Valuation of Electricity Swing Options by Multistage Stochastic Programming

Gido Haarbrücker Daniel Kuhn*

gido.haarbruecker@unisg.ch daniel.kuhn@unisg.ch

Institute for Operations Research and Computational Finance

University of St. Gallen

Bodanstrasse 6, 9000 St. Gallen, Switzerland

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Abstract. Electricity swing options are American-style path-dependent power derivatives. We consider an electricity market driven by several exogenous risk factors and formulate the pricing problem for a class of swing option contracts with energy and power limits as well as ramping constraints. Efficient numerical solution of the arising multistage stochastic program requires aggregation of decision stages, discretization of the probability space, and reparameterization of the decision space. We report on insightful numerical results and discuss analytically tractable limiting cases.

Key words. stochastic programming, swing option, energy, aggregation, discretization

1 Introduction

The ongoing deregulation of electricity markets worldwide has a major impact on the power industry. New price risks require new risk management tools and new methods for the valuation of generation and transmission assets as well as physical and financial electricity contracts. As far as risk management is concerned, many derivative instruments have been designed to hedge against spot price risk or different types of liability risk exposure. For instance, one observes the emergence of markets for standard derivatives such as futures and European call and put options. Examples of such derivative markets comprise the European Energy Exchange (EEX), Amsterdam Power Exchange (APX), Nord Pool with Eltermin and Eloption, and New York Mercantile Exchange (NYMEX). In addition, there is an immense variety of derivative contracts with American-style and path-dependent payoff structure; these contracts are usually traded over-the-counter. Probably the most important examples are swing options, which are sometimes also referred to as take-or-pay contracts, virtual power plants, base-load factor contracts, or flexible nomination contracts. A swing option is an agreement to purchase and or sell electric energy during a fixed period of time and at a predetermined price. The option holder may choose her consumption process freely within certain restrictions stipulated in the contract. Usually, there is some flexibility in the timing and amount of consumption. Swing options are tailored for risk-averse agents willing to mitigate (spot) price risk. As electric energy is storable only in a limited manner, the timing flexibility is appreciated by agents who are unable to control their electricity consumption. While simple futures contracts provide no timing and volumetric flexibility at all, a strip of standard call and or put options could offer more flexibility than needed, thus involving too high risk premia. Therefore, swing options represent ideal hedging instruments, e.g., for risk-averse public utilities facing a stochastic load demand. Moreover, swing options are frequently used for the valuation and hedging of real power plants or as speculative instruments.

Virtually any existing valuation scheme for swing option contracts can be traced back to a stochastic dynamic programming (SDP) approach. For instance, Thompson [29] develops an SDP scheme within a lattice whereas Jaillet, Ronn, and Tompaidis [13] work with a forest of recombining trees. Lari-Lavassani, Simchi, and Ware [19] present a similar discrete forest methodology together with a proof of convergence and various sensitivity analyses. On the other hand, Davison and Anderson [8] consider specific swing options on weekly average on-peak electricity prices. They estimate optimal exercise boundaries and determine the option value via subsequent Monte Carlo simulation. Other promising valuation schemes are based on the Least Squares Monte Carlo approach (LSM) due to Longstaff and Schwartz [20]. The LSM algorithm uses least squares regression to estimate the conditional expectation functions involved in the SDP iterations. A slightly different Monte Carlo method for the valuation of multiple exercise options is discussed in Meinhausen and Hambly [21]. Finally, a linear programming based lower bound on the swing option value is presented by Keppo [16], and a mathematically rigorous analysis of the optimal multiple stopping problem (inherent in the swing option pricing problem) can be found in Carmona and Touzi [7] or Carmona and Dayanik [6].

The present article addresses challenging swing option pricing problems that are difficult to solve with SDP techniques. The reason for this difficulty is threefold. Firstly, we embed the valuation problem in a model economy driven by several risk factors. Multi-dimensionality of risk is a key feature of any realistic market model as it eliminates perfect correlation of the forward price fluctuations corresponding to different maturities. Secondly, we consider swing options involving ramping constraints, i.e. we impose restrictions on the gradient of the consumption rate; see Sect. 3. Notice that standard SDP schemes [13, 19, 29] are designed to deal with swing contracts involving limitations on the consumption rate and its integral, but not on its gradient. However, ramping constraints are important when a swing option (virtual power plant) is used to approximate a real power plant with limited start-up speed. Thirdly, we study contracts with many exercise times such as hourly exercisable swing options over one year. Numerical experiments suggest that SDP and its variants are well adapted for pricing daily exercisable swing options without ramping constraints and given that risk is one-dimensional. For the pricing problems addressed in this article, SDP does not provide sufficiently accurate solutions in reasonable time.

Due to suitable aggregation of decision stages and discretization of the risk space, our stochastic programming approach entails short computation times in the range of only few seconds. Moreover, calculations are based on the current forward price curve. Thus, our valuation scheme meets typical operational requirements of trading and risk management departments in privatized electricity companies: decisions on whether submitting or accepting an offer have usually to be taken within a time frame of some minutes only, and exercise decisions should always take account of the latest market data. Although stochastic programming approaches usually make higher demands on the implementation effort required than SDP approaches do: bringing out the favorable run-time behavior of the stochastic programming approach presented here – even in the presence of multiple risk factors – and its capability to cope with pricing problems of challenging swing options (e.g. including ramping constraints), we deem the implementation efforts to be worth wile.

This article is structured as follows. Section 2 develops a Pilipovic-type model for the forward price dynamics which fully characterizes all relevant aspects of the electricity market under consideration. In Sect. 3, we specify the swing option contracts to be considered in this work and propose an exact valuation scheme. Next, Sect. 4 provides some a priori information on the option's value and optimal exercise strategy in different regimes of parameter space, while Sect. 5 approximates the exact pricing problem by a numerically tractable auxiliary problem. Section 6 reports on numerical results, and Sect. 7 concludes.

2 Forward Price Dynamics

When dealing with the valuation of electricity derivatives, one faces several major difficulties. First, electricity cannot be stored efficiently, and production has to cover demand instantaneously. Therefore, the traditional, storage based no-arbitrage methods of valuing commodity derivatives fail. Moreover, markets for electricity derivatives suffer from insufficient liquidity; there are large bid-offer spreads, and agents face a significant counterparty risk. Next, electricity spot prices are mean-reverting and exhibit strong seasonalities, jumps, spikes, and regime-switching behavior. These complicating features result from the non-storability and the grid bound nature of power. Finally, many of the most wide-spread derivative contracts in the electricity industry are non-standard instruments. Their payoff structure is American-style and path-dependent, which requires sophisticated valuation schemes.

In this article, we will assume the electricity spot and forward markets to be efficient since their liquidity is likely to rise in the future. However, we dismiss the unrealistic assumption that electric energy is storable and traded. Instead, we consider the spot price as a non-traded state variable. A market is arbitrage-free if and only if there exists a martingale measure Q, under which the discounted price processes of all investment goods are driftless [11, 12]. These goods must be traded; i.e. their prices must be determined by matching demand and supply in a liquid market. In particular, it is always assumed implicitly that the investment goods are storable. This requirement is trivially satisfied for securities, but not for electric energy. As there is no problem with storing a financial electricity contract, and since such contracts are traded, there exists a martingale measure Q under which all discounted contingent claims on electric energy are martingales. In contrast, the spot price of electricity — as a non-traded state variable — is not necessarily a martingale under Q. Alternatively, we can postulate that electricity sold at different times must be viewed as different commodities. The corresponding forward contracts are traded and storable; and they constitute a set of basis securities, whose price processes must be given exogenously. If these contracts define a complete market, we can replicate any contingent claim by dynamically trading in the forwards and a riskless instrument. For simplicity of exposition, throughout this article we will assume that the risk-free interest rate is zero. Note that this assumption could easily be relaxed at the cost of additional notation.

Let us now specify the probabilistic model which underlies the swing option pricing problem to be formulated in Sect. 3. Since an electricity traders' decisions are usually based on current (and historic) forward price information, we should provide a model for the forward price dynamics. Our approach is inspired by the popular Pilipovic model [23, Sect. 5.6], which is widely used in the energy sector and has the advantage of analytical tractability. Concretely speaking, we will work with a probability space $(\Omega, \mathcal{F}, Q)$ on which a two-dimensional Brownian motion $\boldsymbol{B}_t = (B_{1,t}, B_{2,t})$ is defined. Recall that the components of $\boldsymbol{B}_t$ are independent one-dimensional Brownian motions. For technical details we refer to Protter [25]. Next, we fix a deterministic forward price curve $\{F(0,T)\}_{T\geq 0}$ at time 0. For any $0 \leq t \leq T$, we will assume that the forward price $F(t,T)$ at time t is represented by the random variable[1]

$$F(t,T) = F(0,T)\,\frac{E(\exp \xi_{1,T} |\, \mathcal{F}_t)}{E(\exp \xi_{1,T})}, \tag{1}$$

where $\boldsymbol{\xi}_t := (\xi_{1,t}, \xi_{2,t})$ is a vector of risk factors subject to the stochastic differential equation (SDE)

$$d\boldsymbol{\xi}_t = A\boldsymbol{\xi}_t dt + \Sigma\, d\boldsymbol{B}_t, \tag{2}$$

and $\mathcal{F}_t := \sigma(\boldsymbol{\xi}_s \,|\, 0 \leq s \leq t)$ is the σ-algebra generated by the history of risk factors up to time t. Furthermore, the matrices appearing in (2) are given by

$$A := \begin{pmatrix} -\alpha_1 & \alpha_1 \\ 0 & -\alpha_2 \end{pmatrix} \quad \text{and} \quad \Sigma := \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix},$$

where $\alpha_1 > \alpha_2 \geq 0$ and $\sigma_1, \sigma_2 > 0$. These parameters have to be estimated empirically. Although the calibration of the forward price model under consideration is beyond the scope of the present article, we should remark that one usually finds $\alpha_1 \gg \alpha_2$ and $\sigma_1 \gg \sigma_2$ in real energy markets. Therefore, the second risk

[1] By construction, for every maturity parameter $T \geq 0$, the process $\{F(t,T)\}_{0\leq t\leq T}$ is in fact a martingale relative to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

factor $\xi_{2,t}$ can conveniently be interpreted as the equilibrium level around which $\xi_{1,t}$ fluctuates. Systems of SDEs of the type (2) with initial data $\boldsymbol{\xi}_s$, $0 \le s$, can systematically be solved. The solution reads

$$\boldsymbol{\xi}_t = \underbrace{\exp(A(t-s))}_{=:H_{t,s}} \boldsymbol{\xi}_s + \underbrace{\int_s^t \exp(A(t-r))\, \Sigma\, d\boldsymbol{B}_r}_{=:\boldsymbol{\varepsilon}_{t,s}}, \quad s \le t, \tag{3}$$

where 'exp' stands for the matrix exponential,[2] i.e.

$$\exp(At) = \begin{pmatrix} e^{-\alpha_1 t} & \frac{\alpha_1}{\alpha_1-\alpha_2}\left(e^{-\alpha_2 t} - e^{-\alpha_1 t}\right) \\ 0 & e^{-\alpha_2 t} \end{pmatrix}.$$

The random variable $\boldsymbol{\varepsilon}_{t,s}$ is independent of the outcome history $\{\boldsymbol{\xi}_r\}_{r\le s}$. From elementary stochastic calculus we know that $\boldsymbol{\varepsilon}_{t,s}$ is normally distributed, i.e.

$$\boldsymbol{\varepsilon}_{t,s} \sim \mathcal{N}\left(0, \int_s^t \exp(A(t-r))\, \Sigma\, \Sigma^\top \exp(A(t-r))^\top dr\right).$$

Plugging the first component of $\boldsymbol{\xi}_T = H_{T,t}\, \boldsymbol{\xi}_t + \boldsymbol{\varepsilon}_{T,t}$ into the formula for the forward prices (1), calculating the conditional expectation value, and applying Itô's lemma we obtain the SDE

$$\frac{dF(t,T)}{F(t,T)} = e^{-\alpha_1(T-t)}\, \sigma_1\, dB_{1,t} + \frac{\alpha_1}{\alpha_1-\alpha_2}\left(e^{-\alpha_2(T-t)} - e^{-\alpha_1(T-t)}\right) \sigma_2\, dB_{2,t}. \tag{4}$$

Observe that the first Brownian motion accounts for the fluctuations of the short-term forward prices ($\alpha_1^{-1} \gg T-t \ge 0$), whereas the second Brownian motion determines the mid-term fluctuations ($\alpha_2^{-1} \gg T-t \gg \alpha_1^{-1}$). Thus, there is no perfect correlation between the the forward price fluctuations across the different maturities, which is an important feature of multifactor models. By (1) the spot price of electricity can be written as[3]

$$S(t) := F(t,t) = F(0,t)\, \frac{\exp \xi_{1,t}}{E(\exp \xi_{1,t})}. \tag{5}$$

[2] Rewrite A as RDR^{-1}, where R is regular and D is a diagonal matrix containing the eigenvalues of A. Recalling the power series representation of the matrix exponential, we find $\exp(At) = R \exp(Dt)\, R^{-1}$.

[3] Note that the spot price is generally no martingale relative to the filtration $\{\mathcal{F}_t\}_{t\ge 0}$.

Plugging $\boldsymbol{\xi}_T = H_{T,0}\,\boldsymbol{\xi}_0 + \boldsymbol{\varepsilon}_{T,0}$ into (1), we can verify that the distribution of the forward prices is independent of the choice of the initial data $\boldsymbol{\xi}_0$. Without loss of generality, the start value $\boldsymbol{\xi}_0$ may thus be set to zero, which will always be assumed from now on. Moreover, by plugging $\boldsymbol{\xi}_T = H_{T,t}\,\boldsymbol{\xi}_t + \boldsymbol{\varepsilon}_{T,t}$ into (1), we find that there is a one-to-one correspondence between forward price curves and risk factors in our model economy. In other words, for fixed t, each vector of risk factors $\boldsymbol{\xi}_t$ uniquely determines a forward price curve $\{F(t,T)\}_{T\geq t}$ and vice versa. There is also a one-to-one correspondence between spot prices and the first components of the risk factors: for fixed t, each $\xi_{1,t}$ uniquely determines a spot price $S(t)$ and vice versa. However, as the spot price represents one single point on the forward price curve, it is not possible to reconstruct $\boldsymbol{\xi}_t$ from only observing $S(t)$. These results will be important when modelling the information structure underlying the swing option pricing problem.

In practice, one might wish to work with more realistic multivariate forward price models involving jumps and spikes, regime switching, and stochastic volatility, as proposed, e.g., by Deng [9]. In these cases, our methodology for pricing swing options to be developed in Sect. 3 still applies. As far as the underlying probabilistic model is concerned, all that matters is to identify some time-dependent vector of risk factors, which parameterizes the forward price curves in a one-to-one fashion. Moreover, one needs to know the conditional law of the risk factors at time t given the the risk factors at time s for $0 \leq s \leq t$.

3 Electricity Swing Options

This section introduces the instrument 'electricity swing option' and sketches the range of swing option types under consideration. In brief, an electricity swing option is an agreement to purchase and or sell electric energy at a predetermined strike price K during a fixed contract period $[\underline{t}, \overline{t}]$, and it provides some flexibility in the quantity to be delivered; see Kaminski et al. [14], Barbieri et al. [2], or Pilipovic [24]. Typically, a swing option appears along with limitations on power at time t and energy cumulated within the interval $[\underline{t}, t]$, where t ranges from $\underline{t}$ to $\overline{t}$. In general practice, limits on energy quantities relate to the whole contract period. Thus, cumulative energy at time $\overline{t}$ is required to lie between the given target values $\underline{e}$ and $\overline{e}$. In case undershooting or exceeding these limits is allowed, penalties c^- or c^+ are imposed to each unit of shortfall or exceedance, respectively. Note that energy constraints corresponding to shorter subperiods can also be handled. However, because of their rare occurrence, such constraints will be disregarded in the sequel. Limitations on power hold at any time t in the contract period.

Table 1: Contractual parameters

Param.	Value	Unit
$\underline{t}$	start of the delivery period	(h)
$\overline{t}$	end of the delivery period (expiry)	(h)
Δ	length of a rebalancing interval	(h)
K	strike price	(€/MWh)
$\{\underline{p}_i\}_{i\in\mathcal{I}}$	lower power limits	(MW)
$\{\overline{p}_i\}_{i\in\mathcal{I}}$	upper power limits	(MW)
p_{start}	start power	(MW)
$\underline{e}$	lower energy limit at expiry $\overline{t}$	(MWh)
$\overline{e}$	upper energy limit at expiry $\overline{t}$	(MWh)
$\{\varrho_i\}_{i\in\mathcal{I}}$	ratchets	(MW)
c^+	penalty costs for exceeding energy limit $\overline{e}$	(€/MWh)
c^-	penalty costs for undershooting energy limit $\underline{e}$	(€/MWh)

In practice, rebalancing is restricted to discrete time points $t_i = i\Delta$, $i \in \mathcal{I} = \{\underline{i}, \ldots, \overline{i}\} \subset \mathbb{Z}$, at which new forward price curves are observed. Concretely speaking, power may be chosen freely at each time t_i, based on past and present forward price information, and remains fixed at some level p_i in the interval from t_i to t_{i+1}, $i \in \mathcal{I}$. The constant parameter Δ characterizes the length of a *rebalancing interval*. Typically, Δ lies between 15 minutes and several hours. We require the start and end dates of the contract period to be compatible with the subdivision into rebalancing intervals, i.e. $\underline{t} = \underline{i}\Delta$ and $\overline{t} = (\overline{i}+1)\Delta$. Furthermore, swing options are assigned a *quality*. One usually distinguishes *base*, *peak*, and *off-peak* quality, allowing the option rights to be exercised only in base, peak, or off-peak hours within the contract period, respectively. This feature is conveniently captured by introducing time-dependent power limits:

$$\underline{p}_i \leq p_i \leq \overline{p}_i \quad \forall i \in \mathcal{I}.$$

Sometimes, additional ramping constraints are specified. These constraints limit the slope of the load pattern corresponding to a given exercise strategy. The maximum power difference $\varrho_i \geq 0$ between rebalancing intervals $i-1$ and i is referred to as a *ratchet*, see e.g. [2]. Usually, ratchets are constant over time. However, imagine that the rebalancing interval has length one hour, and one wishes to price a daily exercisable swing option: then, one may set up a model with $\varrho_i = 0$ unless i is a multiple of 24. In the absence of ramping constraints we simply set $\varrho_i = \overline{p}_i - \underline{p}_i$ for all $i \in \mathcal{I}$. Thus, formally speaking, we may always require

$$|p_i - p_{i-1}| \leq \varrho_i \quad \forall i \in \mathcal{I}.$$

Here, the dummy variable $p_{\underline{i}-1}$ is set to the start power p_{start}, which represents another contractual parameter. Typically, swing contracts involving nontrivial ratchets (which are strictly smaller than $\overline{p}_i - \underline{p}_i$) are assigned base quality; hence, the option is exercisable at any time in the contract period. However, ramping constraints can also be meaningful when dealing with 'non-base' quality. The complete parameter set necessary to characterize a swing contract is shown in Table 1. The decision variables at the choice of the option holder are $\{p_i, e_i\}_{i\in\mathcal{I}}$.

As mentioned above, p_i stands for power in the i'th rebalancing interval, whereas e_i denotes the amount of energy exercised by the end of interval i.

Without loss of generality, let us now determine the price of a given swing option at time $t = 0$ (today). It is always assumed that $\bar{t} > 0$,[4] but we will consider problems with $\underline{t} \geq 0$ as well as $\underline{t} < 0$. In the latter case, we determine the residual option value corresponding to the remaining delivery period from 0 to $\bar{t}$ and contingent on the amount of energy exercised in the period from $\underline{t}$ to 0. Valuation is based on the following assumptions: agents are rational and only price driven, they have access to efficient electricity spot and forward markets, the forward price dynamics is governed by the SDE (4), there are no bid-ask spreads, no discounting takes place over time, and electric energy is not storable. We will work directly under the martingale measure Q. Notice that the electricity market under consideration is complete, implying that Q is in fact unique. Consequently, the unique value of the swing contract equals the sum of the expected cash flows transferred from the issuer to the holder [11, 12]. However, these cash flows depend on the option holders' exercise strategy. The challenge of pricing a swing contract is thus to evaluate the option holders' exercise strategy under the premise that he or she is a rational agent trying to maximize the contract value.

For the sake of transparent notation, we define $\mathcal{I}^+$ as the index set of the future rebalancing intervals, $\mathcal{I}^+ = \mathcal{I} \cap \mathbb{N}_0$. Moreover, in the remainder we will always choose time units such that $\Delta = 1$. At time 0, the option holder observes the forward price curve $\{F(0,T)\}_{T \geq 0}$ and, bearing in mind the contractual constraints, decides on the exercise level p_0, which is held constant up to time 1. Thus, the contract issuer is obliged to sell the holder an amount of energy p_0 at the strike price K. As the agents are purely price driven and can not accumulate energy, they square positions instantaneously in the spot market. At time 1, a new forward price curve $\{F(1,T)\}_{T \geq 1}$ is observed, in response to which

[4] Otherwise, the option is obviously worthless.

the contract holder chooses the exercise level p_1, etc. This procedure recurs until the end of the delivery period. Generally speaking, at time $i \in \mathcal{I}^+$, the net cash flow $(S(i) - K)\, p_i$ is transferred from the issuer to the contract holder. The exercise decisions p_i and e_i may depend on forward price information observed at times $j = 0, \ldots, i$; however, they are required to be independent of all information revealed at later rebalancing intervals $j = i+1, \ldots, \bar{i}$. This property is referred to as *non-anticipativity* in literature (see e.g. [5, 27]). In Sect. 2 we have argued that there is a one-to-one correspondence between forward price curves and risk factors. Thus, we may equivalently state that, for $i \in \mathcal{I}^+$, the decisions p_i and e_i depend exclusively on information about the risk factors $\{\boldsymbol{\xi}_j\}_{j=0}^{i}$. As the risk factors are random variables on $(\Omega, \mathcal{F}, P)$, so are the decisions. By non-anticipativity, they are $\mathcal{G}_i$-measurable, where

$$\mathcal{G}_i = \sigma(\boldsymbol{\xi}_j \,|\, j = 0, \ldots, i)$$

is the sub-σ-algebra (or the information set) generated by the risk factors observed in the past.[5] At this point of progress, it is satisfactory to establish the optimization problem for valuing swing options in a formal way. The development of a computationally tractable mathematical model is postponed to Sect. 5. The (residual) value of a swing option at time $t = 0$ is given by the optimal value

[5]The random variable $p_i : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is $\mathcal{G}_i$-measurable if and only if there is a measurable function $\varphi_i : (\mathbb{R}^{2(i+1)}, \mathcal{B}(\mathbb{R}^{2(i+1)})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $p_i = \varphi_i(\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_i)$; see e.g. [1, Theorem 6.4.2 (c)]. Non-anticipativity is obvious from the last expression.

of the following multistage stochastic program (cf. [5, Sect. 3.5]):

$$\max\, E \sum_{i\in\mathcal{I}^+} (S(i)-K)\,p_i \tag{6}$$

$$\text{s.t.} \quad \left.\begin{array}{l} \underline{e} \le e_{\bar{i}} \le \overline{e} \\ \left.\begin{array}{l} e_i - e_{i-1} = p_i \\ \underline{p}_i \le p_i \le \overline{p}_i \\ |p_i - p_{i-1}| \le \varrho_i \\ p_i, e_i \ \mathcal{G}_i\text{-measurable} \end{array}\right\} \forall i \in \mathcal{I}^+ \end{array}\right\} Q\text{-a.s.}$$

Thus, we maximize the option holders' expected (future) profit earned by means of the (future) exercise strategy $\{p_i, e_i\}_{i\in\mathcal{I}^+}$. Observe that the energy balance equation (or continuity equation) determines the relation between power and cumulative energy. If $\underline{t} < 0$ (i.e. 'today' lies within the delivery period), then e_{-1} represents the amount of energy exercised from $\underline{t}$ to 0, and p_{-1} stands for initial power in interval -1. Otherwise, if $\underline{t} \geq 0$ (i.e. the delivery period lies in the future), then the dummy variable $e_{\underline{i}-1}$ is set to zero while $p_{\underline{i}-1}$ is set to p_{start}. All admissible exercise strategies comply with the contractual constraints on power and energy as well as with the ramping constraints. As already mentioned above, we require the decisions p_i and e_i to be non-anticipative with respect to the underlying information process $\{\mathcal{G}_i\}_{i\in\mathcal{I}^+}$.

Note that, although the forward prices (or equivalently: the risk factors) do not show up explicitly in the stochastic program (6), they impact the pricing problem through the important measurability constraint. If the option holder could observe only spot prices instead of the entire forward price curves, we should require the time-i decisions to be measurable with respect to $\sigma(\xi_{1,j}|j=0,\ldots,i)$,[6] which is a real sub-σ-algebra of $\mathcal{G}_i$. This extra restriction would reduce the swing option value. However, since electricity traders do in fact observe the entire

[6]Recall that there is a one-to-one correspondence between the first components of the risk factors and the spot prices.

forward price curves, we necessarily have to work with the 'richer' information process $\{\mathcal{G}_i\}_{i\in\mathcal{I}^+}$.

Notice that the energy limits are strict in (6), i.e. failure to meet the energy targets is penalized with an infinite loss. Passing over to finite penalties is straightforward but omitted for the sake of better readability. We will briefly return the case of finite penalties in the next section.

4 A Priori Analysis

For ease of exposition, consider a swing option contract with strict energy limits and without ramping constraints, i.e. $\varrho_i = \overline{p}_i - \underline{p}_i$ for all $i \in \mathcal{I}$. Without performing any calculations, exact statements are available about the optimal exercise strategy and the option value in specific regions of parameter space. To see this, consider again the stochastic program (6). It has relatively complete recourse [26, 30] if we explicitly take account of the induced constraints

$$\underline{e} - \sum_{j=i+1}^{\bar{i}} \overline{p}_j \le e_i \le \overline{e} - \sum_{j=i+1}^{\bar{i}} \underline{p}_j \quad \forall i \in \mathcal{I}^+. \tag{7}$$

These constraints will play a decisive role in the below discussion. Fig. 1 sketches the time-energy diagram of the swing option under consideration, and the inset at the bottom shows the corresponding quality pattern. Precisely speaking, the grey bars indicate on-quality intervals when the option may principally be exercised (here we show the situation $\overline{p}_i > \underline{p}_i > 0$), and the spaces denote off-quality intervals ($\overline{p}_i = \underline{p}_i = 0$). In all on-quality intervals, the option holder should purchase energy at a rate no lower than $\underline{p}_i$ and no larger than $\overline{p}_i$, whereas in all off-quality intervals energy is held constant. Observe that points outside the shaded region are not accessible when starting with zero energy at the beginning of the contract period.

The time-energy plane is divided into different regimes, where analytical a priori statements about the optimal exercise strategy $\{p_i^*\}_{i\in\mathcal{I}^+}$ (Table 2) and the

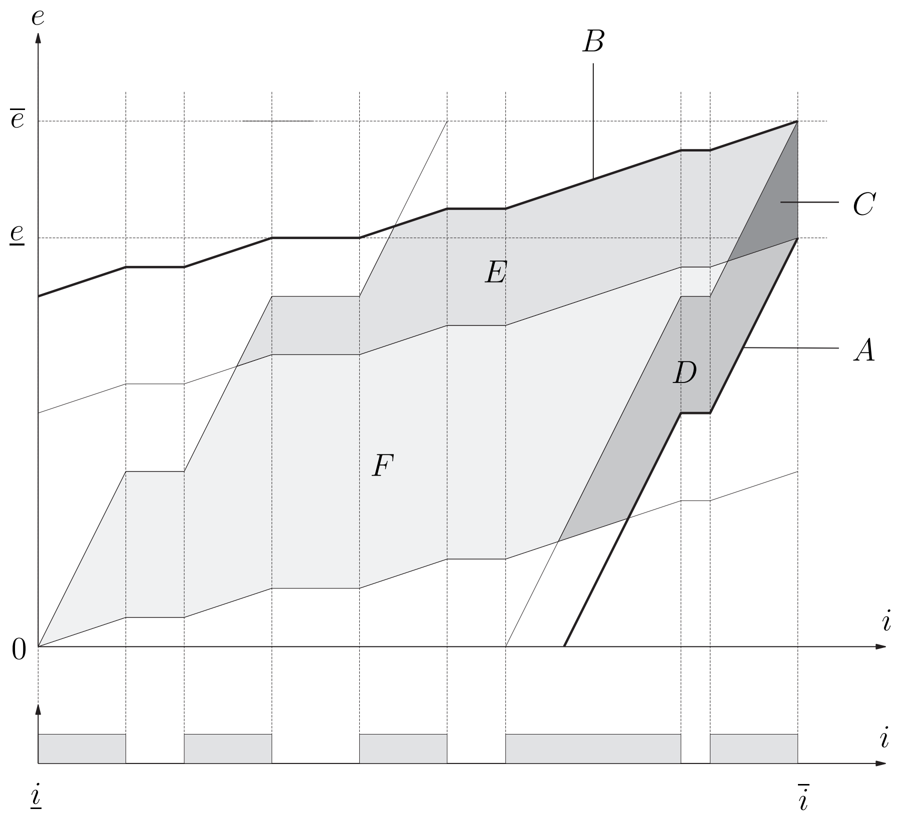

Figure 1: Time-energy diagram

option value (Table 3) can be made. Notice that the 'upper' and 'lower' induced constraints (7) are binding in regimes A and B, respectively. Therefore, the option has to be exercised at maximum power in regime A. Otherwise, the target value $\underline{e}$ would be undershot by the end of the contract period. Conversely, the swing option has to be exercised at minimum power in regime B in order to prevent exceedance of the energy target $\overline{e}$ at expiry. In any case, the option reduces to a portfolio of cash and forward contracts with different maturities.

In regime C, both the upper and lower energy limits become obsolete. The optimal exercise level at time i only depends on the sign of $S(i) - K$, and the swing option reduces to a portfolio of cash, forwards, and European call options with different maturities. Observe that the time-i price of a European call option which pays off $(S(j) - K)^+$ at time $j \geq i$ is denoted by $C(i, j)$.

Table 2: Optimal exercise strategy in different regions of parameter space

Region	Optimal Strategy at time i
A	$p_i^* = \overline{p}_i$
B	$p_i^* = \underline{p}_i$
C	$S(i) \geq K \;\Rightarrow p_i^* = \overline{p}_i$, $S(i) < K \;\Rightarrow p_i^* = \underline{p}_i$
D	$S(i) \geq K \;\Rightarrow p_i^* = \overline{p}_i$, $S(i) < K \;\Rightarrow$ no a priori statement
E	$S(i) \leq K \;\Rightarrow p_i^* = \underline{p}_i$, $S(i) > K \;\Rightarrow$ no a priori statement
F	no a priori statement

Table 3: Option value in different regions of parameter space

Region	Option Value at time i		
A	$=$	$\sum_{j=i}^{\bar{i}} \overline{p}_j \left(F(i,j) - K\right)$	
B	$=$	$\sum_{j=i}^{\bar{i}} \underline{p}_j \left(F(i,j) - K\right)$	
C	$=$	$\sum_{j=i}^{\bar{i}} \underline{p}_j \left(F(i,j) - K\right) + (\overline{p}_j - \underline{p}_j)\, C(i,j)$	
D	$\leq$	"	, $\geq \sum_{j=i}^{\bar{i}} \overline{p}_j \left(F(i,j) - K\right)$
E	$\leq$	"	, $\geq \sum_{j=i}^{\bar{i}} \underline{p}_j \left(F(i,j) - K\right)$
F	$\leq$	"	

In regimes D and E the upper and lower energy limits, respectively, become redundant. Some implications of this partial redundancy are summarized in Tables 2 and 3. Note that the upper bounds on the option value are by relaxation of the energy constraints. In contrast, the lower bound in D is due to feasibility of $\{p_i = \overline{p}_i\}_{i \in \mathcal{I}^+}$, while the lower bound in E is due to feasibility of $\{p_i = \underline{p}_i\}_{i \in \mathcal{I}^+}$.

Next, consider regime F where no precise a priori statement about the optimal decisions is available. Nevertheless, studying the dynamic version (cf. [17, Sect. 3.3]) of the stochastic program (6) provides qualitative insights. The today's shadow price of energy is defined as the rate of change of the option value with respect to small changes of the parameter e_{-1}, i.e. the energy exercised in the past. In the limit of (infinitesimally) short rebalancing intervals one can prove that the option should be exercised at maximum power if the shadow price of energy is larger or equal to the difference of the strike and the spot price. Otherwise, it is recommended that the option be exercised at minimum power. Note that determination of the shadow price of energy requires solution of the pricing problem (6). The upper bound on the option value is again by relaxation of the energy constraints.

If exceeding or undershooting energy limits is allowed, it is principally possible to leave the shaded region, and the above qualitative statements should thoroughly be reconsidered. In case of high penalty costs, the above results can serve as a first approximation. However, we find that $S(i) > K + c^+$ implies $p_i^* = \overline{p}_i$, and $S(i) < K - c^-$ implies $p_i^* = \underline{p}_i$ independent of the volume exercised in the past. In the presence of ramping constraints, the above qualitative analysis has to be generalized. Allowing for nontrivial ratchets leads to a power dependence of the induced constraints (7). To obtain similar results as above, one would have to extend the time-energy diagram of Fig. 1 by adding the dimension of power. However, as it reveals hardly any new insights, there is no need to go through this tedious analysis, here.

5 Approximations

For later reference we will refer to the stochastic program (6) as the *exact pricing problem*. In a complete market, the optimal value of (6) represents the maximum amount of money which can be earned from the underlying swing option contract *without risk*. Realization of this certain profit requires implementation of the optimal exercise strategy and continuous delta hedging.[7]

[7] A market is complete if all risks are traded in the sense that investors can obtain any desired exposure to the shocks to the economy. Notice that continuous delta hedging requires the risks to be traded (and observed) continuously.

Unfortunately, the stochastic program (6) can not be solved numerically. In fact, each decision variable represents a random object with a continuum of realizations. Thus, (6) must be viewed as an optimization problem over an infinite-dimensional function space, which is computationally untractable. The standard approach to tackle this problem is to discretize the underlying probability space. But then, one still fails to deal with the large number of decision stages, which ranges from 500 to 10'000 for typical contract periods between one month and one year. In order to establish an approximate pricing problem, which can be solved numerically, we successively perform three basic approximations:

1) reduction of the information process;

2) discretization of the probability space;

3) reduction of the number of decision variables.

In the remainder of this section, we discuss these basic approximations and assess the errors thereby incurred.

5.1 Reduction of the Information Process

The exact pricing problem (6) has $|\mathcal{I}^+| \gtrsim 500$ decision stages, each of which is assigned two decision variables. By reducing the underlying information process, we effectively establish a new optimization problem with only few decision stages, each of which is assigned several hundred decision variables. Concretely speaking, the future contract period from today's perspective, i.e. the interval from $0 \vee \underline{t}$ to $\overline{t}$, is divided into a number of periods indexed by $s \in \mathcal{S}^+ = \{0, \ldots, \overline{s}\}$. These periods must be of (possibly different) integer lengths. For each $s \in \mathcal{S}^+$, let i_s be the first rebalancing interval belonging to period s, and set $i_{\overline{s}+1} = \overline{i} + 1$. Usually, we have $|\mathcal{S}^+| \ll |\mathcal{I}^+|$ implying that each period covers many rebalancing intervals. The exercise strategy or 'load pattern' within period s is given by the following vector of power decisions

$$\boldsymbol{p}_s = (p_{i_s}, \dots, p_{i_{s+1}-1}) \in \mathbb{R}^{d_s},$$

where $d_s = i_{s+1} - i_s$. With a slight abuse of notation we denote by e_s the amount of energy exercised until the end of stage s. Of course, by virtue of the energy balance equation, e_s is fully determined by the preceding power decisions. For the sake of transparent notation, we define C_s as the set of load patterns in period s which comply with the power and ramping constraints, i.e.

$$\begin{aligned} C_s := \big\{ \boldsymbol{p}_s \in \mathbb{R}^{d_s} : \ & \underline{p}_i \le p_i \le \overline{p}_i \quad \text{for } i_s \le i < i_{s+1} \quad \text{and} \\ & |p_i - p_{i-1}| \le \varrho_i \quad \text{for } i_s < i < i_{s+1} \big\}. \end{aligned}$$

By construction, C_s is convex polyhedral and compact. Next, for the sake of transparent notation, introduce a d_s-dimensional random vector

$$\boldsymbol{S}_s := (S(i_s), S(i_s + 1), \dots, S(i_{s+1} - 1)),$$

and let $\mathbf{1}_m := (1, \dots, 1) \in \mathbb{R}^m$ for all $m \in \mathbb{N}$. With these conventions, the amount of energy exercised within period s is given by $\langle \mathbf{1}_{d_s}, \boldsymbol{p}_s \rangle$, while the profit earned in period s amounts to $\langle \boldsymbol{S}_s - K\mathbf{1}_{d_s}, \boldsymbol{p}_s \rangle$. As a first important approximation, we assume now that spot price updates are no longer observed at all intervals $i \in \mathcal{I}^+$ but only at intervals $i \in \{i_s | s \in \mathcal{S}^+\} \subset \mathcal{I}^+$. Thus, from now on the option holder's decisions $\boldsymbol{p}_s$ and e_s will depend on the information $\mathcal{H}_s$ available at period s.

$$\mathcal{H}_s = \sigma(\boldsymbol{\xi}_{i_r} | \, r = 0, \dots, s) \quad s \in \mathcal{S}^+$$

By definition, $\mathcal{H}_s$ is a real sub-σ-algebra of $\mathcal{G}_i$ for all $i \ge i_s$, and the approximate pricing problem associated with the reduced information process $\{\mathcal{H}_s\}_{s \in \mathcal{S}^+}$ is given by the multistage stochastic program

$$\max E \sum_{s \in \mathcal{S}^+} \langle \boldsymbol{F}_s - K\mathbf{1}_{d_s}, \boldsymbol{p}_s \rangle \tag{8}$$

$$\text{s.t.} \quad \left. \begin{array}{l} \underline{e} \leq e_{\overline{s}} \leq \overline{e} \\ \left. \begin{array}{l} \boldsymbol{p}_s \in C_s \\ \langle \mathbf{1}_{d_s}, \boldsymbol{p}_s \rangle = e_s - e_{s-1} \\ \boldsymbol{p}_s, e_s \ \mathcal{H}_s\text{-measurable} \end{array} \right\} \forall s \in \mathcal{S}^+ \end{array} \right\} Q\text{-a.s.}$$

Here, we used the law of iterated conditional expectations to replace $\boldsymbol{S}_s$ by the random vector

$$\boldsymbol{F}_s := E(\boldsymbol{S}_s | \mathcal{H}_s).$$

Energy target constraints are as in (6), while power and ramping constraints are formulated abstractly by using the compact sets $\{C_s\}_{s \in \mathcal{S}^+}$. However, notice that the approximate pricing problem (8) disregards the ramping constraints at each rebalancing interval i_s for $s \in \mathcal{S}^+$. This deficiency could principally be remedied at the cost of additional notation, but the gain in precision would be modest since $|\mathcal{S}^+| \ll |\mathcal{I}^+|$. As usual, neighboring decision stages are coupled by the energy balance equation, and the dummy variable e_{-1} stands for the amount of energy exercised in the past. In other words, energy purchased (sold) until the end of period s must equal energy purchased (sold) until the end of period $s-1$ and energy purchased (sold) in period s. In contrast to the exact pricing problem (6), the decisions $\boldsymbol{p}_s$ and e_s are required to be non-anticipative with respect to the reduced information process $\{\mathcal{H}_s\}_{s \in \mathcal{S}^+}$.

5.2 Discretization of the Probability Space

The approximate pricing problem (8) remains computationally intractable as it exhibits infinitely many scenarios. The standard approach to overcome this difficulty is to discretize the underlying probability space. In doing so, one effectively approximates the stochastic program (8) by an optimization problem over a finite-dimensional Euclidean space, which is numerically tractable.

The selection of an appropriate discrete probability measure Q^d is referred to as *scenario generation* and represents a primary challenge in the field of stochastic programming. A survey and critical assessment of modern scenario generation techniques is provided in [15]. Once a suitable discrete measure Q^d is identified, we approximate (8) by the stochastic program

$$\max\ E^d \sum_{s\in\mathcal{S}^+} \langle \boldsymbol{F}_s - K\mathbf{1}_{d_s}, \boldsymbol{p}_s\rangle \tag{9}$$

$$\text{s.t.}\quad \left.\begin{array}{l} \underline{e} \leq e_{\overline{s}} \leq \overline{e} \\ \left.\begin{array}{l} \boldsymbol{p}_s \in C_s \\ \langle \mathbf{1}_{d_s}, \boldsymbol{p}_s\rangle = e_s - e_{s-1} \\ \boldsymbol{p}_s, e_s\ \mathcal{H}_s\text{-measurable} \end{array}\right\} \forall s \in \mathcal{S}^+ \end{array}\right\} Q^d\text{-a.s.}$$

Here, $E^d(\cdot)$ denotes expectation with respect to Q^d. There is a vast number of scenario generation methods for which the difference of the optimal values of problems (8) and (9) can be estimated in a quantitative sense. So-called *bounding methods* [5, Chap. 9] allow for a strict estimate of the approximation error whereas *sampling-based methods* [5, Chap. 10] provide a probabilistic estimate. Sometimes, the difference of the optimal solution sets can be estimated, too; see e.g. [22, Sect. 1.3] or [18, Sects. 4.6 and 5.5]. Usually, the approximation can be improved by adding new atoms to the discrete measure Q^d.

The calculations of Sect. 6 are based on a moment matching method proposed in [28, Sect. 7.3]. This particular scenario generation technique approximates the conditional distributions of the risk factors by transformed multinomial distributions with 3 outcomes and k trials, k being a tuning parameter. The approximate conditional distributions have $(k+2)(k+1)/2$ discretization points and preserve the true first and second-order moments. Moreover, the approximate distribution Q^d of the risk factors converges weakly to the true distribution Q upon increase of the number of discrete scenarios.

5.3 Reduction of the Number of Decision Variables

After reduction of the information process and discretization of the probability space, the resulting stochastic program (9) principally allows for numerical solution. However, in order to speed up computation time, we should reduce the number of decision variables. We begin with a suitable reparameterization of the load patterns in the compact set C_s. To this end, let $\{\boldsymbol{p}_s^n\}_{n=1}^{N_s}$ be the vertices of C_s. Then, any vector $\boldsymbol{p}_s \in C_s$ is representable as a convex combination of the form

$$\boldsymbol{p}_s = \sum_{n=1}^{N_s} \boldsymbol{p}_s^n \, \lambda_s^n, \quad \sum_{n=1}^{N_s} \lambda_s^n = 1, \quad \lambda_s^n \geq 0 \quad \forall n = 1, \ldots, N_s. \tag{10}$$

Notice that any $\boldsymbol{p}_s \in C_s$ can be viewed as an *exercise profile* for period s, which satisfies the power and ramping constraints. In this sense, the vertices of C_s constitute extremal exercise profiles. Every general exercise profile can be represented as a weighted sum of the extremal profiles with convex weights $\{\lambda_s^n\}_{n=1}^{N_s}$. Conversely, every convex combination of the extremal profiles yields an exercise profile in C_s, and thus it satisfies the power and ramping constraints. However, the representation (10) is generally not unique, i.e. there may be different sets of convex weights for the same exercise profile.

So far, our reparameterization has no numerical advantages: in fact, we have even increased the number of decision variables since the dimension of C_s is usually much smaller than the number of its vertices ($d_s \ll N_s$). The key observation is that we are not interested in representing all exercise profiles in C_s but only those which are potentially optimal in the approximate pricing problem (9). As we will argue below, it is sufficient to concentrate on few vertices $\{\boldsymbol{p}_s^n\}_{n=1}^{n_s}$ ($n_s \ll d_s$). By this means we can substantially reduce the number of decision variables in period s.

In order to formulate the pricing problem in the proposed 'exercise profile framework', we need the following definitions:

$$
\begin{aligned}
\boldsymbol{c}_s &:= (c_s^1, \ldots, c_s^{n_s}), \quad c_s^n := \langle \boldsymbol{F}_s - K\mathbf{1}_{d_s}, \boldsymbol{p}_s^n \rangle, \\
\boldsymbol{w}_s &:= (w_s^1, \ldots, w_s^{n_s}), \quad w_s^n := \langle \mathbf{1}_{d_s}, \boldsymbol{p}_s^n \rangle, \\
\boldsymbol{\lambda}_s &:= (\lambda_s^1, \ldots, \lambda_s^{n_s}),
\end{aligned}
$$

with $\boldsymbol{c}_s$ an exogenous random variable, $\boldsymbol{w}_s$ a constant vector, and $\boldsymbol{\lambda}_s$ representing a decision variable for every $s \in \mathcal{S}^+$. Using the above conventions, problem (9) can be approximated by the reparameterized stochastic program

$$
\max E^d \sum_{s \in \mathcal{S}^+} \langle \boldsymbol{c}_s, \boldsymbol{\lambda}_s \rangle \tag{11}
$$

$$
\text{s.t.} \quad \left.
\begin{array}{l}
\underline{e} \le e_{\overline{s}} \le \overline{e} \\
\left.
\begin{array}{l}
\langle \boldsymbol{w}_s, \boldsymbol{\lambda}_s \rangle = e_s - e_{s-1} \\
\langle \mathbf{1}_{n_s}, \boldsymbol{\lambda}_s \rangle = 1 \\
\boldsymbol{\lambda}_s \ge \mathbf{0} \\
\boldsymbol{\lambda}_s, e_s \ \mathcal{H}_s\text{-measurable}
\end{array}
\right\} \forall s \in \mathcal{S}^+
\end{array}
\right\} Q^d\text{-a.s.}
$$

If we take account of all vertices associated with the sets $\{C_s\}_{s \in \mathcal{S}^+}$, then (11) is equivalent to (9). Conversely, neglect of certain vertices corresponding to conjecturally suboptimal exercise strategies may nevertheless entail a gap between the optima of (9) and (11).

In order to make our approach work, we need a heuristics to construct suitable exercise profiles. Below, we will assume that period s is much shorter than $\max\{\alpha_1, \alpha_2\}^{-1}$ for all $s \in \mathcal{S}^+$. This assumption is nonrestrictive for problems of practical relevance. Then, the deterministic vector

$$
\hat{\boldsymbol{F}}_s := E(\boldsymbol{F}_s) = E(\boldsymbol{S}_s) = (F(0, i_s), \ldots, F(0, i_{s+1} - 1)),
$$

characterizing the today's forward curve for period s, is almost parallel to the $\mathcal{H}_s$-measurable random vector $\boldsymbol{F}_s$ for any realization of the risk factors at the period start time. Concretely speaking, there is a $\mathcal{H}_s$-measurable normalization constant M_s such that $\boldsymbol{F}_s \approx M_s \hat{\boldsymbol{F}}_s$.[8] Next, denote by L_s the shadow price of energy at the beginning of period s. Note that L_s depends on the risk factors and cumulative energy, thus being $\mathcal{H}_s$-measurable, too. Moreover, assume the swing option price to be approximately linear affine in energy on intervals of length $(\overline{p}_i - \underline{p}_i)\, d_s$ for all $i \in \mathcal{I}^+$. Then, a promising set of candidate exercise profiles for period s is found by solving the parametric linear program

$$\max_{\boldsymbol{p}_s \in C_s} \langle \hat{\boldsymbol{F}}_s - K_s \mathbf{1}_{d_s}, \boldsymbol{p}_s \rangle \tag{12}$$

for all thinkable realizations K_s of the random variable $(K - L_s)/M_s$. For the further argumentation, it proves useful to introduce upper and lower bounds on the today's forward price curve for period s.

$$\begin{aligned} K_s^+ &:= \max\{F(0,i) \,|\, i = i_s, \ldots, i_{s+1} - 1\} \\ K_s^- &:= \min\{F(0,i) \,|\, i = i_s, \ldots, i_{s+1} - 1\} \end{aligned}$$

For any $K_s \leq K_s^-$ the objective function coefficients in (12) are nonnegative, and the trivial 'maximum power strategy' achieves maximal profit. Conversely, for any $K_s \geq K_s^+$ the objective function coefficients in (12) are nonpositive, and the trivial 'minimum power strategy' is optimal. Consequently, in order to generate interesting exercise profiles, it suffices to focus on those values of the control parameter K_s which lie between K_s^- and K_s^+. In practice, one might wish to choose freely the number $n_s \geq 2$ of exercise profiles associated with period s. Subsequently, one should select a set of candidate parameters

$$K_s^- = K_s^1 < K_s^2 < \cdots < K_s^{n_s} = K_s^+,$$

and the exercise profile $\boldsymbol{p}_s^n$ can conveniently be taken to be some optimal solution of the linear program (12) for the parameter choice[9] $K_s = K_s^n$, $n = 1, \ldots, n_s$.

[8] If we let the mean-reversion coefficients α_1 and α_2 tend to zero (with α_1/α_2 held fixed), we have pointwise convergence of $\boldsymbol{F}_s$ to $M_s \hat{\boldsymbol{F}}_s$ on Ω.

[9] If the forward curve exhibits prominent peaks, the choice of equidistant control parameters

Without loss of generality, each exercise profile represents some vertex of C_s. Moreover, we may always postulate that the first and the last exercise profiles are given by the trivial minimum and maximum power profiles, respectively. These profiles prevent the pricing problem from becoming infeasible in the new formulation (11), i.e. the energy targets in problem (11) can always be met, provided the exact pricing problem (6) is feasible.

5.4 Estimation of the Approximation Error

In this section we will argue that our model facilitates the calculation of a tight lower bound on the option value. To this end, we will discuss the effects of the basic approximations that led from the exact pricing problem (6) to the numerically tractable stochastic program (11).

Problem (8) differs from the exact pricing problem (6) only in that the ramping constraints are disregarded at the period start times i_s, $s \in \mathcal{S}^+$, and the decisions are adapted to the reduced information process $\{\mathcal{H}_s\}_{s \in \mathcal{S}^+}$. Neglect of some few ramping constraints, on one hand, slightly increases the option value. This price appreciation can be estimated above by an error bound $\varepsilon > 0$. For ease of exposition, assume that $\underline{p}_i = \underline{p}$ and $\overline{p}_i = \overline{p}$ are constant, while $\varrho_i = \varrho$ is a constant integer fraction of the power span $\overline{p} - \underline{p}$ for all $i \in \mathcal{I}^+$. Then, a valid error bound is given by

$$\varepsilon := \frac{(\overline{p} - \underline{p})^2}{2\varrho} \sum_{s \in \mathcal{S}^+} E(\|\boldsymbol{S}_s - K\mathbf{1}_s\|_\infty). \tag{13}$$

The first factor $(\overline{p} - \underline{p})^2/2\varrho$ estimates the maximum amount of energy which has to be bought or sold to match the profiles at the period start times. Moreover, $\|\boldsymbol{S}_s - K\mathbf{1}_s\|_\infty$ stands for the maximum net cost for buying or selling energy

may lead to unsatisfactory results; especially for small values of n_s. In practice, one needs a suitable heuristics, which should depend both on the contractual specifications and the forward price data.

in period s. In a well-specified model, where $(\overline{p} - \underline{p})/\varrho$ is much smaller than $\min\{d_s | s \in \mathcal{S}^+\}$, the error bound ε may be considered small. Note that (13) can be generalized to the case of time-dependent power limits and ratchets. The requirement that the decisions be adapted to the coarser information process $\{\mathcal{H}_s\}_{s\in\mathcal{S}^+}$, on the other hand, leads to a decrease of the option premium. This is a direct consequence of the fact that the measurability constraint in (8) is more restrictive than in (6). In summary, we have thus found that the optimal value of the exact pricing problem (6) is at least as large as the optimal value of (8) subtracted by ε. Notice that reduction of the information process is also referred to as *aggregation of decision stages* in stochastic programming literature [5, Sect. 11.2]. For a general discussion of aggregation and disaggregation we refer to the concise treatment by Wright [31]; more specialized results are due to Birge [3, 4].

Next, we discuss the discretization of the probability space. The objective of the stochastic program (8) is linear in the decision variables, and the coefficients are (at least) twice continuously differentiable in the risk factors. Thus, following the recipe in [18, Sect. 5.1],[10] one can systematically construct a discrete probability measure Q^d on $(\Omega, \mathcal{F})$ such that the optimal value of (8) is no smaller than the optimal value of the discretized problem (9) subtracted by some nonnegative constant ε', which depends parametrically on Q^d. This bound can be made arbitrarily tight. In fact, by successively refining the discrete measure Q^d, the optimum of (9) approaches the optimum of (8) while ε' converges to zero.

The final approximation step consists in a reparameterization of the decision space and a reduction of degrees of freedom. Thereby, the set of admissible exercise strategies is further restricted implying that the optimal value of (11) is no larger than the optimal value of (9).

[10]This requires truncation of the risk factors outside some compact domain and estimation of the error thus incurred. Note that the error can be kept small if the compact domain is chosen large enough.

In summary, we conclude that the exact option premium is bounded below by the optimal value of the computationally tractable stochastic program (11) subtracted by the small constants ε and ε'. The calculation of a tight upper bound seems to be more involved. According to the general guidelines for aggregation and disaggregation in [31], one should focus on the dual of the exact pricing problem (6) and restrict the *dual* decisions to be measurable with respect to the reduced information process. An upper bound can then be found by further simplifying the partially aggregated dual pricing problem. However, the arising bound might be difficult to evaluate in situations of practical interest.

6 Numerical Results

Our numerical calculations are based on the Pilipovic-type forward price model developed in Sect. 2. The mean-reversion parameter α_1 is set to $6.849\text{E}{-}04\,\text{h}^{-1}$, while α_2 is set to zero (taking appropriate limits). Furthermore, the volatility coefficients corresponding to the short and long term fluctuations are set to $\sigma_1 = 1.068\text{E}{-}02\,\text{h}^{-1/2}$ and $\sigma_2 = 2.671\text{E}{-}03\,\text{h}^{-1/2}$, respectively. The model is initialized with the forward price curve observed at the swing option's valuation date. However, since only few futures contracts are traded in the market, the theoretical hourly forward price curve is observable only in a limited manner. Thus, we use a software package developed by Jens Güssow for constructing high-resolution forward price curves from available market data. This tool is based on a methodology due to Fleten and Lemming [10]: historical EEX[11] spot price data is used to design a candidate term structure, which reflects the typical seasonal pattern. By applying a constrained least squares optimization scheme, the candidate curve is smoothed and adjusted to be consistent with currently observed EEX futures prices, see Fig. 2.

Let us now turn to the valuation of some typical swing option contracts, all

[11]EEX spot and futures price data is downloadable at http://www.eex.de/.

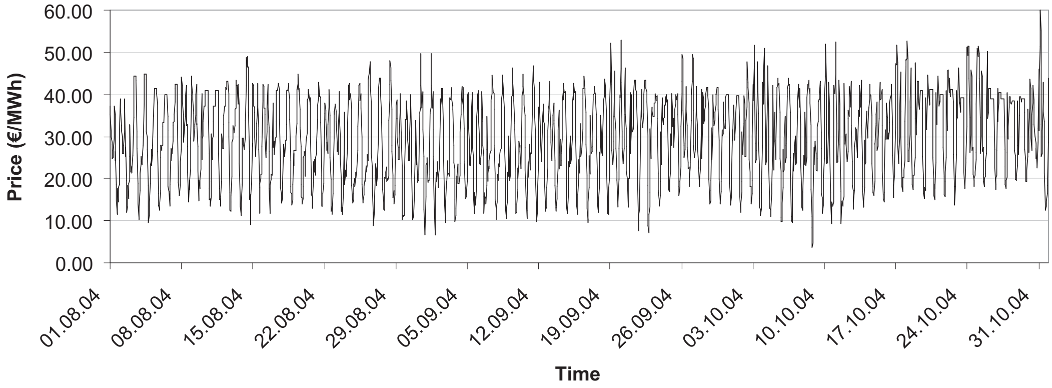

Figure 2: Forward price curve based on EEX futures prices on July 30, 2004

Table 4: Parameters of the Base Contract

$\underline{t}$	$=$	Aug. 1 2004, 00:00	p_{start}	$=$	0 MW
$\overline{t}$	$=$	Oct. 31 2004, 24:00	$\underline{e}$	$=$	0 MWh
Δ	$=$	1 h	$\overline{e}$	$=$	132'480 MWh
K	$=$	20 €/MWh	ϱ_i	$=$	60 MW $\forall i \in \mathcal{I}$
$\underline{p}_i$	$=$	0 MW $\forall i \in \mathcal{I}$	c^+	$=$	$+\infty$ €/MWh
$\overline{p}_i$	$=$	60 MW $\forall i \in \mathcal{I}$	c^-	$=$	$+\infty$ €/MWh

of which are derived from the *base contract* specified in Table 4. By convention, the valuation date corresponds to the contract start date and may be set to $t = 0$ after some suitable 'time translation'. Notice that the base contract's energy limits are redundant (as is the case in regime C in Fig. 1). Thus, it reduces to a portfolio of cash, forwards, and European call options, and its value amounts to

$$\sum_{i \in \mathcal{I}^+} \underline{p}_i \left(F(0,i) - K\right) + \left(\overline{p}_i - \underline{p}_i\right) C(0,i). \tag{14}$$

By assumption, the current forward price curve $\{F(0,t)\}_{t \geq 0}$ is known. Moreover, the time-0 price $C(0,t)$ of a simple European call option with strike price K and maturity t can be calculated analytically. According to (5), the logarithm of the time-t spot price is normally distributed under the risk-neutral measure Q, and its variance can be written as

$$\begin{aligned}\Sigma_t^2 \ := \ & \frac{\sigma_1^2}{2\alpha_1}(1-e^{-2\alpha_1 t}) + \frac{\alpha_1^2}{(\alpha_1-\alpha_2)^2}\left[\frac{\sigma_2^2}{2\alpha_2}(1-e^{-2\alpha_2 t})\right. \\ & \left. + \frac{\sigma_2^2}{2\alpha_1}(1-e^{-2\alpha_1 t}) - \frac{2\sigma_2^2}{\alpha_1+\alpha_2}(1-e^{-(\alpha_1+\alpha_2)t})\right].\end{aligned}$$

Thus, ordinary call options are priced via the Black-Scholes-type formula

$$C(0,t) := F(0,t)\,\Phi\left(\frac{\ln\frac{F(0,t)}{K}+\frac{\Sigma_t^2}{2}}{\Sigma_t}\right) - K\,\Phi\left(\frac{\ln\frac{F(0,t)}{K}-\frac{\Sigma_t^2}{2}}{\Sigma_t}\right), \tag{15}$$

where Φ stands for the standard normal distribution function. Plugging the parameter values of Table 4 into (14) and (15), we can *analytically* evaluate the fair price of the base contract, which amounts to 1.426E+06 €.

If the base contract is assigned restrictive energy limits or nontrivial ratchets, analytical tractability is lost, and a numerical valuation scheme is needed. Numerical calculations are based on the approximations described in Sect. 5. First, the hourly rebalancing intervals within the 3-month contract period are aggregated to at most 12 effective decision stages. The first stage comprises one week, whereas the remaining stages are of equal length. Next, the probability space is discretized by means of the moment matching method proposed in [28, Sect. 7.3]. The corresponding scenario tree has a constant branching factor (number of branches per node) of 1, 3, 6, or 10. Finally, the decision space is reparameterized by concentrating on convex combinations of at most 300 exercise profiles within each decision stage. In practice, it turns out that approximate stochastic programs with 6 effective stages, 3 branches per node, and 6 exercise profiles achieve reasonable accuracy. In this case, we observe solution times of less than 4 seconds,[12] which ensures that our tool can be used by energy traders

[12] All calculations were carried out on a 1 GHz Pentium III PC with 512 MB storage. The respective deterministic equivalent programs (DEQ), that is, instances of problem (11), were generated by an inhouse-developed matrix generator written in C++. As the DEQs are sparse large-scale linear programs, standard linear programming routines of the widely-used solver ILOG CPLEX, Version 9.1, had been employed for their solution.

for real-time price negotiations over the telephone.

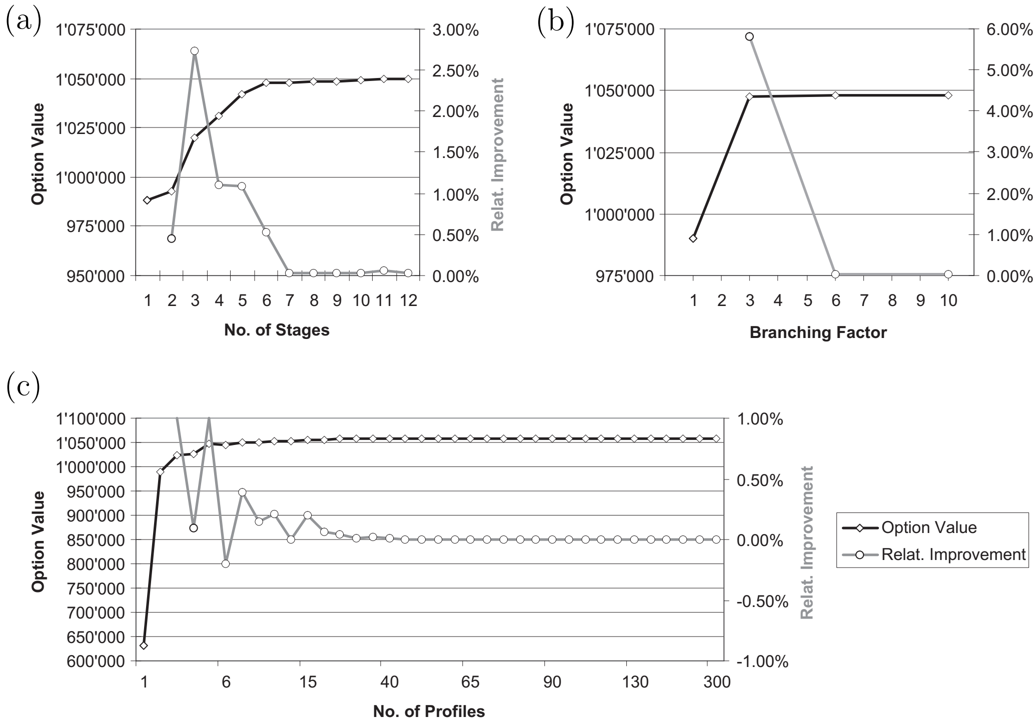

Figure 3: Convergence of the three approximations

The *numerical* value of the base contract amounts to 1.420E+06 €, thus being accurate within -0.4%. This high level of precision in a special case suggests that our numerical valuation scheme generally performs well. Note that we could still improve precision at the cost of additional runtime. If nontrivial energy limits or ramping constraints are introduced, the contract becomes path-dependent and analytically *in*tractable. In order to estimate the accuracy of our valuation scheme for general path-dependent contracts, we perform a numerical convergence analysis based on a modified base contract with $\overline{e} = 52'992\,\mathrm{MWh}$. Note that this contract may be exercised 40% of the time within the planning horizon. Figure 3 shows convergence of the option value as (a) the number of effective decision stages, (b) the branching factor, and (c) the number of exercise profiles are increased. The number of stages is fixed to 6 in (b) and (c), the branching factor is fixed to 4 in (a) and (c), and the number of exercise profiles is fixed to 6 in (a) and (b). We observe convergence in all three cases. Figure 3 suggests that approximations based on 6 effective stages, 3 branches per node, and 6 exercise profiles are reasonably accurate. Problem size grows exponentially with the number of stages and polynomially with the number of branches per node. Thus, any increase of the number of stages or the branching factor is computationally expensive. However, problem size grows only linearly with the number of decision variables per node. Thus, introducing additional exercise profiles is relatively cheap in terms of computation time.

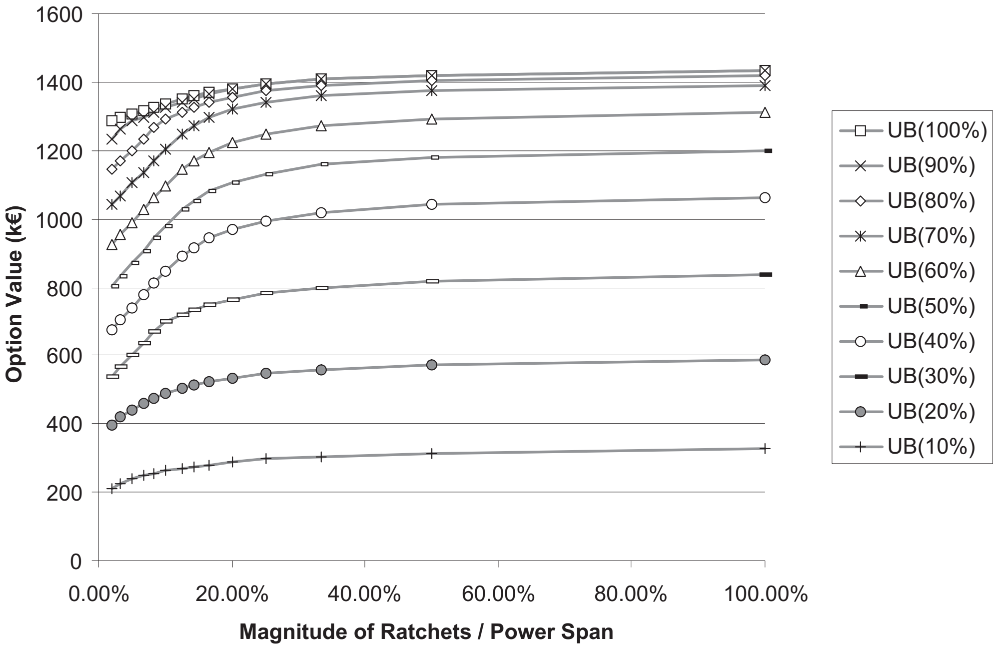

Figure 4: Option value in dependence of the magnitude of the ratchets and for different values of the upper energy limit

For illustrative purposes, we consider several interesting variants of the base contract. On one hand, we reduce the upper energy limit in steps of 10%. On the other hand, we introduce uniform ratchets, which are integer fractions of the power span $\overline{p}_i - \underline{p}_i$. Both modifications entail a successive decrease of the option value, see Fig. 4. If the ratchets tend to zero, the exercise strategy $\{p_i\}_{i\in\mathcal{I}^+}$ becomes constant, and since start power is set to 0 MW, the swing option becomes worthless. However, for very small ratchets our numerical valuation scheme turns obsolete as the ramping constraints are disregarded at the period start times.

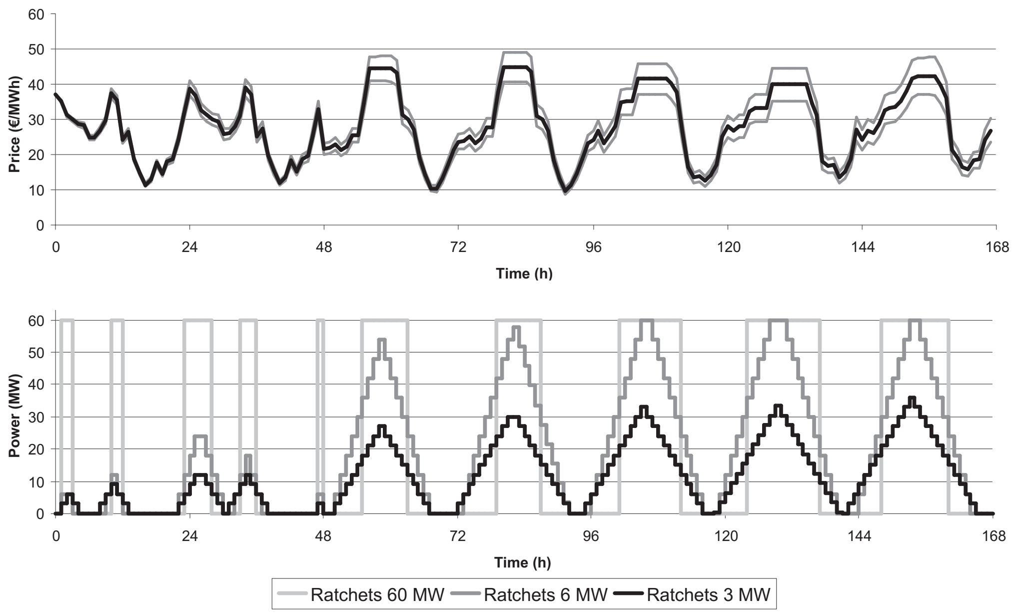

Figure 5: Top: expected value and 67% confidence interval of the risk-neutral spot price distribution in the first contract week; bottom: exercise strategies for three contracts with increasingly restrictive ramping constraints

The forward price curve used in our calculations is visualized in the top diagram of Fig. 5; remember that the forward prices coincide with the expected spot prices under the risk-neutral measure Q. Once having solved the approximate pricing problem (11), an efficient exercise strategy for the first contract week is found by solving the deterministic maximization problem

$$\max \sum_{i=0}^{d_1-1} (F(0,i) - K)\, p_i \tag{16}$$

$$\text{s.t.} \quad \sum_{i=0}^{d_1-1} p_i = e_0^*$$

$$\left.\begin{array}{l} \underline{p}_i \le p_i \le \overline{p}_i \\ |p_i - p_{i-1}| \le \varrho_i \end{array}\right\} \forall i = 0, \ldots, d_1 - 1.$$

Here, the optimal first-stage decision e_0^* of the stochastic pricing problem (11) enters the constraints of (16) as a parameter and specifies the energy target for the first contract week. Moreover, the dummy variable p_{-1} is set to the start power p_{start}. Notice that the short term optimization (16) ensures consistency with the contractual provisions and the energy target provided by (11). Furthermore, it guarantees optimality with respect to the current forward price curve. For practitioners it is indeed desirable to evaluate a *deterministic* short term strategy since exercise schedules must usually be submitted (and thus known) several days ahead.

The bottom diagram in Fig. 5 shows the short term exercise strategies for three modified base contracts with ratchets and a 50% upper energy limit.

7 Conclusions

In this article we study a rich family of electricity swing options. We formulate an exact pricing scheme in a Pilipovic-type model economy driven by two exogenous risk factors. Notice, however, that our pricing scheme is flexible enough to cope easily with a larger number of risk factors. In order to convert the exact pricing model to a computationally tractable stochastic program, we apply three fundamental approximations: aggregation of decision stages, discretization of the probability space, and reparameterization of the decision space. The suggested approach facilitates calculation of a tight lower bound on the option premium. Investigation of analytically tractable limiting cases indicates that our methodology achieves a high degree of precision. As opposed to classical dynamic programming or Least Squares Monte Carlo techniques, our stochastic programming approach performs well in the presence of various risk factors and state variables. It applies even to contracts with a high number of exercise times (such as hourly exercisable swing options with a delivery period of one year). Our model is extremely flexible with respect to new contractual provisions and additional constraints. Future research will be concerned with the determination of a tight upper bound on the option premium and generalization of the presented pricing scheme to real pumped hydropower plants.

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