# Asset Allocation, Longevity Risk, Annuitisation and Bequests

David Schiess

Group for Mathematics & Statistics,
University of St. Gallen,
Bodanstrasse 6, 9000 St. Gallen, Switzerland
david.schiess@unisg.ch
http://www.mathstat.unisg.ch

**Abstract.** Consumption and portfolio optimisation during retirement has not received as much attention in financial research as optimisation prior to retirement. However, retirement planning is becoming more and more relevant for several reasons. The present paper is mainly based on [15] and studies a pensioner deriving utility from a stream of consumption or an annuity and from bequeathing wealth to his heirs in a continuous-time framework. The task of finding the pensioner's optimal consumption, asset allocation and annuity decision rule leads to the interesting interplay of optimal control theory, optimal stopping theory and mortality issues or, technically speaking, to a combined optimal stopping and optimal control problem (COSOCP). [16] solved this problem in an all-or-nothing framework assuming exponential mortality and power utility functions. In this paper we present the extensions of [15] to the model of [16]: The essential inclusion of a bequest motive, the additional study of the economically interesting range of relative risk aversion levels greater than one and a new solution method for the COSOCP via duality arguments. For identical risk aversion levels [16] finds that the pensioner either annuitises immediately or never which means that COSOCP reduces to a trivial or to a pure optimal control problem. In contrast to this, the annuitisation decision rule can become wealth-dependent in our more general model and consequently, a real COSOCP has to be dealt with. The main result is that longevity risk matters very much (quite attractive annuity market) even if we allow for a bequest motive. **Keywords.** optimal control, optimal stopping, optimal annuitisation time, consumption, asset allocation, bequest motive. **M.S.C. classification.** 62L15, 93E20 **J.E.L. classification.** C61, D14

## 1 Introduction

Retirement planning is becoming more and more relevant because of rising conditional life expectancies, growing number of defined contribution pension plans

in many countries and continuing wealth concentration among pensioners. In addition to the afore-mentioned statistical reasons there is of course academic interest in a good understanding of the end of the life-cycle because this can be used as an input to economic models focusing on the labour phase.

A pensioner faces two main sources of uncertainty: Longevity risk and investment risk. We consequently introduce a financial and an annuity market. On the one hand, the pensioner has to find the optimal time to annuitise his wealth (optimal stopping part). On the other hand, he should consume and invest optimally in the pre-annuitisation phase (optimal control part). Furthermore, these two problems are tightly connected through wealth. Thus, we have to solve a combined optimal stopping and optimal control problem (COSOCP).

The pioneering work of [7] led to a vast literature on continuous-time models of the optimal consumption and investment problem. However, most of it assumes either a fixed or an infinite planning horizon and/or does not include an annuity market. Clearly, neither of the approaches is adequate for a pensioner whose remaining lifetime should obviously be treated as a random variable. [17] was the first to include uncertain lifetimes. He showed that a utility maximising individual with mortality risk but no bequest motive should annuitise his entire wealth given that annuities are fair. The work of [17] was followed by [6], [8] and [13]. [8] obtained the result that an individual with exponential mortality behaves as if he were going to live forever with an adjusted discounting parameter. [13] extended the framework in [7] by additionally including freely reversible annuities. However, it is crucial to model the annuitisation decision as irreversible because of adverse selection issues. There are only a few authors providing normative continuous-time models for the retirement phase until now which include both the afore-mentioned uncertainty of remaining lifetime and the irreversibility of annuitisation. Concerning probability minimisation of retirement ruin ths branch of the literature started with [11] and [9]. Concerning utility maximisation this branch of the literature essentially started with [3] and [10]. The main shortcoming of the latter paper is that the pensioner has to adhere to a predetermined annuitisation time no matter what happens after the time of optimisation. [16] has overcome this unrealistic feature by modelling the annuitisation decision as a stopping time controlled by the agent. He provides an appropriate continuous-time model to find the pensioner's optimal consumption, asset allocation and annuitisation decision rule assuming the exponential mortality law and power utility functions. In this paper we present the extensions of [15] to the model of [16]: The essential inclusion of a bequest motive[1], the additional study of the economically interesting range of relative risk aversions greater than one, the introduction of prior life insurance and a corresponding subsistence level of bequests and lastly, a new solution method for the COSOCP via duality arguments.

We assume power utility functions with identical relative risk aversion or subsistence level utility functions for the utility from a stream of consumption or an

[1] People leave a substantial part of their wealth at death. Moreover, annuitisation is in conflict with a potential bequest motive.

annuity and from bequeathing wealth. The studied individual can potentially be younger than some retirement age. However, he must not receive any stochastic income.[2] Moreover, annuitisation is modelled as an all-or-nothing framework: The pensioner annuitises his entire remaining wealth and subsequently consumes his entire annuity.[3] Lastly, we will only consider one riskless asset and one risky asset.[4]

The paper is organised as follows. Section 2 presents the model and formalises the pensioner's COSOCP with general strictly increasing and strictly concave utility functions that satisfy the Inada conditions. We adjust the verification theorem of [16] to our afore-mentioned extensions of his model. This verification theorem reduces the COSOCP to a variational inequality which contains the HJB-equation of pure optimal control as a special case. In Section 3 we exclude any bequest motive and specialise to power utility functions. Depending on the sign of some crucial quantity the pensioner either annuitises immediately (trivial case) or never (pure optimal control case). However, apart from very extreme settings, annuitisation is always optimal for the pensioner. This strong tendency for the annuity market turns into a slight tendency for the financial market with the essential inclusion of a bequest motive in Section 4. The annuitisation decision is again of the now-or-never-type if we assume relative risk aversion levels less than one and an identical level of prior life insurance and subsistence of bequests in Subsection 4.1. In contrast, the annuitisation decision rule becomes wealth-dependent for relative risk aversion levels greater than one in Subsection 4.2. Thus, a real COSOCP (no reduction to a pure optimal control problem) has to be solved. Lastly, Section 5 draws the most important conclusions.

This conference paper is mainly based on [15].

## 2 The Model

### 2.1 Preliminaries

Let $U_i : (0, \infty) \to \mathbb{R}$ for $i = 1, 2, 3$ denote the pensioner's utility from a stream of consumption or an annuity and from bequeathing wealth to his heirs. All utility functions are assumed to be strictly increasing and strictly concave and to satisfy the well-known Inada-conditions. We consider the probability measure space $(\Omega, \mathcal{F}, P)$ with a riskless asset $S_0(t)$ and a risky asset $S_1(t)$ whose price processes are

$$dS_0(t) = S_0(t) r dt, \quad S_0(0) = s_0 \tag{1}$$

and

$$dS_1(t) = S_1(t) (\mu dt + \sigma dB(t)), \quad S_1(0) = s_1 \tag{2}$$

with $\mu > r$. The pensioner's wealth evolution is then given by

$$dW(t) = W(t) [r + \pi(t) (\mu - r)] dt + W(t) \pi(t) \sigma dB(t) - c(t) dt \tag{3}$$

[2] This assumption simplifies the wealth dynamics considerably.
[3] This excludes any control part in the post-annuitisation phase.
[4] The risky asset can of course be thought of to be a broad index.

with initial wealth $W(0) = w > 0$. The consumption and portfolio process $c(t)$ and $\pi(t)$, respectively, are assumed to satisfy the usual admissibility requirements that guarantee the existence and uniqueness of a solution to (3) as well as $W(t) \geq 0$ a.s. $\forall t \geq 0$.

We consider a pensioner aged $x$ at time zero whose remaining lifetime $T_x$ is exponentially distributed. Moreover, we allow for the circumstance that the (subjective) pensioner and the (objective) insurance company will generally have different perceptions towards the pensioner's remaining lifetime.[5] We denote the corresponding mortality rates by $\lambda_x^S$ and $\lambda_x^O$, respectively. The pensioner can by a (lifelong) *annuity* $\overline{a}_x$ for

$$\overline{a}_x = \int_0^\infty e^{-rt} e^{-\lambda_x^O t} dt = \frac{1}{r + \lambda_x^O}. \tag{4}$$

Annuitisation is modelled as an all-or-nothing irreversible decision.[6] The random time of the annuity purchase will be represented by $\tau$. Moreover, we naturally postulate stochastic independence between $T_x$ and any financial variable such as $W(t)$ or $\tau$. Lastly, we will mainly drop the age dependence of quantities for notational ease in the following, since $x$ is given at the time of optimisation.

### 2.2 Optimisation Problem

The pensioner's COSOCP is

$$V(w) = \sup_{(c,\pi,\tau)\in\mathcal{G}(w)} J_{c,\pi,\tau}(w) \qquad \text{for all } w > 0 \tag{5}$$

with

$$\begin{aligned} J_{c,\pi,\tau}(w) = E\Bigg[ \int_0^T e^{-\delta^S t} \left\{ U_1(c(t)) 1_{\{t\leq\tau\}} + U_2\left(\frac{W(\tau)}{\overline{a}_{x+\tau}}\right) 1_{\{t>\tau\}} \right\} dt \\ + \eta e^{-\delta^S T} \left\{ U_3(W(T) + Z^s) 1_{\{T\leq\tau\}} + U_3(Z^s) 1_{\{T>\tau\}} \right\} \Bigg] \end{aligned} \tag{6}$$

with the subjective discount parameter $\delta^S$, the indicator function $1_{\{.\}}$, the bequest parameter $\eta \geq 0$ and where $Z^s$ denotes the difference between the prior decision on life insurance and the subsistence level of bequest, i.e. $Z^s = Z^{prior} - \underline{Z}$. Finally, $\mathcal{G}(w)$ denotes the set of admissible strategies $(c, \pi, \tau)$ in the COSOCP.[7]

[5] See [4] and [5] for some evidence that individuals are likely to be able to estimate their life expectancy beyond the observable mortality risk factors.

[6] The irreversibility of the annuity purchase is due to adverse selection issues. See [1] for an early contribution to the field of adverse selection.

[7] While $(c, \pi)$ satisfy the usual admissibility conditions, we impose $\tau < \tau_G$ where $G = (0, \infty)$ and $\tau_G = \inf\{t > 0 | W(t) \notin G\}$.

**Lemma 1.** *Exploiting the exponential mortality law, the total expected discounted utility in (6) can be written as*

$$J_{c,\pi,\tau}(w) = E^w \left[ \int_0^\tau e^{-\beta^S t} \left\{ U_1(c(t)) + \lambda^S \eta U_3(W(t) + Z^s) \right\} dt \right.$$
$$\left. + e^{-\beta^S \tau} \frac{1}{\beta^S} \left\{ U_2\left(W(\tau)\left(r + \lambda^O\right)\right) + \lambda^S \eta U_3(Z^s) \right\} \right] \tag{7}$$

*with*

$$\beta^S = \delta^S + \lambda^S. \tag{8}$$

The proof mainly exploits Fubini's theorem, the exponential mortality law and the assumed stochastic independence of $T$ and $\tau$. We refer the interested reader to [14].

To save some space we will sometimes use the following general version of the indirect utility function

$$J_{c,\pi,\tau}(w) = E^w \left[ \int_0^\tau e^{-\beta^S t} f(c(t), W(t))\, dt + e^{-\beta^S \tau} g(W(\tau)) \right]. \tag{9}$$

### 2.3 Verification Theorem

The HJB equation of pure optimal control is

$$L^{\text{com}} v(W(t)) = \sup_{(c,\pi) \in \mathcal{G}^\tau(W(t))} \left\{ f(c(t), W(t)) - \beta^S v(W(t)) + Lv(W(t)) \right\} \tag{10}$$

where $\mathcal{G}^\tau(W(t))$ denotes the set of admissible strategies for a given stopping time $\tau$ and where $L$ denotes the partial differential operator

$$Lv(W(t)) = [W(t)(r + \pi(t)(\mu - r)) - c(t)]\, v_W(W(t))$$
$$+ \frac{1}{2}\sigma^2 \pi(t)^2 W(t)^2 v_{WW}(W(t)). \tag{11}$$

The following theorem is crucial to our COSOCP. It is an extension to theorem 2.1 in [16]. As [16] only studies risk aversion levels less than one, he solely has to deal with nonnegative power utility functions. Since we will additionally consider the economically interesting range of risk aversions greater than one, we have to adjust the verification theorem of [16]. In accordance with future needs we therefore also deal with the case where the running and terminal reward function $f$ and $g$ are both negative.

**Theorem 1 (COSOCP Verification Theorem).**

*a) Assume that the following conditions are satisfied.*

*(i)* $v \in C^1(\mathbb{R}^+)$ *with continuous second order derivatives a.e. in* $\mathbb{R}^+$.

*(ii)*

$$v(W(t)) \geq g(W(t)) \quad \text{on } G. \tag{12}$$

*(iii)*

$$L^{com} v(W(t)) \leq 0 \quad \text{on } G. \tag{13}$$

*(iv)* *For all* $(c, \pi, \tau) \in \mathcal{G}(w)$ *with* $w > 0$ *the process* $\int_0^t e^{-\beta^S s} \sigma \pi(s) W(s) v_W(W(s)) dB(s)$ *is a martingale for all* $t \geq 0$.

*Then, we have*

$$V(w) = \sup_{(c,\pi,\tau)\in\mathcal{G}(w)} J_{c,\pi,\tau}(w) \leq v(w) \quad \text{for all } w > 0. \tag{14}$$

*b) Furthermore, we define the continuation region* $D$ *as*

$$D = \{W(t) \in G \mid v(W(t)) > g(W(t))\}. \tag{15}$$

*(v)* *We assume that*

$$L^{com} v(W(t)) = 0 \quad \text{on } D. \tag{16}$$

*(vi)* *Moreover, we make the assumption that* $v(W(t))$ *is strictly concave on* $D$ *and that* $f(c(t), W(t))$ *is strictly concave in its first argument everywhere. Define the strategies*[8]

$$\tau^* = \inf\{t \geq 0 | W^*(t) \notin D\}, \tag{17}$$

$$c^* = I(v_W(W^*(t))) 1_{\{t \leq \tau^*\}} \tag{18}$$

*and*

$$\pi^* = -\frac{\mu - r}{\sigma^2} \frac{v_W(W^*(t))}{W^*(t) v_{WW}(W^*(t))} 1_{\{t \leq \tau^*\}}, \tag{19}$$

*where* $W^*(t)$ *is the solution of (3) corresponding to the control pair* $(c^*, \pi^*)$ *and* $I(.)$ *denotes the inverse function of* $\frac{\partial f}{\partial c}(c(t), W(t))$ *w.r.t. the first argument.*

*(vii)* *We further assume that the transversality condition*

$$\lim_{t\to\infty} e^{-\beta^S t} E^w \left[v(W^*(t)) 1_{\{t<\tau^*\}}\right] = 0 \tag{20}$$

*holds.*

[8] The rules for consumption and investment of course only apply to the pre-annuitisation phase, since we have assumed that the pensioner consumes the entire annuity and as there is no wealth left to be optimally allocated after the annuitisation.

*Then, we have*

$$V(w) = \sup_{(c,\pi,\tau)\in\mathcal{G}(w)} J_{c,\pi,\tau}(w) = v(w) \quad \textit{for all } w > 0 \tag{21}$$

*and* $(c^*, \pi^*, \tau^*)$ *are the optimal strategies.*

The proof is given in [15].

*Remark 1.* 1. It can be shown that the verification theorem 1 reduces the COSOCP to the following variational inequality. We must have

$$\max\left\{L^{\text{com}} v(W(t)), g(W(t)) - v(W(t))\right\} = 0 \quad \text{for } W(t) > 0 \tag{22}$$

subject to the 'smooth paste condition'

$$v(W(t)) = g(W(t)) \quad \text{for all } W(t) \in \partial D \tag{23}$$

and subject to the 'smooth fit condition'

$$v_W(W(t)) = g_W(W(t)) \quad \text{for all } W(t) \in \partial D. \tag{24}$$

2. The optimal stopping time (17) is the first exit time of the crucial continuation region $D$. This is very intuitive, since the value function strictly exceeds the terminal reward on $D$.
3. The optimal consumption level (18) ensures that the marginal instantaneous utility from consumption equals the marginal value of wealth, which is a well-known intertemporal optimality result.
4. The optimal portfolio rule (19) is given by the equity premium times the reciprocal of the relative risk aversion of the value function.
5. Lastly, the optimal strategies (17)-(19) are obviously given in terms of the yet unknown value function. We will get explicit expressions when we specialise to power utility functions. Before doing so we give another general result.

The following lemma provides useful information about the important continuation region $D$.

**Lemma 2.** *Using the operator* $L^{com}$ *given in (10) we define the set* $U$ *as*

$$U = \left\{W(t) \in \mathbb{R}^+ \mid L^{com} g(W(t)) > 0\right\}. \tag{25}$$

*We then have*

$$U \subset D. \tag{26}$$

*Furthermore, the continuation region* $D$ *is an open and connected set.*

The proof is given in [14] and [15].

## 3 Results in the No-Bequest Case

From now on we specialise $U_i(.)$ with $i = 1, 2, 3$ to the power utility functions

$$U_i(\theta) = \frac{\theta^{1-\gamma}}{1-\gamma} \quad \text{with } \gamma > 0 \text{ and } \gamma \neq 1. \tag{27}$$

**Theorem 2.** *In the no-bequest case we have*

$$D = \begin{cases} \mathbb{R}^+ & \text{if } \gamma < 1 \text{ and } M^{nb} > 0 \text{ (case 1) or if } \gamma > 1 \text{ and } M^{nb} < 0 \text{ (case 2)} \\ \emptyset & \text{if } \gamma < 1 \text{ and } M^{nb} \leq 0 \text{ (case 3) or if } \gamma > 1 \text{ and } M^{nb} \geq 0 \text{ (case 4)} \end{cases} \tag{28}$$

*with*

$$\begin{aligned} M^{nb} = {} & \gamma \left(\lambda^S + \delta^S\right)^{\frac{1-\gamma}{\gamma}} \left(r + \lambda^O\right)^{-\frac{(1-\gamma)^2}{\gamma}} - \left(r + \lambda^O\right)^{1-\gamma} \\ & + \left(r + \frac{1}{\gamma}\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right) \frac{\left(r + \lambda^O\right)^{1-\gamma}}{\lambda^S + \delta^S} (1 - \gamma). \end{aligned} \tag{29}$$

The proof exploits lemma 2 and results from optimal stopping theory[9] and is given in [15].

*Remark 2.* Theorem 2 shows that the annuitisation decision is of the now-or-never type: Depending on the parameters of the model, annuitisation either occurs immediately or never. This confirms the results of [16] regarding risk aversion levels less than one (cases 1 and 3). Yet, our findings in the economically interesting case with risk aversion levels greater than one (cases 2 and 4) are novel. As the crucial quantity regarding the annuitisation decision $M^{nb}$ depends on as many as seven model parameters in a non-linear fashion, we mainly have to refuge to numerical analysis.

We can confirm the intuitive qualitative findings of [16] what concerns risk aversion levels less than one: While a higher objective and identical life expectancy decrease the attractiveness of the annuity market, a higher subjective life expectancy increases it. We contribute that these effects hold for risk aversions greater than one, too. Moreover, we get the intuitive result that a higher relative risk aversion level makes annuitisation more likely whereas a higher effective time horizon is an argument for the financial market regardless of whether we assume cheap or expensive annuities. Furthermore, we get an unambiguous effect: The pensioner will never annuitise but always stay in the attractive financial market if the Sharpe ratio $SR = \frac{\mu - r}{\sigma}$ is high enough. However, [15] demonstrates that apart from very extreme settings, annuitisation is always optimal for the pensioner. This very strong tendency for the annuity market will change considerably with the important inclusion of a bequest motive in Section 4.

[9] See f.e. [12].

Lastly, annuitisation either occurs immediately or never as discussed in remark 2. Thus, the COSOCP reduces either to a trivial or to a pure optimal control problem. In the latter case we have to solve the infinite Merton problem with the HJB equation $L^{\text{com}}V(W(t)) = 0$. The corresponding results can be found in [15].

## 4 Results in the Bequest Case

### 4.1 Risk Aversion $\gamma < 1$ without Life Insurance

To keep the analysis simple, we set $Z^s = 0$ in the current subsection. This is equivalent to assuming that the previously bought life insurance matches the subsistence level of bequests.

**Theorem 3.** *Let $\gamma < 1$ and $Z^s = 0$. The continuation region is then given by*[10]

$$D = \begin{cases} \mathbb{R}^+ & \text{if } M^b > 0 \text{ case I} \\ \emptyset & \text{if } M^b \leq 0 \text{ case III} \end{cases} \tag{30}$$

*with*

$$M^b = M^{nb} + \lambda^S \eta. \tag{31}$$

The proof is given in [15].

Clearly, the annuitisation decision is again of the now-or-never type. The bequest parameter solely enters in the additive term $\lambda^S \eta$ where it is directly coupled with the subjective mortality rate. A higher $\lambda^S$ - meaning that death is closer - increases the importance of the bequest motive, as expected. The inclusion of a bequest motive obviously makes annuitisation less likely. This a very natural result because there will be no wealth left after the annuitisation that can be bequeathed in addition to $Z^{prior} = \underline{Z}$. The nature of the effects of the objective life expectancy and the Sharpe ratio on the annuitisation decision clearly remain the same as in the no-bequest case. In contrast, the influence of the subjective and the identical life expectancy as well as the effect of the markup parameter $l$ could potentially be different. However, [14] shows that the nature of these effects does not change either compared to the no-bequest case. Furthermore, [15] demonstrates that the absurdly strong tendency for the annuity market, which we discovered in the no-bequest case of Section 3, now - with the important inclusion of a bequest motive - turns into a slight tendency toward the financial market. He also shows that the pensioner rejects fair and even some favourable annuities if we assume a strong enough bequest motive and/or a high enough effective time horizon.

We know from the continuation region (30) that annuitisation either occurs immediately or never. We skip the trivial case III and refer the interested reader to [15] what concerns case I where a similar infinite Merton problem as in the

[10] Obviously, case I in the bequest case corresponds to case 1 in the no-bequest case, while bequest case III corresponds to no-bequest case 3.

no-bequest case 1 and 2 has to be solved. The investment rule in case I is the same as in case 1 and 2, since we have assumed identical risk aversion levels over consumption and over bequests. In contrast, a pensioner with a bequest motive consumes a lower fraction of his wealth which allows him to bequeath more wealth to his heirs compared to the no-bequest case. Moreover, the consumed fraction of wealth naturally decreases in the bequest parameter.

### 4.2 Risk Aversion $\gamma > 1$ Including Life Insurance

We can see from (7) that we need a positive life insurance net of subsistence ($Z^s > 0$) for $\gamma > 1$ to keep the model well defined.

**Theorem 4.** *Let $\gamma > 1$, $Z^s > 0$. The continuation region $D$ is then given by*
*(i) $D = (0, \infty)$ if $M^{nb} \leq 0$ Case II*
*(ii) $D = (\underline{W}, \infty)$ with $\underline{W} \leq W^U$ if $M^{nb} > 0$ Case IV*
*where $\underline{W}^U$ represents the smallest root of $L^{com} g(W(t))$.*

The proof is given in [15].

Comparing theorem 2 of the no-bequest case to the above theorem 4, we see that the inclusion of a bequest motive naturally makes the annuity market less attractive. Moreover, we get a real COSOCP for the first time in case IV:[11] The pensioner annuitises as soon as his wealth falls below the yet unknown threshold $\underline{W}$. [15] exploits the important verification theorem 1 in case IV, which reduces the current real COSOCP to the problem of finding a solution $v$ to the variational inequality (22) subject to the smooth paste and smooth fit condition (23) and (24), respectively. Furthermore, he uses duality arguments to simplify the resulting ODE for the value function. Finally, [15] has to solve a free boundary value problem for the dual function. He developes a numerical solution algorithm that simultaneously solves for the unknown boundaries and the dual function given the model parameters. This algorithm is based on the bisection method. Moreover, it exploits the idea that the investment rule must be the Merton portfolio in the limit as wealth approaches infinity, since the probability of annuitisation then tends to zero.

[15] shows that the additional option of the annuity market makes the pensioner more aggressive compared to the Merton investor for whom no annuity market exists at all. Furthermore, he obtains the intuitive result that a stronger bequest motive decreases the threshold and therefore the probability of annuitisation while a higher life insurance net of subsistence has the opposite effect. Lastly, [15] finds that the option of annuitisation leads to heavy consumption smoothing which is in contrast to the constant Merton consumption fraction.

[11] In contrast, [16] does not get a real COSOCP with identical risk aversion levels. He shows that the annuitisation decision becomes wealth-dependent with the assumption that the relative risk aversion coefficient is higher in the post-annuitisation phase.

## 5 Conclusions

The task of finding optimal consumption, asset allocation and the optimal annuitisation time for a pensioner leads to a combined optimal stopping and optimal control problem (COSOCP). Assuming power utility with identical relative risk aversion and the exponential mortality law, the COSOCP normally reduces to a trivial or to a pure optimal control problem as in [16]. We extended the model of [16] by additionally studying the economically interesting range of relative risk aversion levels greater than one and by introducing a bequest motive and consequently, prior life insurance and a subsistence level of bequests. Normally, annuitisation is of the now-or-never type: Depending on the sign of some quantity the pensioner either annuitises immediately or never. Moreover, the model parameters affect this decision in an intuitive manner. In contrast to [16], we also get a wealth-dependent annuitisation rule without imposing a higher relative risk aversion coefficient in the post-annuitisation phase. This real COSOCP occurs for relative risk aversion levels greater than one when we include a bequest motive. [15] simplified the problem via duality arguments and provided a numerical solution algorithm. He reports the natural effects of the bequest parameter and life insurance. The main result is that the essential inclusion of a bequest motive turns the very strong tendency for the annuity market into a slight tendency for the financial market. But even in the bequest case there are many realistic situations where the pensioner chooses the annuity market. This highlights the importance of longevity risk and therefore provides an additional legitimation for pension funds besides the intergenerational risk transfer argument discussed in [2].

## References

1. G. A. Akerlof, The Market for Lemons: Quality Uncertainty and the Market Mechanism, Quarterly Journal of Economics **84** (1970) 488-500
2. R. T. Baumann and H. H. Müller, Pension Funds as Institutions for Intertemporal Risk Transfer, Insurance: Mathematics and Economics (2008)
3. N. Charupat and M. A. Milevsky, Optimal Asset Allocation in Life Annuities: A Note, Insurance: Mathematics and Economics **30** (2002) 199-209
4. M. D. Hurd and K. McGarry, Evaluation of the Subjective Probabilities of Survival in the Health and Retirement Study, Journal of Human Resources **30** (1995) 268-291
5. M. D. Hurd and K. McGarry, The Predictive Validity of Subjective Probabilities of Survival, NBER Working Paper 6193 (1997)
6. N. Hakansson, Optimal Investment and Consumption Strategies under Risk, an Uncertain Lifetime, and Insurance, International Economic Review (1969)
7. R. C. Merton, Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case, Review of Economics and Statistics **51** (1969) 247-257
8. R. C. Merton, Optimum Consumption and Portfolio Rules in a Continuous-Time Model, Journal of Economic Theory **3** (1971) 373-413
9. M. A. Milevsky, K. S. Moore and V. R. Young, Asset Allocation and Annuity-Purchase Strategies to Minimize the Probability of Financial Ruin, Mathematical Finance **16(4)** (2006) 647-671

10. M. A. Milevsky and V. R. Young, Annuitisation and Asset Allocation, Journal of Economic Dynamics and Control **31(9)** (2007) 3138-3177
11. M. A. Milevsky and V. R. Young, Optimal Asset Allocation Towards the End of the Life Cycle: To Annuitize or Not to Annuitize? Journal of Risk and Insurance **65(3)** (1998) 401-426
12. B. Øksendal, Stochastic Differential Equations, Springer, Berlin (2003)
13. S. F. Richard, Optimal Consumption, Portfolio and Life Insurance Rules for an Uncertain Lived Individual in Continuous Time Model, Journal of Financial Economics **2** (1975) 187-203
14. D. Schiess, Consumption and Portfolio Optimisation at the End of the Life-Cycle, Doctoral Thesis, University of St. Gallen, Switzerland
15. D. Schiess, Optimal Strategies During Retirement, Center for Finance Working Paper No. 70, University of St. Gallen, Switzerland
16. G. Stabile, Optimal Timing of the Annuity Purchase: A Combined Stochastic Control and Optimal Stopping Problem, International Journal of Theoretical and Applied Finance **9(2)** (2006)
17. M. Yaari, Uncertain Lifetime, Life Insurance, and the Theory of the Consumer, Review of Economic Studies **2** (1965) 137-150