Portfolio and Consumption Decisions under Mean-Reverting Returns and Mean-Reverting Labor Income Growth*

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Comments welcome.

Abstract

This papers focus is on consumption/portfolio optimization under time-varying investment opportunities and dynamic non-financial (labor) income using analytical methods. Previous papers either require approximations or numerical methods. This paper overcomes these issues by assuming that markets are complete in continuous time. The analytical solution leads to a clearer understanding of optimal consumption and investment over horizon and states of the economy. Given a state of the economy, the most striking result is that counter-cyclical non-financial income growth (income growth is low when expected returns are high) can lead to a strong reduction of optimal risky investment and consumption. Given a time horizon, counter-cyclical income growth can lead to falling consumption and lower risky investment even if expected returns are increasing.

Key Words: consumption/portfolio optimization, labor income, time-varying investment opportunities, time-varying labor income growth, HARA-utility, HJB-equation, closed-form solutions

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Consumption and portfolio optimization under time-varying investment opportunities and dynamic non-financial income has not received much attention until now. Three exceptions are Benzoni et al. (2007), Lynch and Tan (2009)[1] and Munk and Sørensen (2010). These papers focus on rather complicated life-cycle models and have to rely on numerical methods. Their results imply that dynamic labor income matters for consumption and portfolio decisions. In this paper we approach the optimization problem with analytical methods. We are able to reproduce results from the aforementioned studies and offer considerably more theoretical insights and implications for empirical research.

As a first main result we state that the inclusion of time-varying labor income growth leads to highly individual optimal policies. Compared to myopic and classical state variable hedging demand, the magnitude of new hedging demand, which arises under dynamic labor income, is important. Similar to Lynch and Tan (2009), we show that for realistic parameter values risky investment can be considerably reduced. In some cases, individuals do not want to hold a positive amount of risky assets at all. Hence, the fact that some individuals do not hold any equity[2] can be confirmed. As a second main result we stress the importance of the sensitivity of the optimal policies across states of the economy. Previous studies neglect this dimension. Most importantly, we show that the magnitude of new labor hedging demand and the sensitivity of the optimal policies are in a close relation. Specifically, a high magnitude implies a high sensitivity across states. In our basic framework, which is close to Lynch and Tan (2009), with time variation in expected returns and labor income growth, unrealistically sensitive optimal policies result. In a modification with constant investment opportunities we find less sensitive and thus more realistic policies.

[1]The paper by Lynch and Tan has been accepted for future publication in the *Journal of Financial Economics*, `http://jfe.rochester.edu/forth.htm` (10th January 2011).

[2]See, for example, Figure 2 of Campbell (2006).

This paper contributes to the literature of consumption and portfolio selection in stochastic environment and to life-cycle models in the presence of labor income. In addition to the aforementioned main results, our further contributions are as follows. Firstly, under the assumptions of perfect correlation between the risky asset and labor income or locally riskfree labor income, the separation of the Hamilton-Jacobi-Bellmann (HJB) equation into ordinary differential equations is still possible[3]. Secondly, we derive a closed-form solution for the value of one unit of future income. With the help of this formula we are able to interpret the value of future income across states. Thirdly, the inclusion of time variation in labor income leads to an adaption of state variable hedging demand. In fact, state variable hedging demand can be separated into the usual part that arises in the absence of labor income and a new part. This part grows monotonically with the planning horizon and can have either sign. Fourthly, a negative sensitivity of labor income growth on the state variable can induce a falling consumption-wealth ratio even if expected returns are increasing in the state variable. This is in contrast to similar models of Wachter (2002) and Campbell et al. (2004) without labor income. Moreover, the level of risky investment can be reduced. Finally, from a technical point of view, the valuation of the labor income stream involves solving ordinary differential equations. Certain combinations of state variable and financial market parameters lead to solutions which do not converge for long horizons. In these cases, the valuation of the income stream leads to extreme results even if the sensitivity of labor income growth to the state variable is low. This result must be considered as a warning for numerical studies of the consumption-investment problem that are calibrated on empirical results.

The importance of non-financial income as labor income was recognized early. Merton (1971) introduced a constant wage and his work was extended in several dimensions. Duffie et al. (1997) discuss properties of the value function and the optimal policies under

[3]The conditions for models without labor income are extensively discussed in Liu (2007).

stochastic labor income. Koo (1998) and He and Pagès (1993) rule out short selling and/or impose borrowing constraints under non-insurable labor income risk. Heaton and Lucas (2000a, b) point out the importance of entrepreneurial risk. Munk (2000) and Viceira (2001) calibrate labor income models to realistic financial and labor market data. Bodie et al. (1992) introduce flexible labor supply. Cocco et al. (2005) examine the effect of different labor income risk on the optimal portfolio policies of a life-cycle investor under a realistic calibration of the labor income process. Benzoni et al. (2007) introduce a model where dividends and labor income are cointegrated. More recent work of Lynch and Tan (2009) and Munk and Sørensen (2010) allows for time variation in the moments of financial assets and labor income.

To be more precisely with respect to our model specification, we solve the consumption investment problem of an individual facing a dynamic financial market and dynamic non-financial income (labor income). The financial market consists of two assets. One is a riskless bond and a risky asset with mean-reverting returns. Thus, the financial market setting is up to an invariant affine transformation identical to Kim and Omberg (1996), Wachter (2002) and Campbell et al. (2004). In addition, it is assumed that the individual faces outside labor income that has a mean-reverting growth rate. There is a single state variable that drives both the risky asset and labor income. Our model is closest to the model by Lynch and Tan (2009)[4]. Their model is more realistic with respect to the model assumptions, but has to rely on numerical methods. For this reason, the reported effects can only be interpreted with a certain depth and sensitivity analysis over states is neglected. In fact, Lynch and Tan focus exclusively on the development of the optimal policies over the life-cycle and omit the sensitivity of the results over states. It is shown

[4]As mentioned above, another related model is that of Munk and Sørensen (2010) but their focus is on combined bond-stock problems and they assume that only interest rates and not stock returns are subject to time-variation.

that the magnitude of new labor hedging demand is monotonically decreasing with the time horizon and hence the development of the optimal policies over the horizon does not show any surprising patterns.

For the sake of closed-form solutions more restrictive assumptions have to be taken. In fact, it has to be assumed that labor income is either locally riskfree or perfectly correlated with the risky asset. Nevertheless, these assumptions come with an advantage. In particular, it is shown that the assumption allows the inclusion of a subsistence level of consumption (HARA preferences) without having to assume that initial financial wealth exceeds the value of the future subsistence consumption stream. Furthermore, and similarly to Wachter (2002) it must be assumed that the state variable and the risky asset are perfectly correlated. However, for the dividend yield this assumption is not too problematic as the correlation is close to -1.

The assumptions of a single state variable and complete markets ease the mathematical derivation and the interpretation of the results. There are several reasons that suggest that the assumption of a single state variable is not too restrictive. Firstly, the use of only one state variable that drives both financial assets and labor income is motivated by the empirical estimation in Lynch and Tan who use the dividend yield as state variable[5]. Secondly, empirical macroeconomic literature as Stock and Watson (1999) shows clear variation in wages/employment and of the financial market at business-cycle frequency. Finally, from a theoretical point of view, factors that drive capital and labor markets simultaneously are reasonable since most output is produced by a combination of labor and capital.

Moreover, the primary intention of the paper is to evaluate whether dynamic labor income matters. For this reason, the model is kept as simple as possible. Nevertheless, the model allows for extensions in several directions. Lifetime uncertainty models with

[5]See Table 1 in Lynch and Tan (2009, p. 44).

simple law of mortality as, for example, Pliska and Ye (2007). More realistic growth pattern for labor income over the life-cycle as, for example, Cocco et al. (2005). The inclusion of a stochastic short rate as, for example, Munk et al. (2004) or Munk and Sørensen (2010). On the other hand, the inclusion of more realistic restrictions as borrowing and/or short selling constraints as in Lynch and Tan (2009) make closed-form solutions impossible.

The rest of this paper is organized as follows. In Section 1, the model with preferences over intermediate consumption is introduced. Section 2 discusses the analytical properties of the optimal policies and illustrates the result for numerically realistic parameter values. Section 3 modifies the model that only labor income growth and not the investment opportunities are time-varying. The final section concludes. Mathematical derivations as the solution of the HJB-equation and other non-trivial derivations are provided in the Appendices A - D.

1 Model with Utility over Consumption

For the sake of simplicity, we keep the model simple. The conditional expected utility over the remaining lifetime for an individual at t is

$$E_t\left[\int_t^T \frac{e^{-\delta s}}{1-\gamma}\left(c\left(s\right)-\bar{c}\right)^{1-\gamma} ds\right],\ \gamma > 1$$

where $\bar{c} > 0$ is the subsistence level of consumption, $\delta \geq 0$ is the time discount parameter and $\tau \equiv T - t$ is the fixed and certain time horizon. In this section, we assume that the risky assets' expected return is affine in a state variable and has constant volatility. In particular, we assume that

$$\frac{dS_1\left(t\right)}{S_1\left(t\right)} = \left(\lambda_1 X\left(t\right) + r_0\right) dt + \sigma_s dW_s\left(t\right) \tag{1}$$

where $\lambda_1 > 0$ and $\sigma_s > 0$. r_0 is the short rate and the riskless asset follows

$$\frac{dS_0(t)}{S_0(t)} = r_0 dt$$

It should be noted that in this framework, the market price of risk is linear in $X(t)$

$$\theta(t) \equiv \frac{\lambda_1}{\sigma_s} X(t)$$

The state variable dynamics are given by

$$dX(t) = -\kappa_x \left(X(t) - \bar{X}\right) dt + \sigma_x dW_x(t) \tag{2}$$

where $\kappa_x \geq 0$, $\bar{X} \geq 0$ and $\sigma_x > 0$. (2) is a well-known Ornstein-Uhlenbeck process.

The specification of the investment opportunities is in accordance with Liu (2007), who analyzes in a general model consumption-investment problems without labor income in a stochastic opportunity set. Moreover, the financial market setting is one-to-one similar[6] to Kim and Omberg (1996), Wachter (2002) and Campbell et al. (2004). Hence, the effect of the inclusion of a stochastic labor income can be directly compared with the results of Wachter[7].

$Y(t)$ denotes risky labor income and follows

$$\frac{dY(t)}{Y(t)} = (y_0 + y_1 X(t))\, dt + \sigma_y dW_y(t) \tag{3}$$

where y_0 is the constant part of labor income growth, y_1 is the sensitivity of labor income growth on the state variable and $\sigma_y \geq 0$.

We assume that the life of an individual consists of a phase of employment and a phase of retirement. The individual is active in the labor market from time 0 till time T_r. Afterwards she retires. During the phase of employment the individual earns a dynamic labor income (3), consumes and invests in a riskless and a risky asset. During

[6]Of course, there are some changes in notation.

[7]We would like to thank Jessica Wachter for providing us with her original Matlab code. With the help of her code we were able to verify our results and our code.

retirement, $T_r < t \leq T$, the individual has no non-financial income and has to ensure consumption from accumulated financial wealth.

1.1 Phase of Retirement

For $t \geq T_r$, it has to be noticed that during retirement consumption $c(t)$ must exceed the subsistence level $\bar{c}$. For this reason, feasible consumption plans over the phase of retirement exist under the assumption

$$\hat{A}(T) = A(T) - \frac{\bar{c}}{r_0} \geq 0 \tag{c.1}$$

As will become more clear below, (c.1) ensures that at the beginning of the phase of retirement financial wealth and pension benefits are sufficient to afford the future subsistence consumption.

During the phase of retirement, the financial wealth dynamics of an investor are as follows

$$dA(t) = [\pi(t) A(t) \lambda_1 X(t) + A(t) r_0 - c(t)] dt + \pi(t) A(t) \sigma_s dW_s(t) \tag{4}$$

For the phase of retirement, the following results hold.

Proposition 1 *For $T_r < t \leq T$, under condition (c.1) and $\rho_{sx} \in \{-1, 1\}$ and one obtains*

$$J(t, X, A) = \frac{e^{-\delta(T-\tau)} \left[\int_0^\tau e^{\frac{1}{\gamma}\left(c_0(s)+c_1(s)X+\frac{1}{2}c_2(s)X^2\right)} ds\right]^\gamma (A - R(\tau))^{1-\gamma}}{1-\gamma}$$

where $\tau \equiv T - t$. The reserves for covering subsistence consumption follow

$$R(\tau) = \frac{\bar{c}}{r_0}\left(1 - e^{-r_0\tau}\right)$$

The solutions of $c_0(s)$, $c_1(s)$ and $c_2(s)$ are identical to Wachter (2002).

Optimal consumption and risky investment are given by

$$c_t^* = \frac{A - R(\tau)}{\int_0^\tau e^{C(s,X)} ds} + \bar{c} \tag{5}$$

$$
\begin{aligned}
A\pi_t^* &= \frac{1}{\gamma}\frac{\lambda_1}{\sigma_s^2}X\left(A-R\left(\tau\right)\right) \\
&\quad +\frac{1}{\gamma}\frac{\rho_{sx}\sigma_x}{\sigma_s}\frac{\int_0^\tau\left(c_1\left(s\right)+c_2\left(s\right)X\right)e^{C(s,X)}ds}{\int_0^\tau e^{C(s,X)}ds}\left(A-R\left(\tau\right)\right) \qquad (6)
\end{aligned}
$$

where $C\left(s,X\right)\equiv c_0\left(s\right)+c_1\left(s\right)X+\frac{1}{2}c_2\left(s\right)X^2$

Since the proof of Proposition 1 is a special case of the proof of Proposition 2, it is omitted.

It should be noticed that the solution is almost identical to the solution of Wachter (2002) with the exception of subsistence consumption and the corresponding reserves. If $\bar{c}=0$ the solutions are identical[8].

1.2 Phase of Employment

For the phase of employment, $t\leq T_r$, and the specified income, financial wealth follows

$$
\begin{aligned}
dA\left(t\right) &= \left[\pi\left(t\right)A\left(t\right)\lambda_1X\left(t\right)+A\left(t\right)r_0+Y\left(t\right)-c\left(t\right)\right]dt \\
&\quad +\pi\left(t\right)A\left(t\right)\sigma_s dW_s\left(t\right) \qquad (7)
\end{aligned}
$$

The HJB is given by

$$
\begin{aligned}
0 &= J_t+\sup_c\left[e^{-\delta t}\frac{\left(c_t-\bar{c}\right)^{1-\gamma}}{1-\gamma}-J_Ac_t\right] \\
&\quad +\sup_\pi\left[\begin{array}{l} J_A\pi\left(t\right)A\left(t\right)\lambda_1X\left(t\right)+\frac{1}{2}J_{AA}\pi\left(t\right)^2A\left(t\right)^2\sigma_s^2 \\ \\ +J_{AX}\pi\left(t\right)A\left(t\right)\rho_{sx}\sigma_s\sigma_x+J_{AY}\pi\left(t\right)A\left(t\right)Y\left(t\right)\rho_{sy}\sigma_s\sigma_y \end{array}\right] \\
&\quad +J_A\left[A\left(t\right)r_0+Y\left(t\right)\right]-J_X\kappa_x\left(X\left(t\right)-\bar{X}\right)+J_YY\left(t\right)\left(y_0+y_1X\left(t\right)\right) \\
&\quad +\frac{1}{2}J_{XX}\sigma_x^2+\frac{1}{2}J_{YY}Y\left(t\right)^2\sigma_y^2+J_{XY}Y\left(t\right)\rho_{xy}\sigma_x\sigma_y \qquad (8)
\end{aligned}
$$

where $dW_sdW_x=\rho_{sx}dt$, $dW_sdW_y=\rho_{sy}dt$ and $dW_xdW_y=\rho_{xy}dt$.

The first order conditions (FOCs) are given by

$$
c_t^*=\left(e^{\delta t}J_A\right)^{-\frac{1}{\gamma}}+\bar{c} \qquad (9)
$$

[8]More comments on the solutions of the Wachter model can be found in Section 1.2.1.

and

$$A(t)\pi_t^* = -\frac{J_A}{J_{AA}}\frac{\lambda_1}{\sigma_s^2}X(t) - \frac{J_{AX}}{J_{AA}}\frac{\rho_{sx}\sigma_x}{\sigma_s} - \frac{J_{AY}}{J_{AA}}\frac{\rho_{sy}\sigma_y}{\sigma_s}Y(t) \tag{10}$$

Plugging in the FOCs (9) and (10) into the HJB equation (8) yields

$$\begin{aligned} 0 &= J_t + \frac{\gamma}{1-\gamma}e^{-\frac{\delta}{\gamma}t}J_A^{1-\frac{1}{\gamma}} - J_A\bar{c} + J_A A(t) r_0 + J_A Y(t) \\ &\quad -J_X\kappa_x\left(X(t) - \bar{X}\right) + J_Y Y(t)\left(y_0 + y_1 X(t)\right) \\ &\quad +\frac{1}{2}J_A A(t)\pi_t^*\lambda_1 X(t) + \frac{1}{2}J_{AX}A(t)\pi_t^*\rho_{sx}\sigma_x\sigma_s \\ &\quad +\frac{1}{2}J_{AY}A(t)Y(t)\pi_t^*\rho_{sy}\sigma_y\sigma_s \\ &\quad +J_{XY}Y(t)\rho_{xy}\sigma_x\sigma_y + \frac{1}{2}J_{YY}Y(t)^2\sigma_y^2 + \frac{1}{2}J_{XX}\sigma_x^2 \end{aligned} \tag{11}$$

One tries a value function of the following form

$$J(t,X,A,Y) = \frac{e^{-\delta t}\left[\int_0^\tau e^{\frac{1}{\gamma}\left(c_0(s)+c_1(s)X+\frac{1}{2}c_2(s)X^2\right)}ds\right]^\gamma \left(A + k(\tau_r, X)Y - R(\tau)\right)^{1-\gamma}}{1-\gamma} \tag{12}$$

where $\tau \equiv T - t$ and $\tau_r \equiv T_r - t$. $k(X, \tau_r)$ is a function of the state variable and the time horizon until retirement and $R(\tau)$ is a function of the time horizon until the end of the planning horizon. Both will be parameterized below.

For the sake of readability, the solution of the HJB (11) is shown in Appendix A. As we focus on closed-form solutions some assumptions have to be implemented. As in Wachter (2002), it must be assumed that the risky asset and the state variable have to be perfectly correlated.

$$\rho_{sx} \in \{-1, 1\} \tag{c.2}$$

Furthermore, it has to be assumed that either

$$\rho_{sy} \in \{-1, 1\} \Rightarrow \rho_{xy} = \rho_{sx}\rho_{sy} \in \{-1, 1\} \tag{c.3}$$

or

$$\sigma_y = 0 \tag{c.4}$$

Admittedly, these assumptions do not match reality one-to-one. Nevertheless, several papers have shown that the results of exactly solvable special cases are qualitatively similar to cases with non-perfect correlation[9]. Hence, we expect that the qualitative results hold for more general cases.

Furthermore, in Campbell and Viceira (1999), Barberis (2000), Wachter (2002), Campbell et al. (2004) and Lynch and Tan (2009) the dividend yield was chosen as the state variable. As shown below, the dividend yield has a correlation to equity close to -1 and thus the assumption $\rho_{sx} = -1$ is, in this case, not too restrictive.

As shown by Koo (1998) and Duffie et al. (1997), without assumption (c.3) or (c.4), the ratio of financial wealth to labor income becomes essential in order to determine the value of the future income stream and this leads to highly non-linear partial differential equations that have no explicit solution. The importance of the financial wealth-income ratio for the valuation of the income stream is intuitive. Imagine an individual with a labor income stream that is risky and not perfectly correlated with the financial market. In this case, financial wealth of an individual can go to zero and the income stream can still have a high value; then the individual might start to borrow in order to invest in risky assets. Now, the wage could suffer from a severe negative shock and go to zero as well. This leads to a situation where the individual is left without labor income and debts and thus she is not able to afford any consumption, which is clearly not optimal[10]. As shown in, for example, Huang and Milevsky (2008), solutions to labor income problems are available under the assumption of perfect correlation between labor income and the financial market or riskfree labor income. Finally, if labor income volatility is rather low, the locally riskfree labor income case $\sigma_y = 0$ is certainly a reasonable approximation.

[9] See Cocco et al. (2005), Huang et al. (2008), Huang and Milevsky (2008), Bick et al. (2009) and Dybvig and Liu (2010).

[10] Assuming a standard utility function as power utility with a coefficient of risk aversion greater than one. For these utility functions $\lim_{c \to 0_+} u(c) = -\infty$.

Hence, it can be stated that despite these assumptions, the results of the model are not only of theoretical interest, but have implications for realistic cases.

Finally, these assumptions come with an advantage besides the interpretability of closed-form solutions. In fact, in the case of $\rho_{sy} \notin \{-1, 1\}$ and $\sigma_y > 0$, current financial wealth has to be higher than the reserves for the future subsistence consumption. This would be an unrealistic assumption, especially for young individuals who generally have a low financial wealth.

Similarly to the models without labor income, the final HJB (37) of Appendix A can be separated into ordinary differential equations.

1.2.1 Separation of the HJB by A

Separating the HJB (37) of Appendix A by A gives the following equation

$$0 = \int_0^\tau e^{C(s,X)} \left\{ \begin{array}{l} -\delta - \left(\frac{\partial c_0(s)}{\partial s} + \frac{\partial c_1(s)}{\partial s} X + \frac{1}{2} \frac{\partial c_2(s)}{\partial s} X^2 \right) + (1-\gamma) r_0 \\ -\kappa_x \left(c_1(s) X + c_2(s) X^2 \right) + \kappa_x \bar{X} \left(c_1(s) + c_2(s) X \right) \\ +\frac{1}{2} \frac{1-\gamma}{\gamma} \frac{\lambda_1^2}{\sigma_s^2} X^2 + \frac{1-\gamma}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 X \left(c_1(s) + c_2(s) X \right) \\ +\frac{1}{2} \frac{1}{\gamma} \sigma_x^2 \left(\gamma c_2(s) + c_1^2(s) + 2 c_1(s) c_2(s) X + c_2(s)^2 X^2 \right) \end{array} \right\} ds \quad (13)$$

which can be separated by X^2, X and constant terms into three ordinary differential equations.

$$\frac{\partial c_2(s)}{\partial s} = k_0 + k_1 c_2(s) + k_2 c_2(s)^2 \quad (14)$$

$$\frac{\partial c_1(s)}{\partial s} = k_3 c_2(s) + \frac{k_1}{2} c_1(s) + k_2 c_2(s) c_1(s) \quad (15)$$

$$\frac{\partial c_0(s)}{\partial s} = k_5 + k_3 c_1(s) + k_4 c_2(s) + \frac{k_2}{2} c_1(s)^2 \quad (16)$$

with initial conditions $c_2(0) = c_1(0) = c_0(0) = 0$ and

$$k_0 \equiv \frac{1-\gamma}{\gamma} \frac{\lambda_1^2}{\sigma_s^2}, \quad k_1 \equiv 2 \left[-\kappa_x + \frac{1-\gamma}{\gamma} \frac{\rho_{sx}\sigma_x}{\sigma_s} \lambda_1 \right], \quad k_2 \equiv \frac{1}{\gamma} \sigma_x^2$$

$$k_3 \equiv \kappa_x \bar{X}, \quad k_4 \equiv \frac{1}{2} \sigma_x^2, \quad k_5 \equiv -\delta + (1-\gamma) r_0$$

The system of ODEs (14) - (16) is one and the same as in the problems without income. In fact, this is exactly the solution found in Wachter (2002)[11]. A detailed discussion is therefore omitted. Nevertheless, for the sake of completeness, Appendix B contains the results of the Wachter model and the two following important results should be kept in mind:

- Because of the assumption $\gamma > 1$, it follows that $c_2(s) < 0$ and $c_1(s) < 0$ for $s > 0$. As a consequence, the sign of state variable hedging demand can be determined unambiguously for $X > 0$.
- Given $\gamma > 1$, $c_2(s)$ converges to a finite number as $s \to \infty$. In other words, the solution of the Riccati differential equation is well-defined.

1.2.2 Separation of the HJB by Y

For the Y parts[12]

$$
\begin{aligned}
0 \quad = \quad & \int_0^\tau e^{C(s,X)} ds \left\{ \begin{array}{l} -\frac{\partial k}{\partial \tau_r} - r_0 k + 1 + k\left(y_0 + y_1 X\right) - \kappa_x X \frac{\partial k}{\partial X} + \kappa_x \bar{X} \frac{\partial k}{\partial X} \\ -\frac{1}{2}\frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1 X \frac{\partial k}{\partial X} - \frac{1}{2}\frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1 X k - \frac{1}{2}\frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1 X \frac{\partial k}{\partial X} \\ -\frac{1}{2}\frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1 X k + \rho_{xy}\sigma_x\sigma_y \frac{\partial k}{\partial X} + \frac{1}{2}\sigma_x^2 \frac{\partial^2 k}{\partial X^2} \end{array} \right\} \\
& + \int_0^\tau e^{C(s,X)} \left(c_1(s) + c_2(s) X\right) ds \left\{ \begin{array}{l} \sigma_x^2 \left[-\frac{1}{2}\rho_{sx}^2 - \frac{1}{2}\rho_{sx}^2 + 1\right] \frac{\partial k}{\partial X} \\ \sigma_x\sigma_y \left[-\frac{1}{2}\rho_{sx}\rho_{sy} - \frac{1}{2}\rho_{sx}\rho_{sy} + \rho_{xy}\right] k \end{array} \right\}
\end{aligned}
$$

With the assumptions (c.2) and (c.3) or (c.4), the second part on the right hand side vanishes and the equation simplifies to

$$
0 = \int_0^\tau e^{C(s,X)} ds \left\{ \begin{array}{l} -\frac{\partial k}{\partial \tau_r} - r_0 k + 1 + k\left(y_0 + y_1 X\right) - \kappa_x X \frac{\partial k}{\partial X} + \kappa_x \bar{X} \frac{\partial k}{\partial X} \\ -\frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1 X \frac{\partial k}{\partial X} - \frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1 X k + \rho_{xy}\sigma_x\sigma_y \frac{\partial k}{\partial X} + \frac{1}{2}\sigma_x^2 \frac{\partial^2 k}{\partial X^2} \end{array} \right\} \tag{17}
$$

[11] It should be noticed that in Wachter (2002) there are several typographical errors. Most notably, b_1 and b_3 are interchanged. All other differences arise from different notation. More important, despite the typographical errors the presented numerical examples in Wachter are not affected and are correct.

[12] Terms similar to A are directly set to zero because of (13).

It should be noticed that the assumptions enable the complete separation of the solution of the labor income part from the results of the SODE (14) - (16) and this simplifies the solution considerably.

As $\int_0^{\tau_r} e^{C(s,X)}ds > 0$, (17) is zero if the part in the brackets is zero. A function of the form

$$k\left(X,\tau_r\right) = \int_0^{\tau_r} e^{d_0(s)+d_1(s)X}ds \tag{18}$$

with initial conditions $d_1\left(0\right) = d_0\left(0\right) = 0$ will solve equation (17). These initial conditions are the only ones that ensure that (17) is solved and that the solution converges to the one of the constant opportunity set ($\lambda_1 = 0$ and $y_1 = 0$). The relevant partial derivatives are as follows

$$\begin{aligned} k_\tau &= \int_0^{\tau_r} \left(\frac{\partial d_0\left(s\right)}{\partial s} + \frac{\partial d_1\left(s\right)}{\partial s}X\right) e^{d_0(s)+d_1(s)X}ds + 1 \\ k_X &= \int_0^{\tau_r} d_1\left(s\right) e^{d_0(s)+d_1(s)X}ds \\ k_{XX} &= \int_0^{\tau_r} d_1\left(s\right)^2 e^{d_0(s)+d_1(s)X}ds \end{aligned}$$

Plugging in the partial derivatives into (17) leads to

$$0 = \int_0^{\tau_r} e^{d_0(s)+d_1(s)X} \left\{ \begin{array}{l} -\left(\frac{\partial d_0(s)}{\partial s} + \frac{\partial d_1(s)}{\partial s}X\right) - r_0 + \left(y_0 + y_1 X\right) \\ -\kappa_x X d_1\left(s\right) + \kappa_x \bar{X} d_1\left(s\right) - \frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1 X d_1\left(s\right) \\ -\frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1 X + \rho_{xy}\sigma_x\sigma_y d_1\left(s\right) + \frac{1}{2}\sigma_x^2 d_1\left(s\right)^2 \end{array} \right\} ds \tag{19}$$

Matching coefficients on X and the constant term leads to a system of two ordinary differential equations.

$$\frac{\partial d_1\left(s\right)}{\partial s} = l_0 + l_1 d_1\left(s\right) \tag{20}$$

$$\frac{\partial d_0\left(s\right)}{\partial s} = l_2 + l_3 d_1\left(s\right) + l_4 d_1\left(s\right)^2 \tag{21}$$

where

$$l_0 \equiv y_1 - \frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1, \quad l_1 \equiv -\kappa_x - \frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1$$

$$l_2 \equiv y_0 - r_0, \quad l_3 \equiv \kappa_x \bar{X} + \rho_{xy}\sigma_x\sigma_y, \quad l_4 \equiv \frac{1}{2}\sigma_x^2$$

The first equation is a linear differential equation with constant coefficients, the second can be solved by integration. The solution of equation (20) with initial condition $d_1(0) = 0$ is given by

$$d_1(s) = \begin{cases} \frac{l_0}{l_1}\left(e^{l_1 s} - 1\right) & , \ l_1 \neq 0 \\ l_0 s & , \ l_1 = 0 \end{cases} \tag{22}$$

Because of the simple form of $d_1(s)$ and $\frac{\partial d_0(s)}{\partial s} = l_2 + l_3 d_1(s) + l_4 d_1(s)^2$, the solution of $d_0(s)$ is also available in closed-from. Simple integration yields

$$d_0(s) = \begin{cases} \left(l_2 - l_3\frac{l_0}{l_1} + l_4\frac{l_0^2}{l_1^2}\right)s + \left(l_3\frac{l_0}{l_1^2} - 2l_4\frac{l_0^2}{l_1^3}\right)\left(e^{l_1 s} - 1\right) & \\ +\frac{1}{2}l_4\frac{l_0^2}{l_1^3}\left(e^{2l_1 s} - 1\right) & , \ l_1 \neq 0 \\ l_2 s + \frac{1}{2}l_3 l_0 s^2 + \frac{1}{3}l_4 l_0^2 s^3 & , \ l_1 = 0 \end{cases} \tag{23}$$

Remarks

- From (18) it can be easily seen that $k(\tau_r, X) > 0$ for $\tau_r > 0$. This is intuitive because labor income Y cannot become negative and hence, the individual attaches a positive value to the future labor income stream.

- Since (20) is a linear differential equation, in order that the solution $d_1(s)$ converges for long horizon, $l_1 = -\kappa_x - \frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1 < 0$. Thus, the stability of $d_1(s)$ does not depend on parameters of the labor income process, but only on parameters of the risky asset and the state variable. This result should be a warning for numerical studies of the consumption-portfolio problem with dynamic income. If the estimated parameters imply $l_1 > 0$ then the results will be highly sensitive on changes in l_0.

- The term $y_1 - \beta_{sy}\lambda_1$ can be interpreted as a pricing formula for the wage premium similar to the CAPM, with $\beta_{sy} \equiv \frac{\rho_{sy}\sigma_y}{\sigma_s}$. In other words, if the wage compensation

is in accordance with the stock market compensation[13], $y_1 - \frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1 = 0$ and $d_1(s) = 0, \forall s$. As will be shown below, if $y_1 - \frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1 \neq 0$, an adapted state variable hedging demand will arise.

- The risk aversion parameter γ is not involved in the valuation of the income stream. This is intuitive, as the assumption of complete markets allows for a perfect hedge of labor income risk.

- Because of the sensitivity of k with respect to changes in X, the return on stochastic part of human capital $G \equiv kY$ must not be simply the return on labor income

$$d\ln G = d\ln k + d\ln Y \neq d\ln Y$$

as in Jagannathan and Wang (1996, p. 14) or Santos and Veronesi (2005, p. 28).

1.2.3 Separation the HJB by the Constant Terms

Finally, for the constant parts[14],

$$0 = \int_0^\tau e^{C(s,X)} ds \left\{ \frac{\partial R}{\partial \tau} - \bar{c} + r_0 R \right\} \tag{24}$$

By the same arguments as above, the equation is zero if the term in the brackets is zero. The equation in the brackets is a linear differential equation with constant coefficients and initial condition $R(0) = 0$

$$R(\tau) = \frac{\bar{c}}{r_0}\left(1 - e^{-r_0\tau}\right) \tag{25}$$

Since $\frac{\bar{c}}{r_0}$ is the value of a perpetual bond that pays $\bar{c}$ as its coupon, it becomes clear that (24) can be interpreted as the reserves necessary to cover the subsistence level of consumption.

[13] In this case, the solution of $k(X,\tau)$ collapses to $k(\tau) = \frac{1}{y_0 - r_0}\left(e^{(y_0 - r_0)\tau} - 1\right)$ and is similar to the constant labor income growth case.

[14] Terms that are similar to A are directly neglected because they are equal to zero because of (13).

1.2.4 Main Results

In Appendix C it is shown that under assumptions (c.2) - (c.4), optimal total wealth follows a geometric Brownian motion with time-varying coefficients and will stay non-negative in all cases if initial total wealth is positive. The most important results can be summarized in the following proposition.

Proposition 2 *Given the assumptions* $\hat{A}(0) > 0$, $\rho_{sx} \in \{-1, 1\}$ *and either* $\rho_{sy} \in \{-1, 1\}$ *or* $\sigma_y = 0$ *one obtains*

$$J(t, X, A, Y) = \frac{e^{-\delta t}\left[\int_0^\tau e^{\frac{1}{\gamma}\left(c_0(s)+c_1(s)X+\frac{1}{2}c_2(s)X^2\right)}ds\right]^\gamma \left(A + k(\tau_r, X)\, Y - R(\tau)\right)^{1-\gamma}}{1-\gamma}$$

where $\tau \equiv T - t$ *and* $\tau_r \equiv T_r - t$. *The value of one unit of income is given by*

$$k(\tau, X) = \int_0^\tau e^{d_0(s)+d_1(s)X}ds$$

where $d_0(s)$ *and* $d_1(s)$ *are the solution to the following system of ordinary differential equations*

$$\begin{aligned} \frac{\partial d_1(s)}{\partial s} &= l_0 + l_1 d_1(s) \\ \frac{\partial d_0(s)}{\partial s} &= l_2 + l_3 d_1(s) + l_4 d_1(s)^2 \end{aligned}$$

with initial conditions $d_0(0) = 0$ *and* $d_1(0) = 0$ *and where*

$$\begin{aligned} l_0 &\equiv y_1 - \frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1, \quad l_1 \equiv -\kappa_x - \frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1 \\ l_2 &\equiv y_0 - r_0, \quad l_3 \equiv \kappa_x \bar{X} + \rho_{xy}\sigma_x\sigma_y, \quad l_4 \equiv \frac{1}{2}\sigma_x^2 \end{aligned}$$

The reserves follow

$$R(\tau) = \frac{\bar{c}}{r_0}\left(1 - e^{-r_0\tau}\right)$$

The solutions of $c_0(s)$, $c_1(s)$ *and* $c_2(s)$ *are identical to Wachter (2002).*

Optimal consumption and risky investment are given by

$$c_t^* = \frac{\hat{A}}{\int_0^\tau e^{C(s,X)}ds} + \bar{c} \tag{26}$$

$$
\begin{aligned}
A\pi_t^* &= \frac{1}{\gamma}\frac{\lambda_1}{\sigma_s^2}X\hat{A} + \frac{1}{\gamma}\frac{\rho_{sx}\sigma_x}{\sigma_s}\frac{\int_0^\tau \left(c_1(s) + c_2(s)X\right)e^{C(s,X)}ds}{\int_0^\tau e^{C(X,s)}ds}\hat{A} \\
&\quad -\frac{\rho_{sx}\sigma_x}{\sigma_s}\left(\int_0^\tau d_1(s)\,e^{d_0(s)+d_1(s)X}ds\right)Y \\
&\quad -\frac{\rho_{sy}\sigma_y}{\sigma_s}\left(\int_0^\tau e^{d_0(s)+d_1(s)X}ds\right)Y \qquad (27)
\end{aligned}
$$

This section is concluded with a comment on the similar problem for an investor with utility over terminal wealth only. Kim and Omberg (1996) and Liu (2007) show that without labor income, the assumption $\rho_{sx} \in \{-1, 1\}$ is not necessary in order to obtain closed-form solutions. This is not the case in the presence of labor income. Thus, compared to the consumption problem, the set of assumptions (c.2) - (c.4) is the same.

Besides an analytical derivation[15], the necessity of the assumption $\rho_{sx} \in \{1, -1\}$ is intuitive in order to get explicit solutions. In fact, from the definition of total wealth it can be clearly recognized that the valuation of the stochastic part of labor income $k(\tau_r, X)$ depends on X. Now, if total wealth is close to zero and there is a shock to the state variable, $k(\tau_r, X)$ can fall and the individual risks ending up with negative total wealth, which is clearly not optimal. This undesirable situation can only be avoided if state variable risk to the stochastic labor income stream can be hedged perfectly. If not, the level of financial wealth becomes important for the value of the income stream and the described separation of the HJB is not possible.

2 Illustration of the Results

We organize the discussion of the result in four parts. In the first part we discuss the chosen parameter values. In the second to the fourth part we present the properties of optimal total wealth including the value of human capital, the optimal investment policy and optimal consumption. Compared to former work in the field of dynamic

[15] For a detailed derivation the reader is referred to Moos (2011).

labor income, we stress the importance of the interpretation of the state dimension[16].

For the sake of comparability, we work with the parameter values for the financial market of Wachter (2002). Similar to Campbell et al. (2004) and to Lynch and Tan (2009), Wachter (2002) uses the dividend yield as the single state variable[17]. Table 1 shows the parameters in annualized form, adapted to our notation and normalized to $\lambda_1 = 1$ by invariant affine transformation[18].

Wachter (2002)		
$r = 0.0168$		
$\lambda_1 = 1$	$\sigma_s = 0.1510$	
$\kappa_x = 0.2712$	$\bar{X} = 0.0408$	$\sigma_x = 0.0348$
$\rho_{sx} = -1$		

Table 1: Financial Market Parameter Values - Normalized

Remarks

- The estimated correlation $\rho_{sx} = -0.935$ is not exactly $\rho_{sx} = -1$ but close to this.
- The stationary distribution of the state variable is normally distributed $N\left(\mu, \sigma^2\right)$ with $\mu = \bar{X}$ and $\sigma^2 = \frac{\sigma_x^2}{(2\kappa_x)}$, which is for the Wachter dataset $N\left(0.0408,\ 0.0022\right)$. From $\lambda_1 = 1$ and the standard deviation of the normal distribution of $\sigma = 4.69\%$, it can be directly seen that the estimated parameters imply very strong variations in the premium.

[16]Benzoni et al. (2007), Lynch and Tan (2009) and Munk and Sørensen (2010) focus exclusively on the time dimension.

[17]For the predictive power of the dividend yield see, for example, Fama and French (1988), Campbell and Thompson (2007) and Cochrane (2005, 2008). On the other hand, Goyal and Welch (2008) doubt that the dividend yield is a good predictor.

[18]Invariant affine transformations (IATs) are well-known from the term structure literature as, see for example, Dai and Singleton (2000).

Individual		
$\gamma = 5$	$\delta = 0.04$	
$\bar{y} = 0.0150$		
$A(0) = 0$	$Y(0) = 40$	$\bar{c} = 10$
$T_r = 45$	$T = 70$	

Table 2: Parameter Values

The level of risk aversion and the discount rate are chosen identical to Benzoni et al. (2007) and are in accordance to other studies. The parameters for the labor income process (3) are chosen variably in order to show the effects clearly. For the sake of comparability, y_0 and y_1 are chosen, so that the growth rate at the long-run mean $\bar{X}$ is constant. Specifically,

$$y_0 = \bar{y} - y_1 \bar{X} \tag{28}$$

where $\bar{y}$ is the long-run growth rate and given in Table 2.

A critical issue for these kind of models is that the parameters for the financial and the non-financial processes are chosen exogenously. In a general equilibrium model with a production side that is driven by technology and resources, financial and labor markets should be linked in a reasonable way. For this reason, certain parameter choices for the illustrative examples might be unrealistic.

Nevertheless, several studies have shown that the labor market adapts less quickly to changes in the real economy than the stock market[19]. In fact, institutional conditions such as long-term labor contracts, unions and so on suggest that a simple equilibrium relation between the two kinds of income do not exist. Indeed, the empirical estimation of Table 1 in Lynch and Tan (2009) suggests counter cyclical patterns, i.e. low labor income growth when expected returns are high. Hence, it seems an appropriate choice

[19]See, for example, Stock and Watson (1999).

to treat the parameters freely in order to understand the sensitivity of the results for a range of reasonable parameter values.

In order to reduce the dimension of the problem, we start generally by discussing the properties of the solution for the case of locally riskfree labor income $\sigma_y = 0$. In a second step, risky labor income is discussed. In this case, it is assumed that[20] $\rho_{sy} = 1$. For the sake of brevity, all figures of this section display only results for the locally riskfree labor income case.

In all figures with focus on the state dimension, the center of the horizontal axis corresponds to $\bar{X}$, and the grid points show $\left(\bar{X} - 3\sigma,\ \bar{X} - 1.5\sigma,\ \bar{X},\ \bar{X} + 1.5\sigma,\ \bar{X} + 3\sigma\right)$ where σ is the standard deviation of the unconditional normal distribution of the state variable as was defined above. It should be noticed that since λ_1 is normalized to one, the horizontal axis shows the (annualized) equity premium.

2.1 Optimal Total Wealth and Human Capital

In the Wachter (2002) model, total wealth is simply financial wealth, which does not vary over states. As can be seen from Proposition 2, the inclusion of time-varying labor income growth leads to total wealth, which varies over states through the variation in the value of one unit of income. Hence, a closer look at the properties of $k(\tau_r, X)$ should be taken.

The left panels of Figure 1 show the value of one unit of income $k(\tau_r, X)$. The right panels display the sensitivity of one unit of labor income on states $\partial k(\tau_r, X)/\partial X$. In the upper panels we focus on the evolution over the time horizon and the state variable is in the steady state $X = \bar{X}$. In the lower panels we look at the state dimension for a 45 year old individual with time to retirement of $\tau_r = T_r - t$ of 20 years[21]. The solid

[20] The case $\rho_{sy} = -1$ can be derived in analogy.

[21] Compared to the normalized sensitivity $\lambda_1 = 1$ of expected returns on X, the estimates of Lynch and Tan (2009) suggest an y_1 around -0.5.

blue line represents the case without time-variation in labor income growth ($y_1 = 0$). It serves as a benchmark case and eases the comparison. The dashed (dash-dotted, dotted) lines in magenta and light blue display the cases $y_1 = -0.6$ ($y_1 = -0.4,\ y_1 = -0.2$) and $y_1 = 0.6$ ($y_1 = 0.4,\ y_1 = 0.2$), respectively.

Insert Figure 1 about here.

From the panels to the left it should be recognized that at $X = \bar{X}$ the level of total wealth is considerably lower for cases with a negative sensitivity of labor income growth on the state variable (negative values of y_1). Since X has a symmetric distribution around $\bar{X}$ this result is not trivial.

The intuition comes from the desire to have an intertemporally favorable environment. In fact, it is assumed that $\rho_{sx} = -1$ and this implies that high states of X follow a decline in the value of the risky asset. If $y_1 < 0$, labor income has a low growth rate after a decline of the risky asset and this stands in contrast to the aim of intertemporal hedging, i.e. to be in a good state after a negative return and vice versa. Thus, for a negative sensitivity of income growth on X, labor income is from an intertemporal perspective unfavorable and the individual attaches a lower value to such an income profile.

Appendix D.2 shows that strict analytical results with respect to this property are not available. Nevertheless, the risk-neutral valuation of the future income stream gives an intuitive explanation. In Appendix D.1, we derive the value of future stochastic income $G(\tau_r, X) \equiv k(\tau_r, X)\, Y$ by the martingale approach. From equation (42) it can be recognized that the risk-neutral valuation and the no-arbitrage condition imply that $G(\tau_r, X)$ is priced in accordance with the market price of risk θ. It must be noticed that the RHS of (42) gives the expected premium in excess of the riskless rate that $G(\tau_r, X)$

must deliver. Since

$$y_1 < 0 \Rightarrow \frac{\partial G\left(\tau_r, X\right)}{\partial X} = \frac{\partial k\left(\tau_r, X\right)}{\partial X} Y < 0 \quad \wedge \quad \rho_{sx} = -1 < 0$$

imply that the expected excess return of $G\left(\tau_r, X\right)$ must be positive, the value of G must be lower compared to $y_1 = 0$ ($\partial G/\partial X = 0$). Indeed, this is similar to a financial asset that is discounted at a higher rate. The case $y_1 > 0$ can be derived in analogy and a higher value for $G\left(\tau_r, X\right)$ results[22].

The sensitivity of total wealth across states follows immediately from

$$\frac{\partial \hat{A}}{\partial X} = \frac{\partial k}{\partial X} Y = \left(\int_0^{\tau} d_1\left(s\right) e^{d_0(s) + d_1(s)X} ds \right) Y$$

By the positivity of Y and the exponential function, the sign of $\partial \hat{A}/\partial X$ is given by the sign of $d_1\left(s\right)$. Equation (22) reveals that the sign of $d_1\left(s\right),\ s > 0$ is unambiguously determined by the sign of l_0.

Hence, as displayed by the magenta lines, if labor income growth has a negative sensitivity on the state variable, the value of one unit of labor income lowers with higher X. The result is easiest to interpret for the case of locally riskfree labor income. In this case, $l_0 = y_1$, which implies that the sign of the sensitivity of one unit of income is determined exclusively by the sign of the sensitivity of labor income growth. In fact, for an individual with a negative sensitivity of labor income growth on X, higher states of X simply imply lower labor income growth. Thus, the individual anticipates lower future income and as a consequence the value of the future income stream must decline.

In case of risky labor income, l_0 is the difference of labor income growth that varies with X and the premium of the corresponding hedging portfolio. The magnitude of this difference rises with X, which explains the observed patterns. More precisely, for higher X, the premium of the risky asset rises and given the assumption $\rho_{sy} = 1$, the value of

[22] For the sake of readability, if the relation is clear this statement is omitted for the remainder of the paper.

the future income stream must be discounted at a higher rate. On the other hand, the change in the growth rate has an impact as well. If the growth rate of labor income rises less strongly than the premium of the corresponding hedging portfolio ($l_0 < 0$), then the value of the labor income stream declines.

2.2 Optimal Risky Investment

For the sake of clarity, we introduce the following definitions for the components of risky investment

$$\begin{aligned} A\pi_t^* &= \underbrace{\frac{1}{\gamma}\frac{\lambda_1}{\sigma_s^2}X\hat{A}}_{\Pi_t^m\text{: ``myopic''}} \underbrace{+\frac{1}{\gamma}\frac{\rho_{sx}\sigma_x}{\sigma_s}\frac{\int_0^\tau (c_1(s)+c_2(s)X)\,e^{C(s,X)}ds}{\int_0^\tau e^{C(s,X)}ds}\hat{A}}_{\Pi_t^s\text{: ``state variable hedging''}} \\ &\underbrace{-\frac{\rho_{sx}\sigma_x}{\sigma_s}\left(\int_0^\tau d_1(s)\,e^{d_0(s)+d_1(s)X}ds\right)Y}_{\Pi_t^{il}\text{: ``indirect labor hedging''}} \\ &\underbrace{-\frac{\rho_{sy}\sigma_y}{\sigma_s}\left(\int_0^\tau e^{d_0(s)+d_1(s)X}ds\right)Y}_{\Pi_t^{dl}\text{: ``direct labor hedging''}} \end{aligned} \tag{29}$$

The first two terms of the optimal investment rule (29) are identical to Wachter (2002) and for individuals close to the margin of subsistence ($\hat{A} \to 0$) or high risk aversion, these two parts vanish. The properties of myopic and state variable hedging demand of our model can be easily compared to Wachter (in the Wachter model $\hat{A} = A$).

As shown in Wachter (2002), for the empirically relevant range of the state variable $X > 0$, myopic demand is increasing in X. As shown in Figure 2, in the presence of dynamic labor income, this has not to be the case. In fact, the first derivative of myopic demand with respect to X is given by

$$\frac{\partial \pi_t^m}{\partial X} = \frac{\lambda_1}{\sigma_s^2}\hat{A} + \frac{\lambda_1}{\sigma_s^2}X\left(\int_0^{\tau_r} d_1(s)\,e^{d_0(s)+d_1(s)X}ds\right)Y \tag{30}$$

The first term on the RHS of (30) is unambiguously positive and thus the result of Wachter follows. This may not be the case in the presence of labor income as the sign

of the second term on the RHS is determined by the sign of $d_1(s)$. Moreover, the value of total wealth has an impact on the first term on the RHS. From the discussion of the preceding section it is known that total wealth is generally reduced for an individual with negative l_0. Hence, the lower level and the lower slope of myopic demand in Panel (a) compared to Panel (b) can be explained. The statements are valid in analogy for state variable hedging demand.

Insert Figure 2 about here.

The third term of optimal risky investment (29) is state variable hedging demand that arises under dynamic labor income. It does not vanish for individuals close to the margin of subsistence. Furthermore, it even exists if labor income is locally riskfree ($\sigma_y = 0$) or the correlation between labor income and the risky asset is zero ($\rho_{sy} = 0$). Of course, it is necessary that the risky asset and the state variable are correlated ($\rho_{sx} \neq 0$).

Indirect labor hedging demand has a natural interpretation. In fact, partitioning the third term into

$$\pi_t^{il} = \underbrace{-\rho_{sx}}_{i)} \underbrace{\frac{\sigma_x}{\sigma_s}}_{ii)} \underbrace{\frac{\partial k(\tau_r, X)}{\partial X}}_{iii)} Y$$

allows the following interpretation. Most importantly, $iii)$ is the first derivative of the value per unit of labor income on X. In other words, this part gives the change in the value of one unit of labor income when the state variable moves. $ii)$ is a multiplicator that relates the strength of the shocks of the risky asset and the state variable. $i)$ is simply plus or minus one and gives the direction the state variable moves in relation to the risky asset. Thus, it can be summarized that this third term is a hedge for the value of the future income stream to changes in the state of the economy.

As can be seen from Figure 3, compared to myopic and state variable, the properties of indirect labor hedging demand are more distinct. In fact, from the definition of indirect labor hedging demand it can be noticed that the sign is equal to the sign of

$d_1(s)$, which is equal to the sign of l_0.

Insert Figure 3 about here.

The sign of indirect labor hedging demand can be understood as follows. The situation is easiest to understand for an individual close to the margin of subsistence and with locally riskfree labor income (no direct labor hedging demand). In this case, myopic and state variable hedging demand are close to zero and risky investment is determined by indirect labor hedging demand. In the case $l_0 < 0$, the demand is negative. This has to be the case, as shown above, for $l_0 < 0$ a rise in X leads to a decrease of total wealth. The position in the risky asset must compensate the decline in future labor income in order to prevent total wealth from becoming negative.

The sign of indirect labor hedging demand can be further explained in analogy to state variable hedging demand as already described in Kim and Omberg (1996) and Wachter (2002). State variable hedging demand is positive because the individual likes high expected returns after a decline in the risky asset and this is the case for $\rho_{sx} < 0$. Now, in case of $y_1 < 0$ labor income delivers the opposite, a low growth rate after a decline in the risky asset. To compensate for this undesirable situation the individual takes a short position in the risky asset. The combination of the labor income stream and this short position creates the situation she likes. In fact, this position generates high returns followed by low income growth rates and vice versa.

As a last comment on indirect labor hedging demand it should be noticed that the sensitivity with respect to X and the level of indirect labor hedging demand are in close relation. On the one hand, it can be stated that a negative sensitivity of labor income growth on X leads to a strong decrease in risky investment. On the other hand, indirect labor hedging varies considerably over states. This stems from the fact that $d_1(s)$ is important for both measures. In particular, a highly negative (positive) $d_1(s)$ implies

a low (high) level of indirect labor hedging demand and a high sensitivity. Indeed, the crucial term for the sensitivity of indirect labor hedging demand is

$$\partial^2 k\left(\tau_R, X\right)/\partial X^2 = \int_0^{\tau} d_1\left(s\right)^2 e^{d_0(s)+d_1(s)X} ds > 0 \tag{31}$$

and the importance of $d_1\left(s\right)$ can easily be recognized.

This statement is important. In fact, it states that indirect labor hedging demand of high magnitude always comes with a high sensitivity of this hedging demand to changes in the state variable. From the panels to the left of Figures 2 and 3 it can be recognized that because of different sign of indirect labor hedging demand compared to myopic and state variable hedging demand, the level of risky investment is reduced. However, Panel (a) of Figure 4 shows that total risky investment is highly sensitive and varies strongly over states. For example, for the case $y_1 = -0.4$ (dash-dotted line) risky investment is on a reasonable level at the steady state. Nevertheless, the variation within three standard deviation from the steady state is implausibly high. In fact, the unambiguous positivity of the slope of indirect labor hedging demand makes it impossible to reduce the level for risky investment without inducing a positive effect on the slope of risky investment. Because of this, it becomes evident that assuming a more negative y_1 is not a remedy to this issue as indirect labor hedging demand becomes more sensitive on X and has the same slope as myopic and state variable hedging demand[23].

Insert Figure 4 about here.

This issue is even more pronounced in the cases with a positive sensitivity of labor income growth on X as displayed in Panel (b) of Figure 4. Since the effects of all

[23]Moreover, assuming an even more negative y_1 would also lead to an implausible variation in labor income growth over states. As will shown below, the modified model of Section 3 eases the strong sensitivity of optimal policies. Furthermore, in Moos (2011) Chapter 3 it is shown that adding stochastic labor income volatility can also ease this problem.

components of risky investment go in the same direction, both an unrealistically high level of risky investment and a highly sensitive investment policy result, which seems implausible. Thus, we conclude that the case $l_0 < 0$ implies more realistic behavior, but we keep in mind that variation of optimal investment is unreasonably high.

The last term of (29) is hedging demand for labor income risk. This part does not vanish for individuals close to the margin of subsistence. This part is well-known, see, for example, Koo (1998) or Viceira (2001). A detailed discussion is omitted, but it should be noticed that if labor income is locally riskfree, direct labor hedging demand is zero. In the presence of labor income volatility, the sign is equal to the sign of $-\rho_{sy}$.

Finally, in our model, optimal investment in the risky asset can be negative in the steady state or can turn negative in some states. This is not specific to our model and the models of, for example, Kim and Omberg (1996) and Wachter (2002) share this property. An important intention of the paper is to evaluate whether dynamic labor income matters. Our results clearly show that compared to the effects of classical state variable hedging demand and myopic demand, the impact of dynamic labor income is important.

2.3 Optimal Consumption

As shown in Appendix A, plugging in the relevant partial derivatives into the FOC (9) leads to

$$c_t^* = \frac{\hat{A}}{\int_0^\tau e^{C(s,X)} ds} + \bar{c} \tag{32}$$

Optimal consumption (32) consists of two parts. Only the first part varies over time, the subsistence part is constant. For this reason and because of a less risky income stream, consumption varies less strongly than total wealth. Indeed, in the classical Merton (1969) model, consumption has the same variation as financial wealth, which is

implausible[24]. In Wachter (2002), the relation is not one-to-one, but since the variation in $\int_0^\tau e^{C(s,X)}ds$ is low, the relation is close.

As can be seen from (32), optimal consumption exceeding the subsistence level is determined by the numerator total wealth and the denominator

$$\int_0^\tau e^{\frac{1}{\gamma}\left(c_0(s)+c_1(s)X+\frac{1}{2}c_2(s)X^2\right)}ds$$

As total wealth declines (increases) for negative (positive) $d_1(s)$, the effect of total wealth on the amount consumed is straightforward.

For the empirically relevant range of the state variable $X > 0$, $c_1(s) < 0$ and $c_2(s) < 0$ lead to

$$\int_0^\tau \left(c_1(s) + c_2(s) X\right) e^{\frac{1}{\gamma}\left(c_0(s)+c_1(s)X+\frac{1}{2}c_2(s)X^2\right)}ds < 0 \tag{33}$$

for $\tau > 0$. Thus, for $X > 0$ the denominator is unambiguously decreasing[25]. As a consequence, it can be stated that for $d_1(s) > 0$ the numerator is increasing and the denominator decreasing and thus optimal consumption rises with X.

The situation for the case $d_1(s) < 0$ is more interesting. For $X > 0$, both the denominator and the numerator are decreasing and thus, for realistic parameters optimal consumption can even fall in times when X rises. Loosely speaking, optimal consumption is decreasing if the percentage decline in total wealth is stronger than the percentage decline in the denominator. Hence, it can be concluded that the possible decline in consumption is more pronounced for individuals with low financial wealth and/or with a stronger negative sensitivity of income growth on X. Panels (c) and (d) of Figure 4 illustrate the results.

The insight of declining consumption for high states of X is of importance. In the Wachter model, there was a clear relation between the equity premium and the

[24]Cochrane (2007, p.76).

[25]For $X < 0$ the statement is not valid in general since $(c_1(s) + c_2(s) X)$ becomes positive for low X.

consumption-financial wealth ratio[26] for the empirically relevant set of positive X. Specifically, the premium is high when the consumption-wealth ratio is high. This must clearly not be the case under the presence of time-varying labor growth. The absence of a clear relation between the consumption-wealth ratio and expected returns might also have implications for the asset pricing literature and might help to explain the mixed evidence on consumption-based asset pricing models[27]. Moreover, Lynch and Tan (2009) interpret the dividend yield as a proxy for the business-cycle. In their view, states with high X are recessions and thus falling consumption for increasing X seems to be a desirable property.

Finally, the discussion of the optimal policies revealed that the case $l_0 > 0$ leads to highly unrealistic policies. In fact, compared to the benchmark framework with no variation in labor income growth, this case implies even higher risky investment/consumption and strong variation across states. For this reason we focus on the case $l_0 < 0$ for the rest of the paper.

3 Modification - Constant Investment Opportunities

Time variation in the equity premium is still under challenge[28]. Nevertheless, Lynch and Tan (2009) show that the dividend yield seems a good predictor of labor income growth since it is related to business-cycle fluctuations. Moreover, the dividend yield is naturally related to the stock market with a correlation close to -1. For this reason, we will look at a model where $\lambda_1 \to 0$ and labor income is locally riskfree.

[26] Property 2 of Wachter (2002, p. 75).

[27] Consumption based asset pricing goes back to Breeden (1979). On the one hand, Lettau and Ludvigson (2001) provide results in favor of consumption based asset pricing. On the other hand, Brennan and Xia (2005) and Goyal and Welch (2008) challenge the results of Lettau and Ludvigson.

[28] For a general overview see Goyal and Welch (2008); Pástor and Stambaugh (2001) point out the problem of structural breaks in valuation ratios that are used as instruments.

The model can also be interpreted without the connection to the dividend yield. In fact, the perfectly negative correlation of the state variable and the risky asset simply implies that the growth rate of labor income is in close relation to the financial market. Specifically, the growth rate of labor income is low after a decline in the risky asset. If a decline of a stock market is a useful predictor of a recession as suggested by Choi et al. (1999), then the aforementioned relation between the risky asset and labor income growth seems reasonable. Moreover, based on the work of Campbell and Cochrane (1999), Menzly et al. (2004) introduce a model where the relation between the dividend yield and expected returns can be unclear. On the one hand, a rise in expected returns leads to an increase of the dividend yield. On the other hand, low dividend growth leads also to an increase of the dividend yield but a decrease in expected returns. Hence, the predicting power of the dividend yield for expected returns can be low or even be zero.

It should be noticed that IAT allow almost every combination of long-run equity premium and premium sensitivity to be modeled, but the case $\lambda_1 = 0$ also implies a zero long-run premium in the specification of (1). The reason is that as $\lambda_1 \to 0$, $\bar{X} \to \infty$ in order to ensure a non-zero long-run premium. In order to avoid a zero equity premium, (1) is adapted to

$$\frac{dS_1(t)}{S_1(t)} = (\lambda_0 + \lambda_1 X(t) + r_0)\, dt + \sigma_s dW_s(t)$$

Taking into account the modified risky asset dynamics and following the steps described in Appendix A. SODE (14) - (16) changes to

$$\begin{aligned}
\frac{\partial c_2(s)}{\partial s} &= k_0 + k_1 c_2(s) + k_2 c_2(s)^2 \\
\frac{\partial c_1(s)}{\partial s} &= k_6 + k_3 c_2(s) + \frac{k_1}{2} c_1(s) + k_2 c_2(s)\, c_1(s) \\
\frac{\partial c_0(s)}{\partial s} &= k_5 + k_3 c_1(s) + k_4 c_2(s) + \frac{k_2}{2} c_1(s)^2
\end{aligned}$$

with initial conditions $c_2(0) = c_1(0) = c_0(0) = 0$ and

$$k_0 \equiv \frac{1-\gamma}{\gamma}\frac{\lambda_1^2}{\sigma_s^2}, \quad k_1 \equiv 2\left[-\kappa_x + \frac{1-\gamma}{\gamma}\frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1\right], \quad k_2 \equiv \frac{1}{\gamma}\sigma_x^2$$
$$k_3 \equiv \kappa_x \bar{X}, \quad k_4 \equiv \frac{1}{2}\sigma_x^2, \quad k_5 \equiv -\delta + (1-\gamma)\, r_0 + \frac{1}{2}\frac{1-\gamma}{\gamma}\frac{\lambda_0^2}{\sigma_s^2}$$
$$k_6 \equiv \frac{1-\gamma}{\gamma}\frac{\lambda_0\lambda_1}{\sigma_s^2}$$

Because $\lambda_1 = 0$ and the initial conditions,

$$c_2(s) = c_1(s) = 0, \ \forall s$$

Now, the equation for $c_0(s)$ becomes simple.

$$\frac{\partial c_0(s)}{\partial s} = \Delta \equiv -\delta + (1-\gamma)\, r_0 + \frac{1}{2}\frac{1-\gamma}{\gamma}\frac{\lambda_0^2}{\sigma_s^2}$$

Thus,

$$h(t) \equiv \int_0^\tau e^{C(s,X)} ds = \int_0^\tau e^{\frac{1}{\gamma}\Delta s} ds = \gamma\frac{1}{\Delta}\left(e^{\frac{1}{\gamma}\Delta\tau} - 1\right)$$

In fact, the solution of $\int_0^\tau e^{C(s,X)} ds$ is, in this case, the well-known solution from Merton (1969).

It should be noticed that without the initial conditions equal to zero, $c_2(s)$ and $c_1(s)$ would not be zero and the solution would not be equal to the Merton solution. A similar statement is true for $\gamma \to 1$ (log utility)[29]. Hence, the choice of the initial conditions can be justified not only because they are necessary to solve the HJB as described in Appendix A, but for intuitive reasons as well.

More importantly with $c_2(s) = c_1(s) = 0$, state variable hedging demand vanishes and because $\lambda_1 = 0$, myopic demand is constant.

The SODE of the labor income part (20) - (21) changes to

$$\begin{aligned}\frac{\partial d_1(s)}{\partial s} &= l_0 + l_1 d_1(s) \\ \frac{\partial d_0(s)}{\partial s} &= l_2 + l_3 d_1(s) + l_4 d_1(s)^2\end{aligned}$$

[29] A related statement with respect to risk aversion is made in Campbell and Viceira (1999) and Chacko and Viceira (2005).

where

$$l_0 \equiv y_1, \quad l_1 \equiv -\kappa_x$$

$$l_2 \equiv y_0 - r_0 - \frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_0, \quad l_3 \equiv \kappa_x \bar{X} - \frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_0 + \rho_{xy}\sigma_x\sigma_y, \quad l_4 \equiv \frac{1}{2}\sigma_x^2$$

Remarks

- Through the simplification, the sign of l_1 is unambiguously negative and the solution of $d_1(s)$ (22) reveals that this leads to stability of $d_1(s)$ as $s \to \infty$.
- The sensitivity parameter of labor income growth y_1 determines the sign of $d_1(s)$ and thus, the sign of indirect labor income hedging demand.

The basic properties of total wealth and the value of one unit of income are unchanged. In fact, at $X = \bar{X}$ the value of one unit of income is lower for $y_1 < 0$ compared to the constant growth case. Moreover, the value of income declines with higher X.

Optimal investment is displayed in Panel (a) of Figure 5. It should be recognized that the sensitivity of risky investment across states can be low, which is in contrast to the examples presented in Section 2.2. The results can be explained as follows. On the one hand, since myopic demand varies only with total wealth, the slope of myopic demand is negative for $y_1 < 0$. On the other hand, the slope of indirect hedging demand is unambiguously positive because of the same argument as in Section 2.2. Hence, the negative slope of myopic demand compensates for the positive slope of indirect labor hedging demand. We conclude that the sensitivity of optimal risky investment under constant investment opportunities (or, in analogy, for low variation in the risky asset premium) implies more realistic risky investment policies.

Insert Figure 5 about here.

Panel (b) of Figure 5 shows optimal consumption exceeding the subsistence level,

which is given by

$$c_t^* - \bar{c} = \frac{\hat{A}(X,t)}{h(t)}$$

As the denominator of optimal consumption does not vary with the state variable, the amount consumed varies only with total wealth. As a consequence, consumption unambiguously falls with X if the labor income growth has a negative sensitivity on X.

Insert Figure 6 about here.

As a final part, we examine financial wealth and the optimal policies over the life-cycle. We follow Benzoni et al. (2007) and simulate $N = 100'000$ paths of the economy at weekly frequency and discuss the evolution of the mean of financial wealth and optimal policies. Parameter values are given as in Tables 1 and 2 with the exception that $\lambda_1 = 0$ and $\lambda_0 = \bar{X}$.

Panel (a) of Figure 6 shows the evolution of financial wealth over time. As known from the analytical derivation of Section 1.2, total wealth cannot turn negative. However, financial wealth can turn negative. Indeed, individuals take into account the growth in labor income and due to consumption smoothing over time individuals consume a higher share of their income at the beginning of their life-cycle, which is confirmed by a look at Panel (b) of Figure 7. In the presented numerical examples, at the beginning of the phase of employment, individuals consume an amount close to their initial income. For this reason, financial wealth grows only slowly or turns even slightly negative. In the course of time, income becomes higher than consumption and the individual starts to accumulate financial wealth for the phase of retirement. It should be recognized that financial wealth is less likely to turn negative for individuals with a strong negative sensitivity of labor income growth. In these cases individuals attach a lower value to future income and thus consumption at the beginning of the life-cycle is reduced and shows a hump-shaped pattern over time.

Insert Figure 7 about here.

Panel (a) of Figure 7 shows the evolution of risky investment as a proportion of financial wealth α_t if financial wealth is positive. It should be noticed that for young individuals with a strong negative sensitivity of labor income growth, optimal risky investment is negative, which is due to the negative indirect labor hedging demand[30]. In the course of time, the importance of indirect labor hedging demand decreases and the accumulated financial wealth and remaining human capital in myopic demand lead to positive investment. The peak of risky investment is located somewhere in the middle of the phase of employment. After this peak the proportion decreases as the human capital part in myopic demand vanishes.

It should be noticed that investment patterns with negative/low risky investment for young individuals depend crucially on the sensitivity parameter of labor income growth on X. In fact, for y_1 close to zero, indirect labor hedging demand is not strong enough to induce a hump-shaped pattern and the proportion of risky investment is decreasing over horizon as in the Wachter (2002) model.

It can be summarized that despite the simplicity of the model, it is able to reproduce realistic patterns. In particular, falling consumption in times of a high state variable (recession), and low or even negative risky asset exposure for individuals with long horizon. Furthermore, in contrast to the model with prediction in expected excess returns, the sensitivity of the optimal policies across states is more reasonable.

4 Conclusion

We presented a life-cycle consumption/investment decision problem with time variation in labor income growth. The assumption of complete market allowed for the separation of

[30]We guess that under the presence of a borrowing constraints these individuals have no risky investment, which is in line with Figure 11 in Benzoni et al. (2007).

the complicated HJB-equation into systems of ordinary differential equations. We found analytical solutions for the value of one unit of labor income, optimal consumption and risky investment and we could analyze their properties in depth.

The impact of time variation in non-financial income on optimal investment and consumption is important. Assuming time variation in the financial market and ignoring it for non-financial income leads to considerably distinct results. Specifically, the inclusion of time variation in labor income leads to an adaption of state variable hedging demand. In fact, state variable hedging demand can be separated into the usual part that arises in the absence of labor income and a new part. This part grows monotonically with planning horizon and can have either sign. Hence, a reduction in risky investment for individuals with a long planning horizon as reported in Lynch and Tan (2009) could be reproduced.

Furthermore, we stress the importance of the state variable dimension. The inclusion of dynamic labor income can induce a falling consumption-wealth ratio even if expected returns are increasing in the state variable. We showed that in the basic model with time variation in expected returns of the risky asset and calibrated to the parameters of Wachter (2002), the sensitivity of optimal risky investment on the state variable is (unreasonably) high. The modification of the model to constant investment opportunities while still allowing for time variation in labor income growth eases this issue and more realistic policies resulted. In our opinion, a good model should yield realistic policies across both dimensions.

Hence, we conclude that dynamic labor income is important for portfolio and consumption decision, but we are aware that without additional model extensions the model is not able to reproduce all aspects of life-cycle decisions. Most notably, without short sale and borrowing constraints financial wealth can turn negative for young individuals. Moreover, at the beginning of the life-cycle risky investment is (too) high in magnitude

compared to financial wealth.

Appendix

A Solution of the HJB-Equation

First of all, it should be noticed that the solution for the phase of retirement is a special case of the more general solution for the phase of employment. In the phase of retirement, the terms related to labor income vanish.

The relevant partial derivatives of (12) are given by

$$J_\tau = e^{-\delta(T-\tau)} \begin{pmatrix} \frac{1}{1-\gamma}\left[\ldots\right]^{\gamma-1}\left(\ldots\right)^{1-\gamma}\int_0^\tau\left(\frac{\partial c_0(s)}{\partial s}+\frac{\partial c_1(s)}{\partial s}X+\frac{1}{2}\frac{\partial c_2(s)}{\partial s}X^2\right)e^{C(s,X)}ds \\ +\frac{\delta}{1-\gamma}\left[\ldots\right]^{\gamma}\left(\ldots\right)^{1-\gamma}+\frac{\gamma}{1-\gamma}\left[\ldots\right]^{\gamma-1}\left(\ldots\right)^{1-\gamma} \\ +\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma}\left(\frac{\partial k}{\partial \tau}Y-\frac{\partial R}{\partial \tau}\right) \end{pmatrix}$$

$$J_A = e^{-\delta(T-\tau)}\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma}, \quad J_{AA} = -\gamma e^{-\delta(T-\tau)}\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma-1}$$

$$J_Y = e^{-\delta(T-\tau)}\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma}k, \quad J_{YY} = -\gamma e^{-\delta(T-\tau)}\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma-1}k^2$$

$$J_X = e^{-\delta(T-\tau)} \begin{pmatrix} \frac{1}{1-\gamma}\left[\ldots\right]^{\gamma-1}\left(\ldots\right)^{1-\gamma}\int_0^\tau\left(c_1(s)+c_2(s)X\right)e^{C(s,X)}ds \\ +\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma}\frac{\partial k}{\partial X}Y \end{pmatrix}$$

$$Jxx = e^{-\delta(T-\tau)} \begin{pmatrix} -\frac{1}{\gamma}\left[\ldots\right]^{\gamma-2}\left(\ldots\right)^{1-\gamma}\left[\int_0^\tau\left(c_1(s)+c_2(s)X\right)e^{C(s,X)}ds\right]^2 \\ +\frac{1}{1-\gamma}\left[\ldots\right]^{\gamma-1}\left(\ldots\right)^{1-\gamma}\int_0^\tau\frac{1}{\gamma}\begin{pmatrix}\gamma c_2(s)+c_1^2(s)\\ +2c_1(s)c_2(s)X\\ +c_2(s)^2X^2\end{pmatrix}e^{C(s,X)}ds \\ +2\left[\ldots\right]^{\gamma-1}\left(\ldots\right)^{-\gamma}\frac{\partial k}{\partial X}Y\int_0^\tau\left(c_1(s)+c_2(s)X\right)e^{C(s,X)}ds \\ -\gamma\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma-1}\left(\frac{\partial k}{\partial X}Y\right)^2+\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma}\frac{\partial^2 k}{\partial X^2}Y \end{pmatrix}$$

$$J_{AX} = e^{-\delta(T-\tau)} \begin{pmatrix} \left[\ldots\right]^{\gamma-1}\left(\ldots\right)^{-\gamma}\int_0^\tau\left(c_1(s)+c_2(s)X\right)e^{C(s,X)}ds \\ -\gamma\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma-1}\frac{\partial k}{\partial X}Y \end{pmatrix}$$

$$J_{AY} = -\gamma e^{-\delta(T-\tau)}\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma-1}k$$

$$J_{XY} = e^{-\delta(T-\tau)} \begin{pmatrix} \left[\ldots\right]^{\gamma-1}\left(\ldots\right)^{-\gamma}k\int_0^\tau\left(c_1(s)+c_2(s)X\right)e^{C(s,X)}ds \\ -\gamma\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma-1}k\frac{\partial k}{\partial X}Y+\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma}\frac{\partial k}{\partial X} \end{pmatrix}$$

where for the sake of brevity we define

$$[\ldots] \equiv \left[\int_0^{\tau} e^{\frac{1}{\gamma}\left(c_0(s)+c_1(s)X+\frac{1}{2}c_2(s)X^2\right)}ds\right]$$

$$(\ldots) \equiv (A + k(\tau_r, X)\, Y - R(\tau))$$

and

$$C(s, X) \equiv \frac{1}{\gamma}\left(c_0(s) + c_1(s)\, X + \frac{1}{2}c_2(s)\, X^2\right)$$

It should be noted that for J_τ, the following rule was applied:

$$f(a, b) = \int_b^a g(x)\, dx = G(a) - G(b)$$

$$\Rightarrow$$

$$\frac{\partial f(a, b)}{\partial a} = \frac{\partial G(a)}{\partial a} = g(a) - g(b) + g(b) = \int_b^a \frac{\partial g(x)}{\partial x}dx + g(b)$$

Moreover, only the terminal conditions $c_0(0) = c_1(0) = c_2(0) = 0$ ensure that J_τ contains $\frac{\gamma}{1-\gamma}[\ldots]^{\gamma-1}(\ldots)^{1-\gamma}$ and this term is inevitable to find a solution for the HJB[31].

Plugging in the relevant partial derivatives into the FOCs (9) and (10) leads to

$$c_t^* = \frac{(\ldots)}{\int_0^{\tau} e^{C(s,X)}ds} + \bar{c} \tag{34}$$

and

$$\begin{aligned} A\pi_t^* &= \frac{1}{\gamma}\frac{\lambda_1}{\sigma_s^2}X(\ldots) + \frac{1}{\gamma}\frac{\rho_{sx}\sigma_x}{\sigma_s}\frac{\int_0^{\tau}(c_1(s) + c_2(s)\, X)\, e^{C(s,X)}ds}{\int_0^{\tau} e^{C(s,X)}ds}(\ldots) \\ &\quad -\frac{\rho_{sx}\sigma_x}{\sigma_s}\frac{\partial k}{\partial X}Y - \frac{\rho_{sy}\sigma_y}{\sigma_s}kY \end{aligned} \tag{35}$$

The solution of the HJB equation is tedious but leads to simple and interpretable results. Plugging in the relevant partial derivatives, (34) and (35) into the HJB and multiplying

[31] See also Wachter (2010, p. 195).

by $e^{\delta(T-\tau)}$ yields

$$
\begin{aligned}
0 \quad = \quad & -\left(\begin{array}{l} \frac{1}{1-\gamma}\left[\ldots\right]^{\gamma-1}\left(\ldots\right)^{1-\gamma}\int_0^\tau\left(\frac{\partial c_0(s)}{\partial s}+\frac{\partial c_1(s)}{\partial s}X+\frac{1}{2}\frac{\partial c_2(s)}{\partial s}X^2\right)e^{C(s,X)}ds \\ \\ +\frac{\delta}{1-\gamma}\left[\ldots\right]^{\gamma}\left(\ldots\right)^{1-\gamma}+\frac{\gamma}{1-\gamma}\left[\ldots\right]^{\gamma-1}\left(\ldots\right)^{1-\gamma}+\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma}\left(\frac{\partial k}{\partial\tau}Y-\frac{\partial R}{\partial\tau}\right) \end{array}\right) \\
& -\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma}\bar{c}+\frac{\gamma}{1-\gamma}\left[\ldots\right]^{\gamma-1}\left(\ldots\right)^{1-\gamma}+\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma}Ar_0+\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma}Y \\
& -\left(\frac{1}{1-\gamma}\left[\ldots\right]^{\gamma-1}\left(\ldots\right)^{1-\gamma}\int_0^\tau\left(c_1(s)+c_2(s)X\right)e^{C(s,X)}ds+\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma}\frac{\partial k}{\partial X}Y\right)\kappa_x\left(X-\bar{X}\right) \\
& +\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma}kY\left(y_0+y_1X\right) \\
& +\frac{1}{2}\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma}\lambda_1X\left(\begin{array}{l} \frac{1}{\gamma}\left(\ldots\right)\frac{\lambda_1}{\sigma_s^2}X+\frac{1}{\gamma}\left(\ldots\right)\frac{\int_0^\tau(c_1(s)+c_2(s)X)e^{C(s,X)}ds}{\int_0^\tau e^{C(s,X)}ds}\frac{\rho_{sx}\sigma_x}{\sigma_s} \\ \\ -\frac{\partial k}{\partial X}Y\frac{\rho_{sx}\sigma_x}{\sigma_s}-kY\frac{\rho_{sy}\sigma_y}{\sigma_s} \end{array}\right) \\
& +\frac{1}{2}\rho_{sx}\sigma_x\sigma_s\left[\begin{array}{l} \left(\begin{array}{l} \left[\ldots\right]^{\gamma-1}\left(\ldots\right)^{-\gamma}\int_0^\tau\left(c_1(s)+c_2(s)X\right)e^{C(s,X)}ds \\ \\ -\gamma\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma-1}\frac{\partial k}{\partial X}Y \end{array}\right). \\ \left(\begin{array}{l} \frac{1}{\gamma}\left(\ldots\right)\frac{\lambda_1}{\sigma_s^2}X+\frac{1}{\gamma}\left(\ldots\right)\frac{\int_0^\tau(c_1(s)+c_2(s)X)e^{C(s,X)}ds}{\int_0^\tau e^{C(s,X)}ds}\frac{\rho_{sx}\sigma_x}{\sigma_s} \\ \\ -\frac{\partial k}{\partial X}Y\frac{\rho_{sx}\sigma_x}{\sigma_s}-kY\frac{\rho_{sy}\sigma_y}{\sigma_s} \end{array}\right) \end{array}\right] \\
& -\frac{1}{2}\gamma\rho_{sy}\sigma_y\sigma_s\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma-1}kY\left(\begin{array}{l} \frac{1}{\gamma}\left(\ldots\right)\frac{\lambda_1}{\sigma_s^2}X+\frac{1}{\gamma}\left(\ldots\right)\frac{\int_0^\tau(c_1(s)+c_2(s)X)e^{C(s,X)}ds}{\int_0^\tau e^{C(s,X)}ds}\frac{\rho_{sx}\sigma_x}{\sigma_s} \\ \\ -\frac{\partial k}{\partial X}Y\frac{\rho_{sx}\sigma_x}{\sigma_s}-kY\frac{\rho_{sy}\sigma_y}{\sigma_s} \end{array}\right) \\
& +\rho_{xy}\sigma_x\sigma_yY\left(\begin{array}{l} \left[\ldots\right]^{\gamma-1}\left(\ldots\right)^{-\gamma}k\int_0^\tau\left(c_1(s)+c_2(s)X\right)e^{C(s,X)}ds \\ \\ -\gamma\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma-1}k\frac{\partial k}{\partial X}Y+\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma}\frac{\partial k}{\partial X} \end{array}\right) \\
& -\frac{1}{2}\gamma\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma-1}k^2Y^2\sigma_y^2 \\
& +\frac{1}{2}\sigma_x^2\left(\begin{array}{l} -\frac{1}{\gamma}\left[\ldots\right]^{\gamma-2}\left(\ldots\right)^{1-\gamma}\left[\int_0^\tau\left(c_1(s)+c_2(s)X\right)e^{C(s,X)}ds\right]^2 \\ \\ +\frac{1}{1-\gamma}\left[\ldots\right]^{\gamma-1}\left(\ldots\right)^{1-\gamma}\int_0^\tau\frac{1}{\gamma}\left(\begin{array}{l} \gamma c_2(s)+c_1^2(s) \\ \\ +2c_1(s)c_2(s)X \\ \\ +c_2(s)^2X^2 \end{array}\right)e^{C(s,X)}ds \\ \\ +2\left[\ldots\right]^{\gamma-1}\left(\ldots\right)^{-\gamma}\frac{\partial k}{\partial X}Y\int_0^\tau\left(c_1(s)+c_2(s)X\right)e^{C(s,X)}ds \\ \\ -\gamma\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma-1}\left(\frac{\partial k}{\partial X}Y\right)^2+\left[\ldots\right]^{\gamma}\left(\ldots\right)^{-\gamma}\frac{\partial^2k}{\partial X^2}Y \end{array}\right)
\end{aligned}
$$

Multiplying by $\left[\ldots\right]^{-(\gamma-1)}\left(\ldots\right)^{\gamma}$ gives

$$
\begin{aligned}
0 \;=\; & -\frac{\delta}{1-\gamma}\left(\ldots\right)\left[\ldots\right]-\frac{1}{1-\gamma}\left(\ldots\right)\int_0^\tau\left(\frac{\partial c_0\left(s\right)}{\partial s}+\frac{\partial c_1\left(s\right)}{\partial s}X+\frac{1}{2}\frac{\partial c_2\left(s\right)}{\partial s}X^2\right)e^{C(s,X)}ds\\
& -\left(\frac{\partial k}{\partial \tau}Y-\frac{\partial R}{\partial \tau}\right)\left[\ldots\right]-\bar{c}\left[\ldots\right]+Ar_0\left[\ldots\right]+Y\left[\ldots\right]+\left(y_0+y_1X\right)kY\left[\ldots\right]\\
& -\frac{1}{1-\gamma}\kappa_x\left(X-\bar{X}\right)\left(\ldots\right)\int_0^\tau\left(c_1\left(s\right)+c_2\left(s\right)X\right)e^{C(s,X)}ds-\kappa_x\left(X-\bar{X}\right)\frac{\partial k}{\partial X}Y\left[\ldots\right]\\
& +\frac{1}{2}\frac{1}{\gamma}\frac{\lambda_1^2}{\sigma_s^2}X^2\left(\ldots\right)\left[\ldots\right]+\frac{1}{2}\frac{1}{\gamma}\frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1X\left(\ldots\right)\int_0^\tau\left(c_1\left(s\right)+c_2\left(s\right)X\right)e^{C(s,X)}ds\\
& -\frac{1}{2}\frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1X\frac{\partial k}{\partial X}Y\left[\ldots\right]-\frac{1}{2}\frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1XkY\left[\ldots\right]\\
& +\frac{1}{2}\frac{1}{\gamma}\frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1X\left(\ldots\right)\int_0^\tau\left(c_1\left(s\right)+c_2\left(s\right)X\right)e^{C(s,X)}ds\\
& \underbrace{+\frac{1}{2}\frac{1}{\gamma}\rho_{sx}^2\sigma_x^2\left(\ldots\right)\left[\ldots\right]^{-1}\left[\int_0^\tau\left(c_1\left(s\right)+c_2\left(s\right)X\right)e^{C(s,X)}ds\right]^2}_{i)}\\
& -\frac{1}{2}\rho_{sx}^2\sigma_x^2\frac{\partial k}{\partial X}Y\int_0^\tau\left(c_1\left(s\right)+c_2\left(s\right)X\right)e^{C(s,X)}ds-\frac{1}{2}\rho_{sx}\sigma_x\rho_{sy}\sigma_ykY\int_0^\tau\left(c_1\left(s\right)+c_2\left(s\right)X\right)e^{C(s,X)}ds\\
& -\frac{1}{2}\frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1X\frac{\partial k}{\partial X}Y\left[\ldots\right]-\frac{1}{2}\rho_{sx}^2\sigma_x^2\frac{\partial k}{\partial X}Y\int_0^\tau\left(c_1\left(s\right)+c_2\left(s\right)X\right)e^{C(s,X)}ds\\
& \underbrace{+\frac{1}{2}\gamma\rho_{sx}^2\sigma_x^2\left(\ldots\right)^{-1}\left(\frac{\partial k}{\partial X}Y\right)^2\left[\ldots\right]}_{ii)}\underbrace{+\frac{1}{2}\gamma\rho_{sx}\sigma_x\rho_{sy}\sigma_y\left(\ldots\right)^{-1}kY\frac{\partial k}{\partial X}Y\left[\ldots\right]}_{iii)}\\
& -\frac{1}{2}\frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1XkY\left[\ldots\right]-\frac{1}{2}\rho_{sx}\sigma_x\rho_{sy}\sigma_ykY\int_0^\tau\left(c_1\left(s\right)+c_2\left(s\right)X\right)e^{C(s,X)}ds\\
& \underbrace{+\frac{1}{2}\gamma\rho_{sx}\sigma_x\rho_{sy}\sigma_y\left(\ldots\right)^{-1}kY\frac{\partial k}{\partial X}Y\left[\ldots\right]}_{iii)}\underbrace{+\frac{1}{2}\gamma\rho_{sy}^2\sigma_y^2\left(\ldots\right)^{-1}k^2Y^2\left[\ldots\right]}_{iv)}\\
& +\rho_{xy}\sigma_x\sigma_ykY\int_0^\tau\left(c_1\left(s\right)+c_2\left(s\right)X\right)e^{C(s,X)}ds\underbrace{-\gamma\rho_{xy}\sigma_x\sigma_y\left(\ldots\right)^{-1}kY\frac{\partial k}{\partial X}Y\left[\ldots\right]}_{iii)}\\
& +\rho_{xy}\sigma_x\sigma_y\frac{\partial k}{\partial X}Y\left[\ldots\right]\underbrace{-\frac{1}{2}\gamma\sigma_y^2\left(\ldots\right)^{-1}k^2Y^2\left[\ldots\right]}_{iv)}\\
& \underbrace{-\frac{1}{2}\frac{1}{\gamma}\sigma_x^2\left(\ldots\right)\left[\ldots\right]^{-1}\left[\int_0^\tau\left(c_1\left(s\right)+c_2\left(s\right)X\right)e^{C(s,X)}ds\right]^2}_{i)}\\
& +\frac{1}{2}\frac{1}{1-\gamma}\sigma_x^2\left(\ldots\right)\int_0^\tau\frac{1}{\gamma}\left(\gamma c_2\left(s\right)+c_1^2\left(s\right)+2c_1\left(s\right)c_2\left(s\right)X+c_2\left(s\right)^2X^2\right)e^{C(s,X)}ds\\
& +\sigma_x^2\frac{\partial k}{\partial X}Y\int_0^\tau\left(c_1\left(s\right)+c_2\left(s\right)X\right)e^{C(s,X)}ds\\
& \underbrace{-\frac{1}{2}\gamma\sigma_x^2\left(\ldots\right)^{-1}\left(\frac{\partial k}{\partial X}Y\right)^2\left[\ldots\right]}_{ii)}+\frac{1}{2}\sigma_x^2\frac{\partial^2 k}{\partial X^2}Y\left[\ldots\right]
\end{aligned}
\tag{36}
$$

To our knowledge, closed-form solutions for this general PDE are not available[32]. The highlighted terms make it impossible to separate the equation into a system of ODEs. However, the highlighted terms $i)$ and $ii)$ vanish under the assumption of $\rho_{sx} \in \{-1, 1\}$,

[32] See Huang and Milevsky (2008), Huang et al. (2008), Munk and Sørensen (2010).

which is the assumption in Wachter (2002). Furthermore, if $\rho_{sy} \in \{-1, 1\}$ then $\rho_{xy} = \rho_{sx}\rho_{sy} \in \{-1, 1\}$ or if $\sigma_y = 0$, the terms indicated by $iii)$ and $iv)$ vanish.

Without the terms highlighted by $i)$, $ii)$, $iii)$ and $iv)$, the HJB simplifies to

$$
\begin{aligned}
0 \quad = \quad & -\frac{\delta}{1-\gamma}\left(\ldots\right)\left[\ldots\right] - \frac{1}{1-\gamma}\left(\ldots\right)\int_0^\tau \left(\frac{\partial c_0\left(s\right)}{\partial s} + \frac{\partial c_1\left(s\right)}{\partial s}X + \frac{1}{2}\frac{\partial c_2\left(s\right)}{\partial s}X^2\right)e^{C(s,X)}ds \\
& -\left(\frac{\partial k}{\partial \tau}Y - \frac{\partial R}{\partial \tau}\right)\left[\ldots\right] - \bar{c}\left[\ldots\right] + r_0\left(\ldots\right)\left[\ldots\right] - r_0\left(kY - R\right)\left[\ldots\right] \\
& +Y\left[\ldots\right] + \left(y_0 + y_1 X\right)kY\left[\ldots\right] \\
& -\frac{1}{1-\gamma}\kappa_x\left(X - \bar{X}\right)\left(\ldots\right)\int_0^\tau \left(c_1\left(s\right) + c_2\left(s\right)X\right)e^{C(s,X)}ds - \kappa_x\left(X - \bar{X}\right)\frac{\partial k}{\partial X}Y\left[\ldots\right] \\
& +\frac{1}{2}\frac{1}{\gamma}\frac{\lambda_1^2}{\sigma_s^2}X^2\left(\ldots\right)\left[\ldots\right] + \frac{1}{2}\frac{1}{\gamma}\frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1 X\int_0^\tau \left(c_1\left(s\right) + c_2\left(s\right)X\right)e^{C(s,X)}ds\left(\ldots\right) \\
& -\frac{1}{2}\frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1 X\frac{\partial k}{\partial X}Y\left[\ldots\right] - \frac{1}{2}\frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1 XkY\left[\ldots\right] \\
& +\frac{1}{2}\frac{1}{\gamma}\frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1 X\left(\ldots\right)\int_0^\tau \left(c_1\left(s\right) + c_2\left(s\right)X\right)e^{C(s,X)}ds \\
& -\frac{1}{2}\rho_{sx}^2\sigma_x^2\frac{\partial k}{\partial X}Y\int_0^\tau \left(c_1\left(s\right) + c_2\left(s\right)X\right)e^{C(s,X)}ds \\
& -\frac{1}{2}\rho_{sy}\sigma_y\rho_{sx}\sigma_x kY\int_0^\tau \left(c_1\left(s\right) + c_2\left(s\right)X\right)e^{C(s,X)}ds \\
& -\frac{1}{2}\frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1 X\frac{\partial k}{\partial X}Y\left[\ldots\right] - \frac{1}{2}\rho_{sx}^2\sigma_x^2\frac{\partial k}{\partial X}Y\int_0^\tau \left(c_1\left(s\right) + c_2\left(s\right)X\right)e^{C(s,X)}ds \\
& -\frac{1}{2}\frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1 XkY\left[\ldots\right] - \frac{1}{2}\rho_{sx}\sigma_x\rho_{sy}\sigma_y kY\int_0^\tau \left(c_1\left(s\right) + c_2\left(s\right)X\right)e^{C(s,X)}ds \\
& +\rho_{xy}\sigma_x\sigma_y kY\int_0^\tau \left(c_1\left(s\right) + c_2\left(s\right)X\right)e^{C(s,X)}ds + \rho_{xy}\sigma_x\sigma_y\frac{\partial k}{\partial X}Y\left[\ldots\right] \\
& +\frac{1}{2}\frac{1}{1-\gamma}\sigma_x^2\left(\ldots\right)\int_0^\tau \frac{1}{\gamma}\left(\gamma c_2\left(s\right) + c_1^2\left(s\right) + 2c_1\left(s\right)c_2\left(s\right)X + c_2\left(s\right)^2X^2\right)e^{C(s,X)}ds \\
& +\sigma_x^2\frac{\partial k}{\partial X}Y\int_0^\tau \left(c_1\left(s\right) + c_2\left(s\right)X\right)e^{C(s,X)}ds + \frac{1}{2}\sigma_x^2\frac{\partial^2 k}{\partial X^2}Y\left[\ldots\right] \qquad (37)
\end{aligned}
$$

Moreover, it should be noticed that the trivial expansion

$$\left(kY - R\right)r_0\left[\ldots\right] - \left(kY - R\right)r_0\left[\ldots\right]$$

was made in the second line of (37). This equation can now be separated into a system of ODEs.

B Solution of the Wachter Model

For the sake of completeness, this appendix contains the main results of the part of the model that is identical to Wachter (2002).

The solutions to the SODE (14) - (16) with initial conditions $c_2(0) = c_1(0) = c_0(0) = 0$ are given by

$$c_2(s) = \frac{2k_0\left(1 - e^{-\eta s}\right)}{2\eta - (k_1 + \eta)\left(1 - e^{-\eta s}\right)}$$

$$c_1(s) = \frac{4k_0k_3\left(1 - e^{-\eta s/2}\right)^2}{\eta\left[2\eta - (k_1 + \eta)\left(1 - e^{-\eta s}\right)\right]}$$

$$c_0(s) = \int_0^\tau k_7 + k_4c_1(s) + \frac{k_2}{2}c_1(s)^2 + k_5c_2(s)\ ds$$

where $\eta \equiv \sqrt{q}$ and $q = k_1^2 - 4k_0k_2$.

The negativity and the convergence of $c_2(s)$ can be derived analytically[33].

C The Dynamics of Total Wealth

From the definition $\hat{A} \equiv A + k(t, X)Y - R(t)$, application of Ito's lemma yields the dynamics of total wealth

$$d\hat{A} = dA + \frac{\partial k}{\partial X}YdX + \frac{1}{2}\frac{\partial^2 k}{\partial X^2}YdX^2 + \frac{\partial k}{\partial t}Ydt + kdY + \frac{\partial k}{\partial X}dXdY - \frac{\partial R}{\partial t}dt$$

[33] See Wachter (2002, pp. 87-88). For a more illustrative derivation with the help of phase plane analysis of the ordinary differential equations, see Moos (2011).

Plugging in (2) - (7) and the optimal policies (26) and (27) leads to

$$\begin{aligned}
d\hat{A}^* &= \left[\begin{array}{l} \frac{1}{\gamma}\frac{\lambda_1^2}{\sigma_s^2}X^2\hat{A}^* + \frac{1}{\gamma}\frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1\frac{\int_0^\tau\left(c_1(s)X+c_2(s)X^2\right)e^{C(s,X)}ds}{\int_0^\tau e^{C(s,X)}ds}\hat{A}^* \\ -\frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1 X\frac{\partial k}{\partial X}Y - \frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1 XkY \\ +r_0\hat{A}^* - r_0\left(kY-R\right) + Y - \frac{1}{\int_0^\tau e^{C(s,X)}ds}\hat{A}^* - \bar{c} \end{array}\right]dt \\
&+ \left(\begin{array}{l} \frac{1}{\gamma}\frac{\lambda_1}{\sigma_s}X\hat{A}^* + \frac{1}{\gamma}\rho_{sx}\sigma_x\frac{\int_0^\tau(c_1(s)+c_2(s)X)e^{C(s,X)}ds}{\int_0^\tau e^{C(s,X)}ds}\hat{A}^* \\ -\rho_{sx}\sigma_x\frac{\partial k}{\partial X}Y - \rho_{sy}\sigma_y kY \end{array}\right)dW_s(t) \\
&-\frac{\partial k}{\partial X}Y\kappa_x\left(X-\bar{X}\right)dt + \frac{\partial k}{\partial X}Y\sigma_x dW_x(t) + \frac{1}{2}\sigma_x^2\frac{\partial^2 k}{\partial X^2}Ydt - \frac{\partial k}{\partial \tau}Ydt \\
&+kY\left(y_0+y_1X\right)dt + kY\sigma_y dW_y(t) + \frac{\partial k}{\partial X}\rho_{xy}\sigma_x\sigma_y Ydt + \frac{\partial R}{\partial \tau}dt
\end{aligned}$$

Arranging in proper order

$$\begin{aligned}
d\hat{A}^* &= \left(\begin{array}{l} r_0 + \frac{1}{\gamma}\frac{\lambda_1^2}{\sigma_s^2}X^2 - \frac{1}{\int_0^\tau e^{C(s,X)}ds} \\ +\frac{1}{\gamma}\frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1\frac{\int_0^\tau\left(c_1(s)X+c_2(s)X^2\right)e^{C(s,X)}ds}{\int_0^\tau e^{C(X,s)}ds} \end{array}\right)\hat{A}^*dt \\
&+\left(\frac{1}{\gamma}\frac{\lambda_1}{\sigma_s}X + \frac{1}{\gamma}\rho_{sx}\sigma_x\frac{\int_0^\tau\left(c_1(s)+c_2(s)X\right)e^{C(s,X)}ds}{\int_0^\tau e^{C(s,X)}ds}\right)\hat{A}^*dW_s(t) \\
&+\left[\begin{array}{l} -\frac{\partial k}{\partial \tau} + 1 - r_0 k - \frac{\partial k}{\partial X}\kappa_x\left(X-\bar{X}\right) + k\left(y_0+y_1X\right) \\ -\frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1\frac{\partial k}{\partial X}X - \frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1 kX + \frac{\partial k}{\partial X}\rho_{xy}\sigma_x\sigma_y + \frac{1}{2}\sigma_x^2\frac{\partial^2 k}{\partial X^2} \end{array}\right]Ydt \\
&+\left[\frac{\partial R}{\partial \tau} + r_0 R - \bar{c}\right]dt \\
&+\left[dW_y(t) - \rho_{sy}dW_s(t)\right]\sigma_y kY + \left[dW_x(t) - \rho_{sx}dW_s(t)\right]\sigma_x\frac{\partial k}{\partial X}Y
\end{aligned} \tag{38}$$

The last line is equal to zero due to the assumptions about perfect dependence (c.2) - (c.3) and locally riskfree labor income (c.4), i.e. $dW_x(t) = \rho_{sx}dW_s(t)$ and $dW_y(t) = \rho_{sy}dW_s(t)$ or $\sigma_y = 0$. Inspecting the parts in the square brackets one can identify (17) and (24) which are also equal to zero. The dynamics of (39) follow directly.

$$\begin{aligned}
\frac{d\hat{A}^*}{\hat{A}^*} &= \left(r_0 + \frac{1}{\gamma}\frac{\lambda_1^2}{\sigma_s^2}X^2\right)dt \\
&+\left(\frac{1}{\gamma}\frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1\frac{\int_0^\tau\left(c_1(s)X+c_2(s)X^2\right)e^{C(X,s)}ds}{\int_0^\tau e^{C(s,X)}ds} - \frac{1}{\int_0^\tau e^{C(s,X)}ds}\right)dt \\
&+\frac{1}{\gamma}\left(\frac{\lambda_1}{\sigma_s}X + \rho_{sx}\sigma_x\frac{\int_0^\tau\left(c_1(s)X+c_2(s)X^2\right)e^{C(s,X)}ds}{\int_0^\tau e^{C(s,X)}ds}\right)dW_s(t)
\end{aligned} \tag{39}$$

D Valuation of the Labor Income Stream

D.1 Valuation of the Labor Income Stream with the Martingale Approach

The risk-neutral valuation of the future labor income stream is given by

$$G(\tau_r, X, Y) = E_0^Q \left[\int_0^{\tau_r} e^{-r_0 s} Y(s)\, ds \right]$$

Since it is assumed that there is only one shock that drives the economy, complete markets are implied. As stated in Pliska (1986) and He and Pearson (1991), the market price of risk is unique in complete markets. With a change in measure from the risk-neutral to the standard probability law, the value of the future income stream can be written as.

$$G(\tau_r, X, Y) = \frac{1}{\phi(0)} E_0 \left[\int_0^{\tau_r} \phi(s)\, Y(s)\, ds \right] \tag{40}$$

where

$$\frac{d\phi(t)}{\phi(t)} = -r_0 dt - \theta(t)\, dW_s(t) = -r_0 dt - \frac{\lambda_1}{\sigma_s} X(t)\, dW_s(t) \tag{41}$$

The process $\phi(t)$ can be interpreted as a system of Arrow-Debreu prices. Indeed, the value of $\phi(t)$ in each state gives the price per unit probability of a dollar in that state.

From (40) and (41) it can be recognized that the value of the labor income stream must be a function of X, Y and τ_r. Furthermore, from Cox et al. (1985) it is known that every asset must obey the following no-arbitrage condition

$$\begin{aligned} & Y + \frac{\partial G}{\partial t} - \frac{\partial G}{\partial X} \kappa_x \left(X - \bar{X} \right) + \frac{\partial G}{\partial Y} (y_0 + y_1 X) Y \\ & + \frac{1}{2} \left[\frac{\partial^2 G}{\partial X^2} \sigma_x^2 + \frac{\partial^2 G}{\partial Y^2} \sigma_y^2 Y^2 + 2 \frac{\partial^2 G}{\partial X \partial Y} \rho_{xy} \sigma_x \sigma_y \right] - r_0 G \\ & = \left[\frac{\partial G}{\partial X} \rho_{sx} \sigma_x + \frac{\partial G}{\partial Y} \rho_{sy} \sigma_y Y \right] \frac{\lambda_1}{\sigma_s} X \end{aligned} \tag{42}$$

The first term Y is the instantaneous wage and is analogous to a dividend paying asset. The RHS shows the expected return every asset must deliver in order to comply with

the premium of the tradable risky asset. Now, a function of the form

$$G(\tau_r, X, Y) = k(\tau_r, X)\, Y$$

implies the following partial derivatives

$$\begin{aligned}
\frac{\partial G}{\partial t} &= -\frac{\partial G}{\partial \tau} = -\frac{\partial k}{\partial \tau} Y \\
\frac{\partial G}{\partial X} &= \frac{\partial k}{\partial X} Y, \quad \frac{\partial^2 G}{\partial X^2} = \frac{\partial^2 k}{\partial X^2} Y \\
\frac{\partial G}{\partial Y} &= k, \quad \frac{\partial^2 G}{\partial Y^2} = 0 \\
\frac{\partial^2 G}{\partial X \partial Y} &= \frac{\partial k}{\partial X}
\end{aligned}$$

Plugging in the partial derivatives into (42) and dividing by Y leads to

$$\begin{aligned}
0 \;=\; & 1 - \frac{\partial k}{\partial \tau} - \frac{\partial k}{\partial X} \kappa_x \left(X - \bar{X}\right) + k\left(y_0 + y_1 X\right) \\
& + \frac{1}{2} \frac{\partial^2 k}{\partial X^2} \sigma_x^2 + \frac{\partial k}{\partial X} \rho_{xy} \sigma_x \sigma_y - r_0 k \\
& - \frac{\rho_{sx} \sigma_x}{\sigma_s} \lambda_1 X \frac{\partial k}{\partial X} - \frac{\rho_{sy} \sigma_y}{\sigma_s} \lambda_1 X k
\end{aligned}$$

which is identical to the term in the brackets of (17).

D.2 Valuation of the Labor Income Stream at $X = \bar{X}$

From

$$k(X, \tau_r) = \int_0^{\tau_r} e^{d_0(s) + d_1(s) X} ds$$

it is clear that the $d_0(s) + d_1(s) X$ is crucial for the valuation of the income stream. It should be kept in mind that $d_1(s)$ and $d_0(s)$ are given by equation (22) and (23)

respectively. For the sake of comparability we focus on $X = \bar{X}$

$$
\begin{aligned}
& d_0(s) + d_1(s)\bar{X} \\
= \; & \left(l_2 - l_3\frac{l_0}{l_1} + l_4\frac{l_0^2}{l_1^2}\right)s + \left(l_3\frac{l_0}{l_1^2} - 2l_4\frac{l_0^2}{l_1^3}\right)\left(e^{l_1 s} - 1\right) \\
& + \frac{1}{2}l_4\frac{l_0^2}{l_1^3}\left(e^{2l_1 s} - 1\right) + \frac{l_0}{l_1}\left(e^{l_1 s} - 1\right)\bar{X} \\
= \; & (\bar{y} - r_0)s - y_1\bar{X}s + \frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1\bar{X}s - \frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1\bar{X}s + l_3\frac{l_0}{l_1}\left[-s + \frac{1}{l_1}\left(e^{l_1 s} - 1\right)\right] \\
& + l_4\frac{l_0^2}{l_1^2}\left[s - 2\frac{1}{l_1}\left(e^{l_1 s} - 1\right) + \frac{1}{2l_1}\left(e^{2l_1 s} - 1\right)\right] + l_0\frac{1}{l_1}\left(e^{l_1 s} - 1\right)\bar{X} \\
= \; & (\bar{y} - r_0)s + \left\{\begin{array}{l} l_0\left[\frac{1}{l_1}\left(e^{l_1 s} - 1\right) - s\right]\left(\frac{l_3}{l_1} + \bar{X}\right) \\ + l_4\frac{l_0^2}{l_1^2}\left[s - 2\frac{1}{l_1}\left(e^{l_1 s} - 1\right) + \frac{1}{2l_1}\left(e^{2l_1 s} - 1\right)\right] \\ -\frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1\bar{X}s \end{array}\right\}
\end{aligned}
\tag{43}
$$

for[34] $l_1 \neq 0$.

It should be noticed that after the second equals sign the relation $y_0 = \bar{y} - y_1\bar{X}$ was used and the trivial expansion $\frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1\bar{X}s - \frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1\bar{X}s = 0$ was made. Furthermore, after the third equals sign $l_0 = y_1 - \frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1$ was used.

It should be kept in mind that $\lambda_1 > 0$ and $\bar{X} \geq 0$. The first part on the RHS of (43), $(\bar{y} - r_0)s$ corresponds to the value of the income stream under a constant growth rate ($y_1 = 0$). Hence, the term in the brackets determines whether the income stream is valued higher or lower than the constant counterpart.

Under locally riskfree labor income, the last term in the brackets of (43) vanishes. Under risky labor income, a positive correlation between the risky asset and labor income leads to a lower valuation. This is the impact of the higher discount rate rate as described in the text.

[34] For the case $l_1 = 0$ the solution is given by $d_0(s) + d_1(s)\bar{X} = (\bar{y} - r_0)s + y_1\bar{X}s + \frac{1}{2}l_0l_3s^2 + \frac{1}{3}l_4l_0^2s^3 - \frac{\rho_{sy}\sigma_y}{\sigma_s}\lambda_1\bar{X}s$.

The term in the square brackets of the second line of (43) is positive for[35] $s > 0$. The positivity of $l_4 = \frac{1}{2}\sigma_x^2$ reveals that through this term, state variable volatility has an unambiguously positive effect on the valuation of the income stream. Moreover, for $l_1 > 0$ (instable cases) this term grows fastest and hence has a strong impact. However, if $l_1 < 0$ this term grows only linearly in the horizon similar to the first term. It becomes obvious that given $l_1 < 0$, if l_4 is low (as it is the case) the impact of this term is low. Hence, for the parameter values as chosen in Table 2, $l_1 < 0$ and the second term is small in magnitude.

Thus, the term on the first line in the brackets of (43) becomes the key for the valuation of the income stream at $X = \bar{X}$ under time-varying income growth. As above, it is easier to start with the discussion of the locally riskfree labor income case. In this case, the term on the first line can be rewritten,

$$\psi \equiv \left[\frac{1}{l_1}\left(e^{l_1 s} - 1\right) - s\right]\left(1 + \frac{l_3/\bar{X}}{l_1}\right)\bar{X} = \left[\frac{1}{l_1}\left(e^{l_1 s} - 1\right) - s\right]\left(1 - \frac{\kappa_x}{\kappa_x + \frac{\rho_{sx}\sigma_x}{\sigma_s}\lambda_1}\right)\bar{X}$$

From Table 3 it should be noticed that ψ is positive if $\rho_{sx} < 0$ (second and third columns) and negative if $\rho_{sx} > 0$ (first column).

	$l_1 < \kappa_x$	$\kappa_x < l_1 < 0$	$l_1 > 0$
$sign\left(\frac{1}{l_1}\left(e^{l_1 s} - 1\right) - s\right)$	$-$	$-$	$+$
$sign\left(1 + \frac{\kappa_x}{l_1}\right)$	$+$	$-$	$+$
$sign\left(\psi\right)$	$-$	$+$	$+$

Table 3: Sign of ψ

It can be seen that the sign of the effect depends crucially on the opportunity set of the financial market. In fact, if low returns on the financial asset are followed by high

[35] This follows from the fact that at $s = 0$ the term is zero and the first derivative with respect to s is given by $\left(e^{l_1 s} - 1\right)^2 > 0,\ s > 0$.

expected returns on the risky asset

$$\rho_{sx} = -1 < 0 \Rightarrow l_1 > \kappa_x \Rightarrow \psi > 0$$

and, in addition, the individual faces low labor income growth ($y_1 < 0$), the individual attaches a lower value to the income stream.

The case of risky labor income is more difficult. Specifically, there is a second term in l_3 and the negativity of $\rho_{xy} = \rho_{sy}\rho_{sx}$ leads to a lower l_3 for higher σ_y. As a consequence, ψ becomes smaller (and can even turn negative).

This result is not intuitive as a negative ρ_{xy} and $y_1 > 0$ imply that a decline in labor income is followed by high income growth and this seems to be a desired feature from an intertemporal point of view. An answer can be found by looking at the dynamics of total wealth. As already described, by the valuation of the income stream the individual compensates the dynamics of the non-financial income stream in order to end up with total wealth, which behaves as in a setting without labor income. The critical term $\rho_{xy}\sigma_x\sigma_y$ in l_3 can be clearly identified in the dynamics of total wealth (38) as $\frac{\partial k}{\partial X}\rho_{xy}\sigma_x\sigma_y$. Since this part does not originate from a first order condition but simply from the cross product of labor income and state variable diffusion, it is comprehensible that there is no connection to intertemporal hedging. Furthermore,

$$l_0 < 0 \Rightarrow \frac{\partial k}{\partial X} < 0, \quad \left(l_0 > 0 \Rightarrow \frac{\partial k}{\partial X} > 0 \right)$$

in combination with $\rho_{xy} = -1$ implies a positive (negative) drift for total wealth. This additional drift has to be taken into account by valuing the income stream[36].

As an alternative to Appendix D.2, the absence of unambiguous results can be verified by looking at the equation that determines $d_0(s)$. In fact, for extreme σ_x $d_0(s)$ can become very large because of the unambiguously positive term $l_4 d_1(s)^2$ in equation (21). Hence, the level of future income at $X = \bar{X}$ could be higher compared to the

[36]This effect will become more clear in the numerical example of Section 3.

constant income growth case even for $y_1 < 0$.

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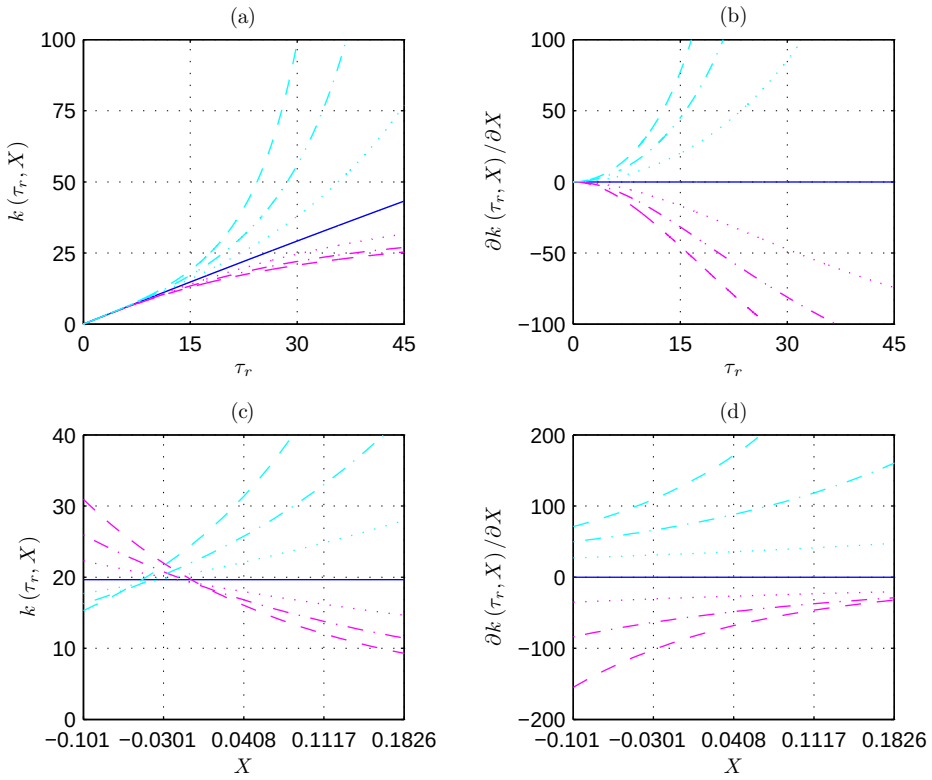


Figure 1: The Valuation of one Unit of Income

Panel (a) shows the value of one unit of income $k(\tau_r, X)$ over time horizon until retirement τ_r at $X = \bar{X}$. Panel (b) shows the sensitivity of the value of one unit of income on states $\partial k(\tau_r, X)/\partial X$ over time horizon until retirement τ_r at $X = \bar{X}$. Panel (c) shows the value of one unit of income $k(\tau_r, X)$ across states X for a 45 year old individual ($\tau_r = 20$). Panel (d) shows the sensitivity of the value of one unit of income on states $\partial k(\tau_r, X)/\partial X$ across states X for a 45 year old individual ($\tau_r = 20$). The solid blue line represents the case without time-variation in labor income growth ($y_1 = 0$), the dashed (dash-dotted, dotted) lines in magenta and light blue display the cases $y_1 = -0.6$ ($y_1 = -0.4,\ y_1 = -0.2$) and $y_1 = 0.6$ ($y_1 = 0.4,\ y_1 = 0.2$), respectively.

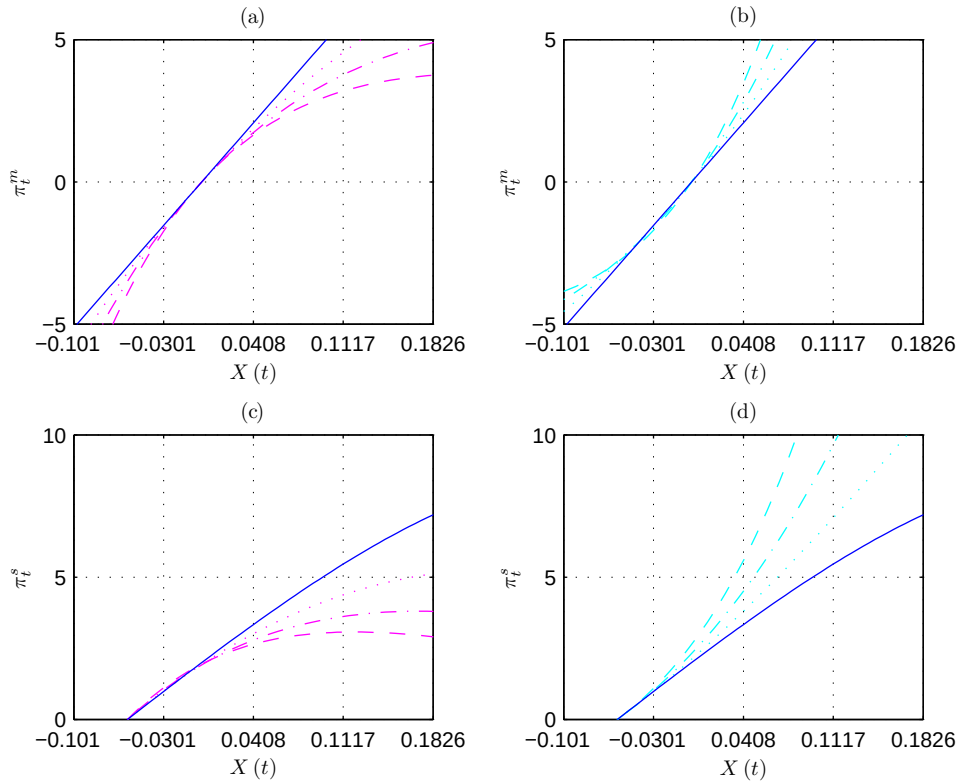


Figure 2: Myopic and State Variable Hedging Demand

Panels (a) and (b) show myopic demand in proportion of financial wealth $\pi_t^m \equiv \Pi_t^m / A(t)$ as described in equation (29) over states. Panels (c) and (d) show state variable hedging demand $\pi_t^s \equiv \Pi_t^s / A(t)$. The individual is assumed to be 45 years old ($\tau_r = 20$) with a financial wealth of 180 (thousand) and an income of 60 (thousand). All other parameter values are given as in Table 1 and 2. The solid blue line represents the case without time-variation in labor income growth ($y_1 = 0$), the dashed (dash-dotted, dotted) lines in magenta and light blue display the cases $y_1 = -0.6$ ($y_1 = -0.4,\ y_1 = -0.2$) and $y_1 = 0.6$ ($y_1 = 0.4,\ y_1 = 0.2$), respectively.

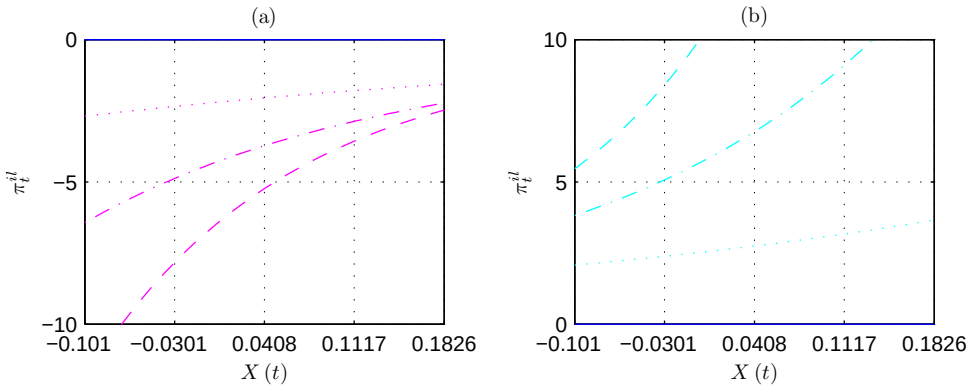


Figure 3: Indirect Labor Hedging Demand

Panels (a) and (b) show indirect labor hedging demand $\pi_t^{il} \equiv \Pi_t^{il}/A\,(t)$ as described in equation (29) over states. The individual is assumed to be 45 years old ($\tau_r = 20$) with a financial wealth of 180 (thousand) and an income of 60 (thousand). All other parameter values are given as in Table 1 and 2. The solid blue line represents the case without time-variation in labor income growth ($y_1 = 0$), the dashed (dash-dotted, dotted) lines in magenta and light blue display the cases $y_1 = -0.6$ ($y_1 = -0.4,\ y_1 = -0.2$) and $y_1 = 0.6$ ($y_1 = 0.4,\ y_1 = 0.2$), respectively.

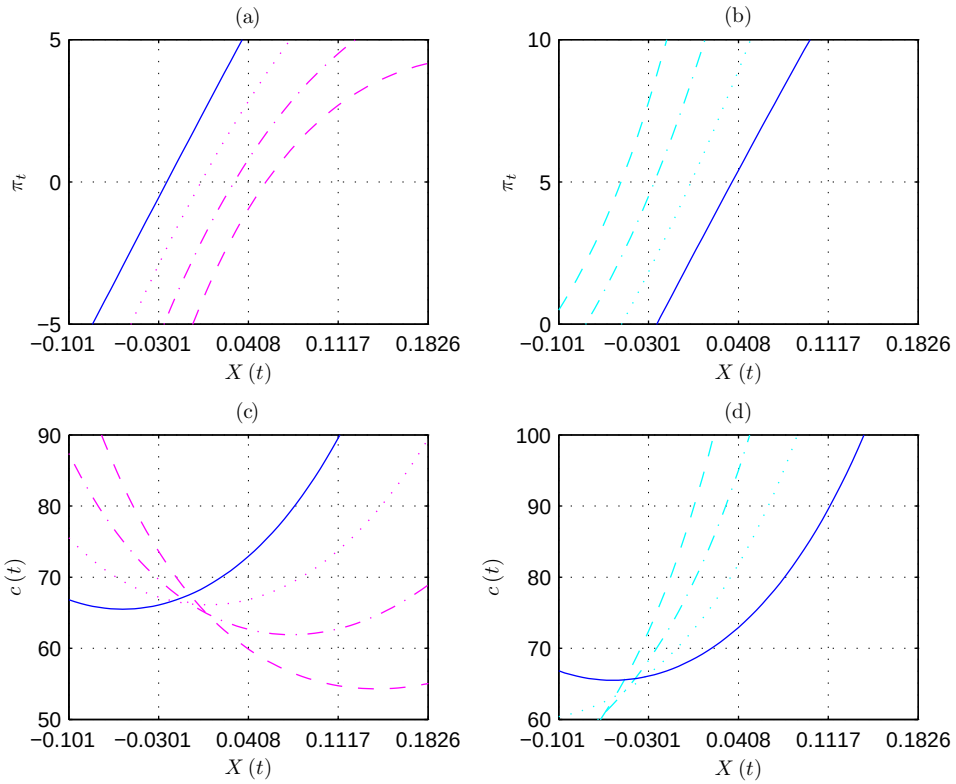


Figure 4: Optimal Policies over States

Panels (a) and (b) show the proportion of financial wealth invested in the risky asset π_t over states. Panels (c) and (d) show optimal consumption. The individual is assumed to be 45 years old ($\tau_r = 20$) with a financial wealth of 180 (thousand) and an income of 60 (thousand). All other parameter values are given as in Table 1 and 2. The solid blue line represents the case without time-variation in labor income growth ($y_1 = 0$), the dashed (dash-dotted, dotted) lines in magenta and light blue display the cases $y_1 = -0.6$ ($y_1 = -0.4,\ y_1 = -0.2$) and $y_1 = 0.6$ ($y_1 = 0.4,\ y_1 = 0.2$), respectively.

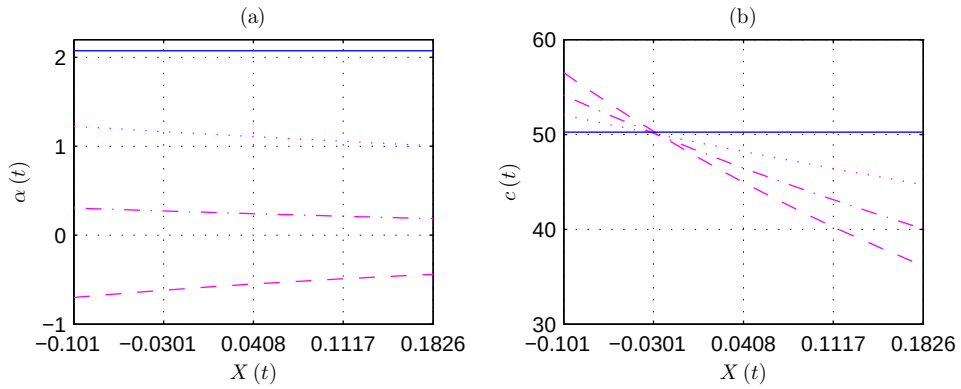


Figure 5: Optimal Policies over States

Panels (a) and (b) show the optimal policies across states X for an individual at the age of 45 with a financial wealth of 180 and an income of 60. Panel (a) shows the proportion of financial wealth invested in the risky asset α_t if financial wealth is positive ($A(t) > 0$). Panel (b) displays optimal consumption c_t. The solid blue line represents the case without time-variation in labor income growth ($y_1 = 0$), the dashed (dash-dotted, dotted) lines in magenta display the cases $y_1 = -0.6$ ($y_1 = -0.4,\ y_1 = -0.2$).

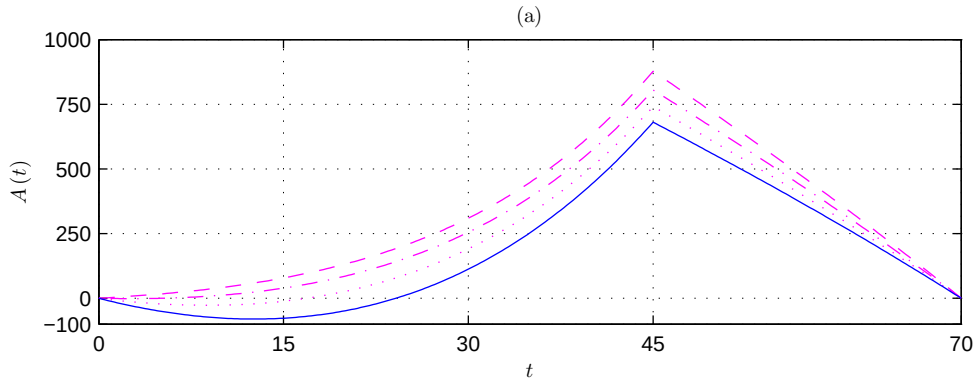


Figure 6: Financial Wealth over Time

Panel (a) shows the evolution of financial wealth $A(t)$ until the end of the planning horizon. The solid blue line represents the case without time-variation in labor income growth ($y_1 = 0$), the dashed (dash-dotted, dotted) lines in magenta display the cases $y_1 = -0.6$ ($y_1 = -0.4,\ y_1 = -0.2$).

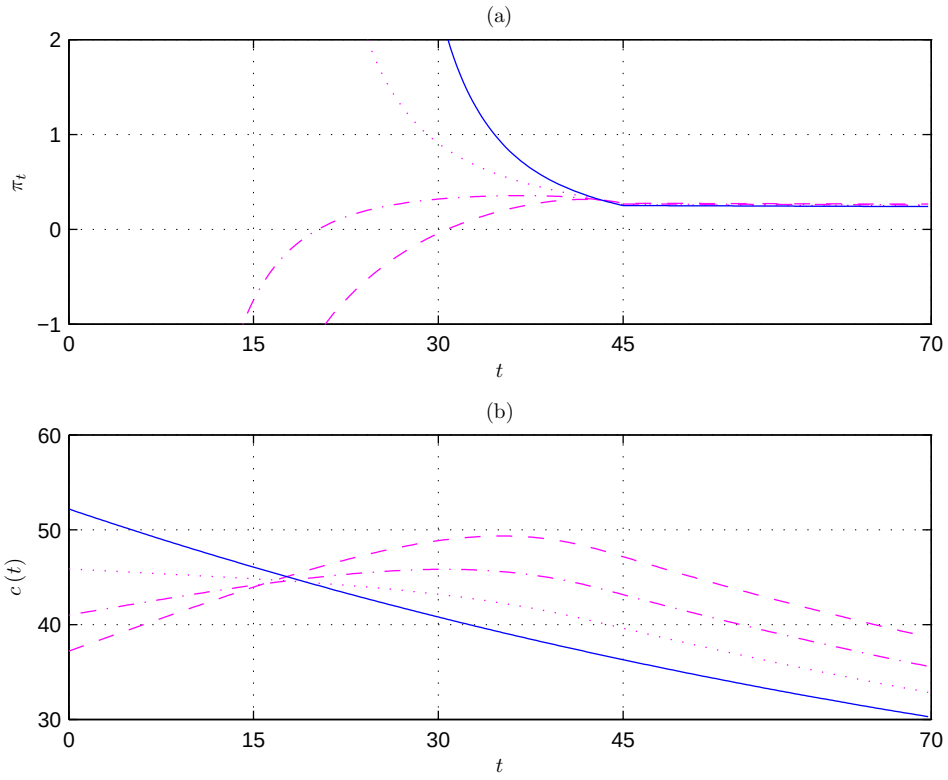


Figure 7: Optimal Policies over Time

Panels (a) and (b) show the evolution of the optimal policies until the end of the planning horizon. Panel (a) shows the proportion of financial wealth invested in the risky asset π_t if financial wealth is positive ($A\,(t) > 0$). Panel (b) displays optimal consumption c_t. The solid blue line represents the case without time-variation in labor income growth ($y_1 = 0$), the dashed (dash-dotted, dotted) lines in magenta display the cases $y_1 = -0.6$ ($y_1 = -0.4,\ y_1 = -0.2$).