentially provide a passage between the empirical and the theoretical universes of economic theory.

To characterize the duality mappings, recall that the theorizing process of a decision theorist can be roughly classified as being deductive or inductive, primarily depending on the starting universe of the analysis, that is whether one begins with an axiomatic theory or with observed regularities. In one direction, suppose one has an axiomatic theory $\mathcal{A}$ and finds a collection of choice structures $\sigma(\mathcal{A})$ consistent with the theory. Now, one can analyze a subset $\mathcal{C}^*$ of $\sigma(\mathcal{A})$ by generating the axiomatic theory $\tau(\mathcal{C}^*)$ whose axioms are semantically consistent with $\mathcal{C}^*$. Then, for semantic

[10]The interested reader is referred to Denecke, Erne, and Wismath (2004), who give an introduction to the history of Galois connections in mathematics and review some applications. Lawvere (1969) uses Galois connections to study a similar relationship between sets of axioms and classes of models in the mathematical foundations of algebraic theories and categorical logic.

consistency to be preserved in this process, the axioms of the original theory $\mathcal{A}$ must be included in the more restricted axiomatization $\tau(\mathcal{C}^*)$. More generally, and to link to the process of theorizing (using $\mathcal{A}$), generating observables (given by $\sigma(\mathcal{A})$), and then theorizing on the observables $\tau(\sigma(\mathcal{A}))$, duality requires that the original axioms of $\mathcal{A}$ must be generated in the theory $\tau(\sigma(\mathcal{A}))$, which may or may not contain more axioms, for semantic consistency to be preserved. In the other direction, suppose one begins with some observed regularities $\mathcal{C}^*$ in the real world and distills an axiomatic theory $\tau(\mathcal{C}^*)$. For semantic consistency to hold, the structures consistent with any necessarily less restrictive subtheory $\mathcal{A}$ of $\tau(\mathcal{C}^*)$ must at least include the original collection $\mathcal{C}^*$. The following formalizes and proves this relationship between the theoretical and the real universes of choice, and the dual processes of moving from one to the other.

Theorem 4.3 (Duality of Axiomatics and Observed Structures)**.** *For all collections of choice structures $\mathcal{C}^* \in \mathbf{Obs}$ and all axiomatizations $\mathcal{A} \in \mathbf{Ax}$, $\mathcal{C}^* \subseteq \sigma(\mathcal{A})$ if and only if $\mathcal{A} \subseteq \tau(\mathcal{C}^*)$.*

Proof. Suppose $\mathcal{C}^* \subseteq \sigma(\mathcal{A})$ and let $P' \in \mathcal{A}$. To show that $P' \in \tau(\mathcal{C}^*)$, recall that $\tau(\mathcal{C}^*) = \{P \mid \forall \mathcal{C} \in \mathcal{C}^*, \mathcal{C} \models P\}$. Since $\mathcal{C}^* \subseteq \sigma(\mathcal{A})$ by assumption, this means that theorems P in $\tau(\mathcal{C}^*)$ also satisfy $\mathcal{C} \models P$ where $\mathcal{C} \in \sigma(\mathcal{A})$. By definition of $\sigma(\mathcal{A})$, its elements $\mathcal{C}$ satisfy $\mathcal{C} \models \mathcal{A}$, or, equivalently, $\mathcal{C} \models P$ for all $P \in \mathcal{A}$. Since $P' \in \mathcal{A}$, we get that $\mathcal{C} \models P'$, and hence $P' \in \tau(\mathcal{C}^*)$. The proof of the reverse direction follows similarly: suppose $\mathcal{A} \subseteq \tau(\mathcal{C}^*)$ and let $\mathcal{C}' \in \mathcal{C}^*$. To show that $\mathcal{C}' \in \sigma(\mathcal{A})$, recall that $\sigma(\mathcal{A}) = \{\mathcal{C} \mid \mathcal{C} \models \mathcal{A}\} = \{\mathcal{C} \mid \mathcal{C} \models P, \forall P \in \mathcal{A}\}$. Since $\mathcal{A} \subseteq \tau(\mathcal{C}^*)$ by assumption, structures $\mathcal{C}$ of $\sigma(\mathcal{A})$ satisfy $\mathcal{C} \models P$, where $P \in \tau(\mathcal{C}^*)$, and those theorems P in return satisfy $\mathcal{C}' \models P$ for all $\mathcal{C}' \in \mathcal{C}^*$. Hence, $\mathcal{C}' \in \sigma(\mathcal{A})$. □

5. Evaluating consistency

In this section, the process of evaluating semantic consistency is examined more closely in terms of the duality mappings and the relationship between theory and reality of choice is revisited. In particular, the duality mappings are used to show that (i) an observed choice is semantically consistent with a given theory if and only if it is a consequence of the theory's axioms; (ii) a consistent axiomatization, even though based on observed data, implicitly also axiomatizes what is unobserved; (iii) the universes of theory and reality are not equivalent; (iv) one strand of rationality capturing logical quality of reasoning can be defined in terms of internal consistency.

5.1. **Consistent consequences.** Suppose an economic theorist formulates an axiomatic theory of choice $\mathcal{A}$ and observes a particular choice made in practice. The question now is whether this choice made is consistent with his theory. Based on the definition of semantic consistency, he can compare the corresponding revealed preference-in-context P with the theory. For semantic consistency to hold, P should coincide with one of the theorems of $\mathcal{A}$ and thus essentially be a consequence of the axioms of $\mathcal{A}$. The concept of consequence or implication is hence fundamental within the discussion of consistency of choice. On the axiomatic level, a preference-in-context is a consistent consequence if it follows from the premises of the theory. This definition refers to proof theoretic derivation, in the sense that there is a formal proof of the consequence from its premises. On the semantic level, a preference-in-context can be thought of as a valid consequence for an economic agent if it does not contradict his observed behavior.

Given an axiomatic theory $\mathcal{A} \in \mathbf{Ax}$, how does one generate all preferences-in-context which are consistent consequences of $\mathcal{A}$? Towards this end, recall that $\sigma(\mathcal{A})$ by definition contains all structures $\mathcal{C}$ which are semantically consistent with $\mathcal{A}$ and hence satisfy $\mathcal{C} \models \mathcal{A}$. Also by definition, $\tau(\sigma(\mathcal{A}))$ is the theory whose axioms are all preferences-in-context P that are satisfied by all structures $\mathcal{C} \in \sigma(\mathcal{A})$. In other words, $P \in \tau(\sigma(\mathcal{A}))$ if and only if any structure $\mathcal{C}$ which satisfies $\mathcal{A}$ also satisfies P. This precisely means that P is a consistent consequence of an axiomatic theory $\mathcal{A} \in \mathbf{Ax}$ if P is an axiom of the theory $\tau(\sigma(\mathcal{A}))$. Therefore, the axiomatization $\tau(\sigma(\mathcal{A})) \in \mathbf{Ax}$ is just the set of all consistent consequences of $\mathcal{A}$.

This argument uses the composite mapping $\tau\sigma := \tau \circ \sigma : \mathbf{Ax} \to \mathbf{Ax}$, which maps an axiomatic theory $\mathcal{A}$ to the theory $\tau\sigma(\mathcal{A})$ whose axioms are all logical consequences of $\mathcal{A}$. It is a *closure* or *consequence operator*[11] satisfying the following properties for all $\mathcal{A}, \mathcal{A}' \in \mathbf{Ax}$:

(i) *Extensivity*: $\mathcal{A} \subseteq \tau\sigma(\mathcal{A})$, since all axioms of a theory $\mathcal{A}$ must be contained in the set of all consistent consequences of $\mathcal{A}$;
(ii) *Monotonicity*: $\mathcal{A} \subseteq \mathcal{A}' \implies \tau\sigma(\mathcal{A}) \subseteq \tau\sigma(\mathcal{A}')$, since all consequences of $\mathcal{A}$ are by definition also consequences of any larger theory containing $\mathcal{A}$;
(iii) *Idempotence*: $\tau\sigma(\tau\sigma(\mathcal{A})) = \tau\sigma(\mathcal{A})$, since any consistent consequence of $\tau\sigma(\mathcal{A})$ is by definition already included in $\tau\sigma(\mathcal{A})$.

These properties essentially imply that the theory $\tau\sigma(\mathcal{A})$ is closed under consistent consequence, as all of its theorems $\mathcal{P}_{\tau\sigma(\mathcal{A})}$ are just included in its axioms, and hence for all structures $\mathcal{C}$ with $\mathcal{C} \models \tau\sigma(\mathcal{A})$, if there is a preference-in-context P with $\mathcal{C} \models P$, then $P \in \tau\sigma(\mathcal{A})$. Moreover, not only does the closure operator $\tau\sigma$ generate a theory closed under consequence, but it generates the *smallest* such theory. To see this, suppose there is an axiomatic theory $\mathcal{A}'$ which is closed under consistent consequence and contains $\mathcal{A}$, and let $P \in \mathcal{P}_{\mathcal{A}}$, that is $P \in \tau\sigma(\mathcal{A})$. Since $\tau\sigma$ is monotone, then $\tau\sigma(\mathcal{A}) \subseteq \tau\sigma(\mathcal{A}')$. But by assumption, $\mathcal{A}'$ is already closed under consequence, and therefore $\tau\sigma(\mathcal{A}')$ is just $\mathcal{A}'$. Hence, $\tau\sigma(\mathcal{A}) \subseteq \mathcal{A}'$.

The gist of this discussion can be summarized as follows: to evaluate whether an observed choice made is consistent with some given axiomatic theory $\mathcal{A}$, one can generate, say computationally, the theory $\tau\sigma(\mathcal{A})$ which is guaranteed to contain all consequences of $\mathcal{A}$. If the revealed observed preference-in-context P corresponding to the observed choice is an element of $\tau\sigma(\mathcal{A})$, then the choice made is consistent with $\mathcal{A}$.

5.2. **Consistent axiomatizability of observed and unobserved choices.** Observed economic phenomena and data are limited by their nature. Hence, starting with some observed regularity, evaluating its consistency with a given theory or distilling an axiomatization which completely describes the empirical regularities presents some challenges. Consider, for example, that when an agent makes a sequence of choices, this finite collection of choices is solely a representation of the agent's true preferences, some or perhaps even most of which remain unobserved. Unobserved

[11]The concept of a closure operator in mathematical analysis is due Moore (1910). See also Tarski (1956).

choices may exist either because they have not been tested for or because they are inherently unobservable or untestable. When the economic theorist "axiomatizes" the agent's observed choices, in practice this axiomatization represents only the limited observables rather than the agent's true, mostly unobserved, preferences. An implicit assumption in the analysis of consistency and axiomatizability is that the axiomatic theory's predictions represent the unobserved extension of the given observed data. In other words, the unobserved are also captured by the axiomatic theory. In particular, one assumes that on one hand, none of the theory's predictions contradict the unobserved data extension, and on the other hand that all of the unobserved choices are included in the theory's predictions.

Denote by $\mathcal{C}_{\text{ext}}$ the unobserved extension of a given choice structure $\mathcal{C} = (X, c)$. Formally, it is a choice structure $\mathcal{C}_{\text{ext}} = (X, c_{\text{ext}})$ satisfying the only requirement that c is a subfunction of c_{ext}, which means that, for $\Gamma \subseteq X$, whenever $c(\Gamma)$ is defined, then $c_{\text{ext}}(\Gamma)$ is defined and $c(\Gamma) \subseteq c_{\text{ext}}(\Gamma)$.

Definition 5.1 (Consistent axiomatizability)**.** A collection $\mathcal{C}^* \in \mathbf{Obs}$ of choice structures is *consistently axiomatizable* if there exists an internally consistent axiomatic choice theory $\mathcal{A} \in \mathbf{Ax}$ such that $\mathcal{C} \in \mathcal{C}^*$ implies $\mathcal{C}_{\text{ext}} \models P$ for every $P \in \mathcal{P}_{\mathcal{A}}$.

An economic agent's limited collection of observed choices $\mathcal{C}$ can thereby be axiomatized by an internally consistent theory $\mathcal{A}$ if not only $\mathcal{C}$ is semantically consistent with $\mathcal{A}$, but also if all theorems of $\mathcal{A}$ are satisfied in $\mathcal{C}_{\text{ext}}$. Informally, this means that the agent is "buying into" all of the axiomatic theory's predictions.

Now suppose some regularities $\mathcal{C}^*$ are observed. The elements of $\mathcal{C}^*$ are by nature limited collections of choice data. Consistently axiomatizing the regularities $\mathcal{C}^*$ in practice, one implicitly assumes that the corresponding extensions in $\mathcal{C}^*_{\text{ext}}$ are also consistent with the axiomatic theory. This larger class of structures, necessarily containing $\mathcal{C}^*$ and including all the unobserved data that in some sense complete $\mathcal{C}^*$, is given by $\sigma(\tau(\mathcal{C}^*))$ (see Theorem 5.2). The significance of this larger collection of choice structures $\sigma(\tau(\mathcal{C}^*))$ lies in the fact that it is the smallest class of structures extending $\mathcal{C}^*$ in a way that all theorems of the theory $\tau(\mathcal{C}^*)$ are satisfied. This collection of choice structures $\sigma\tau(\mathcal{C}^*)$ is defined via the composite $\sigma \circ \tau$ of the duality mappings, which generates the smallest unobserved empirical extension of observed structures in $\mathcal{C}^*$ that can be consistently axiomatized. This unobserved information, which is represented by the collection $\sigma\tau(\mathcal{C}^*)\backslash\mathcal{C}^*$, contains the minimal amount of choices that must be added to $\mathcal{C}^*$ to make it consistently axiomatizable.

Theorem 5.2. *$\sigma\tau(\mathcal{C}^*)$ is an extension of $\mathcal{C}^*$ that is consistently axiomatizable by $\tau(\mathcal{C}^*)$. Moreover, if there exists another extension $\mathcal{D}^*$ with $\mathcal{C}^* \subseteq \mathcal{D}^*$, then $\sigma\tau(\mathcal{C}^*) \subseteq \mathcal{D}^*$.*

Proof. For any structure $\mathcal{C} \in \sigma\tau(\mathcal{C}^*)$, $\mathcal{C} \models \tau(\mathcal{C}^*)$, and in the other direction $\mathcal{C} \models \tau(\mathcal{C}^*)$ by definition implies $\mathcal{C} \in \sigma\tau(\mathcal{C}^*)$. Therefore, $\sigma\tau(\mathcal{C}^*)$ is consistently axiomatizable by $\tau(\mathcal{C}^*)$. Next suppose that $\mathcal{D}^* \in \mathbf{Obs}$ is an axiomatizable collection of choice structures containing $\mathcal{C}^*$. Since $\mathcal{C}^* \subseteq \mathcal{D}^*$ by assumption, and τ is monotonically order-reversing, we have $\tau(\mathcal{D}^*) \subseteq \tau(\mathcal{C}^*)$. Similarly

$\sigma\tau(\mathcal{C}^*) \subseteq \sigma\tau(\mathcal{D}^*)$, since σ is monotonically order-reversing. But then, $\sigma\tau(\mathcal{D}^*) \subseteq \mathcal{D}^*$, because $\mathcal{D}^*$ is axiomatizable. Therefore, $\sigma\tau(\mathcal{C}^*) \subseteq \mathcal{D}^*$. □

5.3. **Revisiting duality: from theory to observation and back.** The axiomatic theory for reasoning about choice and actual observed choices made are not equivalent structures. This corresponds to the epistemological viewpoint that the theory and the reality of choice do not exactly coincide. Consider again the scientific methodology common to economic theorists: starting from a hypothetical theory, one can produce predictions, which motivates the specification of, say, an experiment to test the theory's predictions; these experiments result in observations. If one were to axiomatize the resulting observations, the original theory and the resulting theory are not the same. On the other hand, a comparison of observations against predictions confirms the nonequivalence of the two. These processes represent the essence of the duality of axiomatics and semantics of choice. It implies that one would not necessarily end up with the same theory when moving from theory to reality and back. Nevertheless, the duality mappings provide a useful tool for studying properties of a real object (observable choice) based on the properties of another, more theoretical but by construction more known, kind of object (an axiomatic choice theory).

Even though the duality mappings σ and τ are not necessarily inverses of one another, their restrictions to their respective ranges can be thought of as inverses. Indeed, consistently axiomatizable collections of structures, given by $\sigma\tau(\mathbf{Obs})$, correspond bijectively to axiomatizations that are closed under consistent consequence, given by $\tau\sigma(\mathbf{Ax})$.

Before formally verifying this, consider the correspondence between the maximal and minimal elements of $\sigma\tau(\mathbf{Obs})$ and $\tau\sigma(\mathbf{Ax})$. Recall that $\tau\sigma(\mathbf{Ax})$ contains all axiomatizations which are closed under consistent consequence and is partially ordered, just like $\mathbf{Ax}$, by subset inclusion $\subseteq$. The minimum of such closed axiomatizations is simply the empty theory with no axioms, and the maximum is the necessarily *inconsistent* closed axiomatic theory containing all possible axioms. Just under the maximum inconsistent theory will be those axiomatic theories $\mathcal{A}$ such that adding even one more new axiom to $\mathcal{A}$ which is not already in the set takes us back to the maximal, inconsistent theory. Such a theory $\mathcal{A}$ is necessarily consistent and hence does not contain a contradictory choice in its theorems. Moving further down in $\tau\sigma(\mathcal{A})$ takes us through more general necessarily consistent theories containing less and less axioms and theorems, until we reach the empty axiomatic theory.

On the other hand, the collection $\sigma\tau(\mathbf{Obs})$ contains only those sets of structures that are consistently axiomatizable and is also partially ordered. Its minimal element, the empty set, corresponds to the maximal element of $\tau\sigma(\mathbf{Ax})$, which is the inconsistent theory containing all possible axioms. This theory cannot be satisfied using any structures in $\sigma\tau(\mathbf{Obs})$, and hence it can only be mapped to the empty structure. In simpler words: no individual whose behavior is consistent can satisfy an inconsistent theory. On the other end, the maximal element of $\sigma\tau(\mathbf{Obs})$ contains all possible choice structures that are consistently axiomatizable. It is the most general collection of axiomatizable structures, and hence the theory that corresponds to it is the least restrictive one, that is the theory with no axioms.

The following Theorem states this equivalence formally. Its proof is a consequence of the results derived in this and the previous section. More precisely, it follows from the fact that τ and σ are both monotonically order-reversing and that the composite mapping $\tau\sigma$ is a closure operator.

Theorem 5.3. *The restrictions of the semantics and axiomatics mappings, σ and τ, are (reverse-order) isomorphisms*[12] *between the partially ordered collection of axiomatizations $\tau\sigma(\mathbf{Ax})$ closed under consistent consequence and the set of consistently axiomatizable sets of choice structures $\sigma\tau(\mathbf{Obs})$.*

5.4. **Evaluating rationality in terms of consistency.** What is meant by rationality in decision making has been extensively debated in psychology and behavioral economics. Informally, it implies the conformity of one's beliefs with one's reasons to believe, or of one's actions with one's reasons for action. The notion of consistency, both loosely speaking in the traditional sense and more formally as defined in this paper, is closely related to the concept of rationality. As Gilboa (2015) points out, "*the rise of neoclassical mathematical economics in the early 20th century, influenced by logical positivism, could be viewed as taking a step back, and reducing the concept of rationality to consistency.*" Traditionally, preferences are *rational* if they are complete and transitive binary relations. If behavior is generated by such rational preferences, then the choice structure representing this behavior satisfies the consistency requirements embodied in the weak axiom of revealed preference. In the other direction, that is starting from choice behavior, one traditionally says that a choice structure is *rationalizable* if there is a rational (transitive and complete) preference relation which generates that choice structure. In other words, a rational preference rationalizes a choice rule if the optimal choices generated by the preference relation coincide with those of the choice rule.

The exact definition of rationality, however, takes many forms. One strand of rationality that may be evaluated is that of internal consistency. More precisely, consistent axiomatizability of a decision maker's choices may fail, and if his corresponding choice structure or its extension contains a contradictory choice, then an axiomatization, if at all feasible, must be internally inconsistent. Consistent axiomatizability can thus be used to provide a formalization of a particular view of rationality linked to *quality of reasoning*.

There are several alternative theories of the cognitive processes that a decision maker's reasoning is based on. One view is that people rely on a mental logic consisting of formal inference rules similar to those developed by logicians.[13] This form of reasoning, called deductive reasoning, is closely related to the subject matter of this paper. The most classical example of valid deductive reasoning is the *modus ponens* rule: if all men are mortal, and Socrates is a man, then Socrates is mortal. In deductive reasoning, a person thus starts with a known claim or a general belief and from there asks what follows from these foundations or how do these premises relate to others. In other words, the decision maker starts with a hypothesis and examines the possibilities to reach a

[12] A mapping $F : A \to B$ between two partially ordered sets is a *reverse-order isomorphism* if it is bijective and monotonically order-reversing.

[13] Other views contend that people rely on on domain-specific or content-sensitive rules of inference or on mental representations that correspond to imagined possibilities.

conclusion, and deductive reasoning helps him understand why his choices are wrong and indicates whether prior knowledge or beliefs are off track. Deductive reasoning hence concerns the consistent consequences of assumptions or axioms.

That said, reasoning in the context of choice is far from limited to deductive reasoning alone; and human decision making, behavioral logic, habits, beliefs and reasoning do not reduce to formal deductive logic.[14] Similarly, rationality is not confined to internally consistent deductive reasoning, and the formalization proposed next is not an attempt to promote that coherence of a decision maker's set of preferences is the sole criterion for evaluating his consistency.

However, formalizing the notion of internal logical consistency of a choice theory helps evaluate whether actual decisions are logically justified and consistent. If they are not, there may be non-logical behavioral notions or perhaps ethical beliefs driving those decisions. The view taken here follows that of Anand (1993), who argues that there are several strands to rationality in economic decision making: *"firstly, there is the concept of instrumentality, basically the idea that people and organisations are instrumentally rational, that is adopt the best actions to achieve their goals. Secondly, there is an axiomatic concept that rationality is a matter of being logically consistent within your preferences and beliefs. Thirdly, people have focused on accuracy of beliefs and full use of information, in this view a person who is not rational has beliefs that don't fully use the information they have.* And thus, the following formalism of consistency in economic reasoning and rationality is meant to capture the second axiomatic strand of rationality in decision making.

As a starting point, consider the definition of Gilboa (2015) (see also Gilboa, Macheroni, Marinacci, and Schmeidler (2010)), which extends traditional ideas of rationality beyond observed behavior by introducing a sense of regret: "*a mode of behavior is irrational for a decision maker, if, when the latter is exposed to the analysis of her choices, she would have liked to change her decision, or to make different choices in similar future circumstance.*" Such an *analysis of choices* can be provided using the notions of consistency and axiomatizability. This leads to the following proposal for an instance of rationality, that I shall call *axiomatic rationality* following the classification of Anand (1993), in terms of quality of logically consistent deductive reasoning, hence distinguishing it clearly from that of consistency:

Definition 5.4 (Internally consistent behavior and axiomatic rationality)**.** An economic agent exhibits *internally consistent behavior* if his observed behavior (and its empirical extension) can be consistently axiomatized. A person is *axiomatically rational* if, when exposed to a formal analysis proving that his choices are not consistently axiomatizable, he changes the decision in order to attain consistent axiomatizability.

[14]For example, intuition often plays an important role. An intuitive approach is not primarily based on an understanding of the assumptions and logical deduction, but on its perceived similarity to other situations, for which an appropriate behavior in known, which can be transferred to the problem.

6. Implications and related work

The primary purpose of this paper is to disentangle the notion of consistency, an inherently logical concept that is fundamentally void of connotation, from traditional rationality or consistency conditions imposed on choice models. The view taken throughout is that consistency of an axiomatic theory of choice reduces to an analysis of the choice system as a whole and its relationship with observed data as opposed to an analysis of properties of preference relations or choice functions. In studies of choice models, an evaluation of consistency takes one of two forms: internal consistency, which refers to the property that a given theoretical choice system is in itself consistent in the sense that it does not generate contradictory statements; and semantic consistency, which refers to the idea that a given choice theory's preferences are valid with respect to some observed choice data. The evaluation of semantic consistency is formalized in terms of duality mappings, which are transformations between the theoretical universe of axiomatic choice theories and the observed universe of concrete choice structures. The duality mappings are used to gain deeper insights into the process of evaluating the consistency of an axiomatic choice model with respect to some observed phenomena.

There are several reasons making this essentially logical definition of consistency important for the economic theorist in general and the choice theoretic community in particular. First, this paper aims to unify the discussion about internal consistency and promotes the idea that decision makers' preferences may well be internally consistent without having to abide by various axiomatic rules traditionally imposed on choice models. Second, there is an ever ongoing discussion about the purpose of economic models and the nature of the relationship between models and the reality to which they apply. The duality between the axiomatics and semantics of choice is a formalization of the idea that no theoretical choice model is able to capture the entirety of reality, and hence any theory will always be an incomplete model of real data and behavior. One may use this duality setup to prove various other "incompleteness theorems" in economic contexts extending beyond those of decision theory. Third, disentangling classical rationality, which assumes transitivity of choice, from consistent rationality opens the door to an experimental exploration of decision makers' consistent rationality and its relationship various decision making heuristics. If decision makers are not logically rational, then there is little point in attempting to use standard economic models, which are assumed to be internally consistent, to describe their behavior. An understanding of various strands of rational choice behavior, including quality of reasoning, may hence ultimately tell us something about choices we make.

I conclude by discussing how classical revealed preference theory and other formalizations of the relationship between the theory and reality of choice relate to this framework.

6.1. **Revealed preference theory.** This work clearly takes inspiration from revealed preference theory. In this section, classical questions of consistency and rationality of revealed preference theory are related to the paper's setting of semantic consistency between axiomatic and observed choice. In particular, the theory representing rational preferences is just to be thought of as an instance of a specific axiomatic theory of choice, whose semantic consistency with choice structures can be analyzed in the setting of this paper.

Revealed preference theory, as pioneered by Samuelson (1938), is based on the idea that preferences of economic agents are revealed in their choice behavior. Traditionally, it refers to what "consistent" or "rational" preferences should be and what their corresponding choice structure needs to satisfy. The theory is developed by imposing *rationality axioms* on the decision maker's preferences and then analyzing the consequences of these preferences in terms of observed choice behavior. A central assumption in this approach is the weak axiom of revealed preference, which is an imposition on choice behavior paralleling the rationality assumptions of preferences. Revealed preference theory can hence be viewed as an instance of a particular axiomatic theory with some imposed axiomatic structure. The following is a formulation of the traditional revealed preference approach in the context of the present paper's focus on semantic consistency between axiomatics and observed structures.

For a set of alternatives X, define the axiomatic theory $\mathcal{R} = (X, \mathbb{A}_{\mathcal{R}})$ representing what is traditionally known as *rational preferences*. These rational preferences are context-independent and hence satisfy the following axiom:

$$\textit{Context-independence:} \quad \frac{\Gamma \vdash x \succsim y}{X \vdash x \succsim y} \quad \text{and} \quad \frac{X \vdash x \succsim y}{\Gamma \vdash x \succsim y} \quad \text{for all } \Gamma \subseteq X.$$

Moreover, preferences satisfy the following two axioms representing (context-independent) completeness and transitivity:

$$\textit{Completeness:} \quad \frac{x, y \in X}{X \vdash x \succsim y \vee X \vdash y \succsim x}\ .$$

$$\textit{Transitivity:} \quad \frac{X \vdash x \succsim y \wedge X \vdash y \succsim z}{X \vdash x \succsim z}\ .$$

The set $\mathbb{A}_{\mathcal{R}}$ of axioms of the theory $\mathcal{R}$ hence contains context-independence, completeness and transitivity. Revealed preference theory asks what choice structures corresponding to this rational theory $\mathcal{R}$ look like, and in the other direction, which behavior can be *rationalized* by $\mathcal{R}$. The relationship between rational theories and choice structures can be framed in terms of semantic consistency, as follows.

Behavior that is semantically consistent with $\mathcal{R}$ is an element of the set of structures $\sigma(\mathcal{R}) = \{\mathcal{C} \mid \mathcal{C} \models \mathcal{R}\}$. Rational preference theory tells us that any $\mathcal{C} \in \sigma(\mathcal{R})$ must satisfy the weak axiom of revealed preference, which states that if given any context where x is chosen over y, then there can be no other context containing both alternatives x and y for which y is chosen and x is not. This in effect imposes the "internal consistency" condition criticised by Sen (1993). In the other direction, starting with some observed choice behavior $\mathcal{C} = (X, c)$, the traditional notion of *rationalizability* is related to those of semantic consistency and consistent axiomatizability, but with some subtle yet crucial differences. The structure $\mathcal{C}$ is rationalized by a preference relation $\succsim$ if the choice structure $\mathcal{C}_{\succsim} = (X, c_{\succsim})$ generated by $\succsim$, defined by

$$c_{\succsim}(\Gamma) = \{x \in \Gamma \mid \Gamma \vdash x \succsim y \text{ for every } y \in \Gamma\} \ ,$$

exactly coincides with the original choice structure $\mathcal{C}$, meaning that $c(\Gamma) = c_{\succsim}(\Gamma)$ for all $\Gamma \in X_{\Gamma}$. This choice structure $\mathcal{C}_{\succsim}$ generated by the rational preference relation $\succsim$ encapsulates the predictions of the rational theory $\mathcal{R}$; that is $\hat{C}_{\mathcal{R}} = \mathcal{C}_{\succsim}$. Semantic consistency is hence a weaker notion than rationalizability, as it only requires $c(\Gamma) \subseteq c_{\succsim}(\Gamma)$ rather than exact equality of choice sets. Whereas rationalizability allows us to interpret the decision maker's choices as if he were a preference maximizer, semantic consistency only tells us that observed behavior is consistent with and hence does not contradict the rational theory $\mathcal{R}$. More significantly, a structure $\mathcal{C}$ that is rationalizable not only implies that its observed choices are consistently axiomatizable, but also that the structure $\mathcal{C}$ already contains its hypothetical unobserved extension $\mathcal{C}_{\text{ext}}$. In other words, the equality $c(\Gamma) = c_{\succsim}(\Gamma) = \hat{c}_{\mathcal{R}}$ tells us that observed choices in $\mathcal{C}$ contain choice instances of all possible derivable theorems of $\mathcal{R}$. This imposes a form of equivalence between theory and observed data rather than duality in terms of a semantic consistency.

6.2. **Theorizing on observed economic phenomena.** The relationship between theory and reality and the extent of the role of theoretical models in understanding economic phenomena is an ongoing debate and the subject of methodological writings in both the philosophy of science and economic methodology (see for example Gibbard and Varian (1978), Debreu (1991), Sugden (2000), and Rubinstein (2006)).

In the context of choice theories, this paper addresses the question of how axiomatic choice theories and observed choice behavior relate in terms of semantic consistency. As a result, theory and data are to be thought of as dual rather than equivalent. This is an example of a mathematical relationship known as the *Galois connection*. These are pairs of mappings between two universes, the abstract and the concrete, which can provide a useful tool for studying the concrete objects in terms of the abstract objects. Galois connections are ubiquitous in studies of model theory, which is concerned with the analysis of classes of mathematical structures from the perspective of mathematical logic. In this regard, this paper can be seen as an application of model theory in the choice theoretic setting, especially since model theory recognizes and is intimately concerned with a notion of duality. It examines semantical structures that have some concrete meaning in terms of syntactical elements consisting of language and axioms of a corresponding theoretical model. This relationship was studied abstractly in Lawvere (1969), who wrote of *"the familiar Galois connection between sets of axioms and classes of [structures] for a fixed [theory]."* From this abstract viewpoint, semantic consistency and the relationship between axiomatic theories and observed choice can also be regarded as an example of Lawvere's duality of syntax and semantics and may prove to be a useful tool for understanding other theory-reality relationships outside the choice-theoretic context.

In studying this duality in terms of semantic consistency, an implicit assumption made was that an axiomatic theory's predictions represent the unobserved extension of some observed data. The distinction between observed and unobserved data has been extensively studied in the philosophy of science, see for example Craig (1956) and Adams, Fagot, and Robinson (1970). The classical foundations on empirical data and falsifiability of theories in the philosophy of science was laid out by Popper (1959), who viewed the goal of science to prove hypotheses or to induce them from observational data and gave falsifiability an emphasis as a criterion of empirical statements in science. The notion of falsifiability is not directly addressed in this paper, however. Recently, Chambers, Echenique, and Shmaya (2014) study the notions of falsifiability and empirical content, independently of their specific meaning in particular economic theories. They define abstract theories as hypothetical "extensions" of observed data sets, in the sense that they consist of a class of preferences whose consistency with the data is to be tested. Consistency, however, is evaluated in terms of ratioanlizability and not in terms of the weaker concept of semantic consistency: if there is no preference that can rationalize the data in the revealed-preference-theory sense, then the data would falsify the theory.

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