

Online Appendix of

Sharp IV bounds on average treatment effects on the treated and other populations under endogeneity and noncompliance

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A Appendix

A.1 Mean independence within principal strata without further assumptions

A.1.1 Identified set for the proportion of defiers based on linear programming

To shorten notation, let $f(y(1), y(0)|t, z)$ denote the probability density of the counterfactual outcomes $(Y(1), Y(0))$ evaluated at $(y(1), y(0))$ given $T = t$ and $Z = z$ and let $f_Y(y)$ denote the probability density function of the observed outcome Y evaluated at y . Assumption 2 together with compatibility with the probability distribution of observed variables (Y, D, Z) translates into the following restrictions.

Assumption 2 (i)

$$E(Y(d)|T = t, Z = 1) = E(Y(d)|T = t, Z = 0), \quad \forall t \in \{11, 10, 01, 00\}, \forall d \in \{1, 0\} \quad (\text{A.1})$$

We rewrite (A.1) in terms of $f(y(1), y(0)|t, z)$:

$$\begin{aligned} \iint y(1) f(y(1), y(0)|t, 1) dy(1) dy(0) &= \iint y(1) f(y(1), y(0)|t, 0) dy(1) dy(0), \\ \iint y(0) f(y(1), y(0)|t, 1) dy(1) dy(0) &= \iint y(0) f(y(1), y(0)|t, 0) dy(1) dy(0). \end{aligned} \quad (\text{A.2})$$

Assumption 2 (ii) together with compatibility with the distribution of (Y, D, Z)

For $D = 1$ and $Z = 1$, we have

$$\begin{aligned} f_Y(y|Z = 1, D = 1) \Pr(D = 1|Z = 1) &= f_{Y(1)}(y|11, 1) \Pr(T = 11|Z = 1) + f_{Y(1)}(y|10, 1) \Pr(T = 10|Z = 1) \\ &= f_{Y(1)}(y|11, 1) \Pr(T = 11) + f_{Y(1)}(y|10, 1) \Pr(T = 10), \end{aligned}$$

where $f_Y(y|Z = z, D = d)$ denotes the conditional density function of Y given $D = d$ and $Z = z$, $f_{Y(d)}(y|t, z)$ stands for the conditional density function of a potential outcome $Y(d)$ given $T = t$ and $Z = z$. The second equation follows from the Assumption 2 (ii). Similarly,

$$\begin{aligned} f_Y(y|Z = 0, D = 1) \Pr(D = 1|Z = 0) &= f_{Y(1)}(y|11, 0) \Pr(T = 11) + f_{Y(1)}(y|01, 0) \Pr(T = 01), \\ f_Y(y|Z = 1, D = 0) \Pr(D = 0|Z = 1) &= f_{Y(0)}(y|01, 1) \Pr(T = 01) + f_{Y(0)}(y|00, 1) \Pr(T = 00), \\ f_Y(y|Z = 0, D = 0) \Pr(D = 0|Z = 0) &= f_{Y(0)}(y|10, 0) \Pr(T = 10) + f_{Y(0)}(y|00, 0) \Pr(T = 00) \end{aligned}$$

This implies that the following equations must hold for all $y \in \mathcal{Y}$:

$$\begin{aligned} P_{1|1} \cdot f_Y(y|Z = 1, D = 1) &= (P_{1|0} - \pi_{01}) \int f(y, y(0)|11, 1) dy(0) + (P_{1|1} - P_{1|0} + \pi_{01}) \int f(y, y(0)|10, 1) dy(0), \\ P_{1|0} \cdot f_Y(y|Z = 0, D = 1) &= (P_{1|0} - \pi_{01}) \int f(y, y(0)|11, 0) dy(0) + \pi_{01} \int f(y, y(0)|01, 0) dy(0), \\ P_{0|1} \cdot f_Y(y|Z = 1, D = 0) &= \pi_{01} \int f(y(1), y|01, 1) dy(1) + (P_{0|1} - \pi_{01}) \int f(y(1), y|00, 1) dy(1), \\ P_{0|0} \cdot f_Y(y|Z = 0, D = 0) &= (P_{1|1} - P_{1|0} + \pi_{01}) \int f(y(1), y|10, 0) dy(1) + (P_{0|1} - \pi_{01}) \int f(y(1), y|00, 0) dy(1). \end{aligned} \tag{A.3}$$

If there exists a proper density function $f(y(1), y(0)|t, z)$ for all $t \in \{11, 10, 01, 00\}$, $z \in \{1, 0\}$ which satisfies (A.2) and (A.3), then $\pi_{01} \in \mathcal{P}^*$, where $\mathcal{P}^*$ denotes the sharp identified set for the share of defiers. Sharpness of $\mathcal{P}^*$ follows immediately by construction: If, for a certain value of π_{01} , we cannot find a data-generating process that (1) is compatible with the observed data and (2) satisfies the imposed assumptions, then such a value does not lie in the identified set. On the other hand, if such a data-generating process exists, that particular value of π_{01} cannot be refuted.

With discrete Y , the problem is equivalent to the non-emptiness of the linearly constrained feasible set. Consider a discrete y with support $\mathcal{Y}^* = \{y_1, y_2, \dots, y_k\}$ and let $h_t^z(y_i, y_j) = \Pr(Y(1) = y_i, Y(0) = y_j|T = t, Z = z)$. We denote the set of conditional distributions by $\{h_t^z\}$. Then equations (A.2) can be rewritten as

$$\begin{aligned} \sum_{i=1}^k \sum_{j=1}^k y_i h_t^1(y_i, y_j) &= \sum_{i=1}^k \sum_{j=1}^k y_i h_t^0(y_i, y_j), \\ \sum_{i=1}^k \sum_{j=1}^k y_j h_t^1(y_i, y_j) &= \sum_{i=1}^k \sum_{j=1}^k y_j h_t^0(y_i, y_j). \end{aligned} \tag{A.4}$$

Similarly, equations (A.3) become

$$\begin{aligned}
\Pr(Y = y_i, D = 1|Z = 1) &= (P_{1|0} - \pi_{01}) \sum_{j=1}^k h_{11}^1(y_i, y_j) + (P_{1|1} - P_{1|0} + \pi_{01}) \sum_{j=1}^k h_{10}^1(y_i, y_j), \quad \forall i = 1, \dots, k, \\
\Pr(Y = y_i, D = 1|Z = 0) &= (P_{1|0} - \pi_{01}) \sum_{j=1}^k h_{10}^0(y_i, y_j) + \pi_{01} \sum_{j=1}^k h_{01}^0(y_i, y_j), \quad \forall i = 1, \dots, k, \\
\Pr(Y = y_j, D = 0|Z = 1) &= \pi_{01} \sum_{i=1}^k h_{01}^1(y_i, y_j) + (P_{0|1} - \pi_{01}) \sum_{i=1}^k h_{00}^1(y_i, y_j), \quad \forall j = 1, \dots, k, \\
\Pr(Y = y_j, D = 0|Z = 0) &= (P_{1|1} - P_{1|0} + \pi_{01}) \sum_{i=1}^k h_{10}^0(y_i, y_j) + (P_{0|1} - \pi_{01}) \sum_{i=1}^k h_{00}^0(y_i, y_j), \quad \forall j = 1, \dots, k.
\end{aligned} \tag{A.5}$$

At the same time, h_t^z must be a proper probability distribution for all $t \in \{11, 10, 01, 00\}$ and $z \in \{1, 0\}$:

$$\begin{aligned}
\sum_{i=1}^k \sum_{j=1}^k h_t^z(y_i, y_j) &= 1, \quad \forall t \in \{11, 10, 01, 00\}, \quad \forall z \in \{1, 0\}, \\
h_t^z(y_i, y_j) &\geq 0, \quad \forall i = 1, \dots, k, \quad \forall j = 1, \dots, k, \quad \forall t \in \{11, 10, 01, 00\}, \quad \forall z \in \{1, 0\}.
\end{aligned} \tag{A.6}$$

We conclude that the identified set for the share of defiers is a collection of points $\pi_{01} \in \mathcal{P}$ for which a feasible solution exists to a set of constraints given by (A.4)-(A.6) (formally $\mathcal{P}^* = \{\pi_{01} \in \mathcal{P} : \exists \{h_t^z\} \text{ that solves (A.4)-(A.6)}\}$), which can be checked using a linear programming tool.

Lemma 1 *The identified set $\mathcal{P}^*$ for π_{01} is an interval.*

Proof of Lemma 1. Consider the smallest and the largest admissible values $\pi_{01}^{\min}$ and $\pi_{01}^{\max}$ and the corresponding sets of conditional distributions $\{h_t^{z, \min}\}$ and $\{h_t^{z, \max}\}$ that satisfy (A.4) – (A.6). For a fixed $0 < \lambda < 1$, let $\pi_{01}^\lambda = \lambda \pi_{01}^{\min} + (1 - \lambda) \pi_{01}^{\max}$.

Consider $h_t^{z, \lambda}(y_i, y_j) = \theta^t h_t^{z, \min}(y_i, y_j) + (1 - \theta^t) h_t^{z, \max}(y_i, y_j)$ for all $t \in \{11, 10, 01, 00\}$, $z \in \{1, 0\}$, and $i, j \in \{1, \dots, k\}$, where

$$\begin{aligned}
\theta^{11} &= \frac{\lambda(P_{1|0} - \pi_{01}^{\min})}{\lambda(P_{1|0} - \pi_{01}^{\min}) + (1 - \lambda)(P_{1|0} - \pi_{01}^{\max})}, \\
\theta^{10} &= \frac{\lambda(P_{1|1} - P_{1|0} + \pi_{01}^{\min})}{\lambda(P_{1|1} - P_{1|0} + \pi_{01}^{\min}) + (1 - \lambda)(P_{1|1} - P_{1|0} + \pi_{01}^{\max})}, \\
\theta^{01} &= \frac{\lambda \pi_{01}^{\min}}{\lambda \pi_{01}^{\min} + (1 - \lambda) \pi_{01}^{\max}}, \\
\theta^{00} &= \frac{\lambda(P_{0|1} - \pi_{01}^{\min})}{\lambda(P_{0|1} - \pi_{01}^{\min}) + (1 - \lambda)(P_{0|1} - \pi_{01}^{\max})}.
\end{aligned}$$

Then, $\{h_t^{z, \lambda}\}$ solves (A.4) and (A.6) because these constraints do not depend on π_{01} . It is left to prove that the set of distributions $\{h_t^{z, \lambda}\}$ satisfies (A.5) for π_{01}^λ . We demonstrate it for the first equality in (A.5) and the validity of

the remaining equalities follows similarly. We would like to show that $\forall i \in \{1, \dots, k\}$:

$$\Pr(Y = y_i, D = 1 | Z = 1) = (P_{1|0} - \pi_{01}^\lambda) \sum_{j=1}^k h_{11}^{1,\lambda}(y_i, y_j) + (P_{1|1} - P_{1|0} + \pi_{01}^\lambda) \sum_{j=1}^k h_{10}^{1,\lambda}(y_i, y_j).$$

Consider the first term and the second term on the right hand side of this equation separately:

$$\begin{aligned} & (P_{1|0} - \pi_{01}^\lambda) \sum_{j=1}^k h_{11}^{1,\lambda}(y_i, y_j) \\ &= \left[P_{1|0} - (\lambda \pi_{01}^{\min} + (1 - \lambda) \pi_{01}^{\max}) \right] \sum_{j=1}^k \left[\theta^{11} h_{11}^{1,\min}(y_i, y_j) + (1 - \theta^{11}) h_{11}^{1,\max}(y_i, y_j) \right] \\ &= \lambda (P_{1|0} - \pi_{01}^{\min}) \sum_{j=1}^k h_{11}^{1,\min}(y_i, y_j) + (1 - \lambda) (P_{1|0} - \pi_{01}^{\max}) \sum_{j=1}^k h_{11}^{1,\max}(y_i, y_j), \\ & (P_{1|1} - P_{1|0} + \pi_{01}^\lambda) \sum_{j=1}^k h_{10}^{1,\lambda}(y_i, y_j) = \\ &= \left[P_{1|1} - P_{1|0} + (\lambda \pi_{01}^{\min} + (1 - \lambda) \pi_{01}^{\max}) \right] \sum_{j=1}^k \left[\theta^{10} h_{10}^{1,\min}(y_i, y_j) + (1 - \theta^{10}) h_{10}^{1,\max}(y_i, y_j) \right] \\ &= \lambda (P_{1|1} - P_{1|0} + \pi_{01}^{\min}) \sum_{j=1}^k h_{10}^{1,\min}(y_i, y_j) + (1 - \lambda) (P_{1|1} - P_{1|0} + \pi_{01}^{\max}) \sum_{j=1}^k h_{10}^{1,\max}(y_i, y_j). \end{aligned}$$

Summing and rearranging the two terms results in

$$\begin{aligned} & (P_{1|0} - \pi_{01}^\lambda) \sum_{j=1}^k h_{11}^{1,\lambda}(y_i, y_j) + (P_{1|1} - P_{1|0} + \pi_{01}^\lambda) \sum_{j=1}^k h_{10}^{1,\lambda}(y_i, y_j) \\ &= \lambda \left[(P_{1|0} - \pi_{01}^{\min}) \sum_{j=1}^k h_{11}^{1,\min}(y_i, y_j) + (P_{1|1} - P_{1|0} + \pi_{01}^{\min}) \sum_{j=1}^k h_{10}^{1,\min}(y_i, y_j) \right] \\ &+ (1 - \lambda) \left[(P_{1|0} - \pi_{01}^{\max}) \sum_{j=1}^k h_{11}^{1,\max}(y_i, y_j) + (P_{1|1} - P_{1|0} + \pi_{01}^{\max}) \sum_{j=1}^k h_{10}^{1,\max}(y_i, y_j) \right] \\ &= \lambda \cdot \Pr(Y = y_i, D = 1 | Z = 1) + (1 - \lambda) \cdot \Pr(Y = y_i, D = 1 | Z = 1) = \Pr(Y = y_i, D = 1 | Z = 1), \end{aligned}$$

where the last equality follows from the fact that $\{h_t^{z,\min}\}$ and $\{h_t^{z,\max}\}$ satisfy (A.5) for $\pi_{01}^{\min}$ and $\pi_{01}^{\max}$, respectively.

We have therefore shown that $\{h_t^{z,\lambda}\}$ satisfies (A.4)-(A.6) for π_{01}^λ so that $\mathcal{P}^* = [\pi_{01}^{\min}, \pi_{01}^{\max}]$. We note that the proof easily extends to the case with a continuous Y , where the sums would be replaced by integrals. ■

In a similar manner, one can obtain sharp bounds on various ATEs or potential outcomes using the linear programming tool whenever Y is discrete. Instead of searching for a *feasible* solution to (A.4)-(A.6), we would search for a solution that minimizes (or maximizes) a linear functional of a set of conditional distributions $\{h_t^z\}$ corresponding to the object of our interest (e.g. $\Delta_t, \Delta_{D=1}, \Delta, E(Y(d))$).

A.1.2 Proof of the validity and the sharpness of the bounds on the ATEs within principal strata

The proof consists of two steps. In the first step, we show that the bounds on the ATEs are valid. Secondly, we show that these bounds are sharp.

Validity

We start by deriving the valid bounds on the mean potential outcomes within principal strata for π_{01} fixed, which contain the respective actual mean potential outcomes with probability one under the imposed assumptions. For the sake of brevity, we omit conditioning on π_{01} in the mean potential outcomes and write $E(Y(d)|T = t)$ rather than $E(Y(d)|T = t)(\pi_{01})$ (for $d = 1, 0, t = 11, 10, 01, 00$). Secondly, we use our bounds on the mean potential outcomes given π_{01} to construct bounds on $\Delta_t^{UB}(\pi_{01})$ and $\Delta_t^{LB}(\pi_{01})$, the ATEs within principal strata given π_{01} . Thirdly, since π_{01} is unknown, we obtain the upper and lower bounds on the ATEs within principal strata by optimizing over π_{01} : $\Delta_t^{UB} = \sup_{\pi_{01} \in \mathcal{P}^*} \Delta_t^{UB}(\pi_{01})$ and $\Delta_t^{LB} = \sup_{\pi_{01} \in \mathcal{P}^*} \Delta_t^{LB}(\pi_{01})$ (for $t = 11, 10, 01, 00$).

Assume, for the moment, that π_{01} is fixed (albeit omitted in the notation in the subsequent discussion). Lemma 1 of Imai (2008), which applies Proposition 4 in Horowitz and Manski (1995) to the case that the upper and lower bounds of the mixing probabilities of the principal strata are known, together with Imai's Proposition 1, which shows the sharpness of the bounds on the mean potential outcomes (and the ATE), implies that $E(Y(1)|Z = 1, T = 11) \leq \bar{Y}_{1,1}(\max |q_{1,1}^{11}|)$ and $E(Y(1)|Z = 0, T = 11) \leq \bar{Y}_{0,1}(\max |q_{0,1}^{11}|)$. Under Assumption 2, $E(Y(1)|Z = 1, T = 11) = E(Y(1)|Z = 0, T = 11) = E(Y(1)|T = 11)$ and therefore,

$$E(Y(1)|T = 11) \leq E(Y(1)|T = 11)^{UB} = \min(\bar{Y}_{1,1}(\max |q_{1,1}^{11}|), \bar{Y}_{0,1}(\max |q_{0,1}^{11}|)).$$

In a symmetric way one can show that

$$\begin{aligned} E(Y(1)|T = 11) &\geq E(Y(1)|T = 11)^{LB} = \max(\bar{Y}_{1,1}(\min |q_{1,1}^{11}|), \bar{Y}_{0,1}(\min |q_{0,1}^{11}|)), \\ E(Y(0)|T = 00) &\leq E(Y(0)|T = 00)^{UB} = \min(\bar{Y}_{1,0}(\max |q_{1,0}^{00}|), \bar{Y}_{0,0}(\max |q_{0,0}^{00}|)), \\ E(Y(0)|T = 00) &\geq E(Y(0)|T = 00)^{LB} = \max(\bar{Y}_{1,0}(\min |q_{1,0}^{00}|), \bar{Y}_{0,0}(\min |q_{0,0}^{00}|)). \end{aligned}$$

Our assumptions do not provide any restrictions on $Y(0)|T = 11$ and $Y(1)|T = 00$, so that

$$E(Y(0)|T = 11)^{LB} = y^{LB} \leq E(Y(0)|T = 11) \leq E(Y(0)|T = 11)^{UB} = y^{UB},$$

and

$$E(Y(1)|T = 00)^{LB} = y^{LB} \leq E(Y(1)|T = 00) \leq E(Y(1)|T = 00)^{UB} = y^{UB}.$$

Note that for any random variable W distributed as a two components mixture:

$$f(w) = p \cdot f_1(w) + (1 - p) \cdot f_2(w) \quad p \in [0, 1],$$

so that $W = AW_1 + (1 - A)W_2$, where f_1 , f_2 , and f are probability density functions of W_1 , W_2 , and W , respectively, and A denotes a random variable such that $A \in \{0, 1\}$, $A \perp (W_1, W_2)$, $A \sim \text{Bernoulli}(p)$, it holds that

$$E(W) = p \cdot E(W_1)^{UB} + (1 - p) \cdot E(W_2)^{LB}, \quad (\text{A.7})$$

where $E(W_1)$ and $E(W_2)$ denote the expected values of the first and second components, respectively. Combining this result with equations (1) to (4) we obtain bounds on the mean potential outcomes that are valid under our assumptions:

$$\begin{aligned} \frac{P_{1|1} \cdot \bar{Y}_{1,1} - (P_{1|0} - \pi_{01}) \cdot E(Y(1)|T = 11)^{UB}}{P_{1|1} - P_{1|0} + \pi_{01}} &\leq E(Y(1)|T = 10) \leq \frac{P_{1|1} \cdot \bar{Y}_{1,1} - (P_{1|0} - \pi_{01}) \cdot E(Y(1)|T = 11)^{LB}}{P_{1|1} - P_{1|0} + \pi_{01}}, \\ \frac{P_{0|0} \cdot \bar{Y}_{0,0} - (P_{0|1} - \pi_{01}) \cdot E(Y(0)|T = 00)^{UB}}{P_{1|1} - P_{1|0} + \pi_{01}} &\leq E(Y(0)|T = 10) \leq \frac{P_{0|0} \cdot \bar{Y}_{0,0} - (P_{0|1} - \pi_{01}) \cdot E(Y(0)|T = 00)^{LB}}{P_{1|1} - P_{1|0} + \pi_{01}}, \\ \frac{P_{1|0} \cdot \bar{Y}_{0,1} - (P_{1|0} - \pi_{01}) \cdot E(Y(1)|T = 11)^{UB}}{\pi_{01}} &\leq E(Y(1)|T = 01) \leq \frac{P_{1|0} \cdot \bar{Y}_{0,1} - (P_{1|0} - \pi_{01}) \cdot E(Y(1)|T = 11)^{LB}}{\pi_{01}}, \\ \frac{P_{0|1} \cdot \bar{Y}_{1,0} - (P_{0|1} - \pi_{01}) \cdot E(Y(0)|T = 00)^{UB}}{\pi_{01}} &\leq E(Y(0)|T = 01) \leq \frac{P_{0|1} \cdot \bar{Y}_{1,0} - (P_{0|1} - \pi_{01}) \cdot E(Y(0)|T = 00)^{LB}}{\pi_{01}}. \end{aligned}$$

We use the following compact notation to refer to these bounds, $E(Y(d)|T = t)^{LB} \leq E(Y(d)|T = t) \leq E(Y(d)|T = t)^{UB}$, $d = 1, 0, t = 10, 01$.

The bounds on the ATE within some stratum t for π_{10} fixed are given by

$$\begin{aligned} \Delta_t^{UB}(\pi_{01}) &= [E(Y(1)|T = t)^{UB} - E(Y(0)|T = t)^{LB}], \\ \Delta_t^{LB}(\pi_{01}) &= [E(Y(1)|T = t)^{LB} - E(Y(0)|T = t)^{UB}], \end{aligned}$$

where conditioning on π_{10} is now made explicit in the bounds.

Finally, since π_{01} is unknown, the bounds on the ATEs within strata over all admissible π_{01} (which are shown to be sharp further below) are obtained by the following optimization:

$$\begin{aligned} \Delta_t^{UB} &= \sup_{\pi_{01} \in \mathcal{P}^*} \Delta_t^{UB}(\pi_{01}) = \sup_{\pi_{01} \in \mathcal{P}^*} [E(Y(1)|T = t)^{UB} - E(Y(0)|T = t)^{LB}], \\ \Delta_t^{LB} &= \inf_{\pi_{01} \in \mathcal{P}^*} \Delta_t^{LB}(\pi_{01}) = \inf_{\pi_{01} \in \mathcal{P}^*} [E(Y(1)|T = t)^{LB} - E(Y(0)|T = t)^{UB}], \end{aligned}$$

which for $t = 10, 01$ are equivalent to the bounds provided in the main text.

Since $\bar{Y}_{1,1}(\max |q_{1,1}^{11}|)$, $\bar{Y}_{0,1}(\max |q_{0,1}^{11}|)$, $\bar{Y}_{0,0}(\max |q_{0,0}^{00}|)$, and $\bar{Y}_{1,0}(\max |q_{1,0}^{00}|)$ are increasing in π_{01} , $\sup_{\pi_{01} \in \mathcal{P}^*} \Delta_{11}^{UB}(\pi_{01}) = \Delta_{11}^{UB}(\pi_{01}^{\max})$ and $\sup_{\pi_{01} \in \mathcal{P}^*} \Delta_{00}^{UB}(\pi_{01}) = \Delta_{00}^{UB}(\pi_{01}^{\max})$. Moreover, since $\bar{Y}_{1,1}(\min |q_{1,1}^{11}|)$, $\bar{Y}_{0,1}(\min |q_{0,1}^{11}|)$, $\bar{Y}_{0,0}(\min |q_{0,0}^{00}|)$, and $\bar{Y}_{1,0}(\min |q_{1,0}^{00}|)$ are decreasing in π_{01} , $\inf_{\pi_{01} \in \mathcal{P}^*} \Delta_{11}^{LB}(\pi_{01}) = \Delta_{11}^{LB}(\pi_{01}^{\max})$ and $\inf_{\pi_{01} \in \mathcal{P}^*} \Delta_{00}^{LB}(\pi_{01}) = \Delta_{00}^{LB}(\pi_{01}^{\max})$.

Sharpness

To demonstrate the sharpness of the proposed bounds, one needs to show that for each $E(Y(d)|T = t) \in [E(Y(d)|T = t)^{LB}, E(Y(d)|T = t)^{UB}]$, $d = 1, 0$, $t = 11, 10, 01, 00$, there exist distributions of T given Z and of $(Y(1), Y(0))$ given T and Z that are compatible with a data generating process that satisfies Assumption 2. As $E(Y(d)|T = t)^{LB}$ and $E(Y(d)|T = t)^{UB}$ are the smallest and largest values of the interval $[E(Y(d)|T = t)^{LB}, E(Y(d)|T = t)^{UB}]$, it is sufficient to prove the existence at those two extremes, because the values of $E(Y(d)|T = t)$ inside this interval can be achieved as convex combinations of the probability distributions of T given Z and the joint distributions of $(Y(1), Y(0))$ given T and Z that generate $E(Y(d)|T = t)^{LB}$ and $E(Y(d)|T = t)^{UB}$. The fact that the bounds conditional on π_{01} are continuous in π_{01} (as shown in Appendix A.1.4) and $\mathcal{P}^*$ is an interval (as shown in Lemma 1) implies that the identified sets for the ATEs are intervals, too. We show that such compatible distributions can indeed be found so that $E(Y(1)|T = t) = E(Y(1)|T = t)^{UB}$, $t = 10, 01, 00$, $E(Y(0)|T = t) = E(Y(0)|T = t)^{UB}$, $t = 11, 10, 01$, $E(Y(1)|T = 11) = E(Y(1)|T = 11)^{LB}$ and $E(Y(0)|T = 00) = E(Y(0)|T = 00)^{LB}$ hold under Assumption 2. To this end, let $h_t^z(y(1), y(0)) = f(y(1), y(0)|T = t, Z = z)$ be the conditional density of $(Y(1), Y(0))$ evaluated at $(y(1), y(0))$ given $T = t$ and $Z = z$. For the sake of brevity, we refer to $h_t^z(y(1), y(0))$ by h_t^z . Consider the following distributions of T given Z and $(Y(1), Y(0))$ given T and Z :

$$\begin{aligned} \Pr(T = 11|Z = 1) &= \Pr(T = 11|Z = 0) = P_{1|0} - \pi_{01}, \\ \Pr(T = 00|Z = 1) &= \Pr(T = 00|Z = 0) = P_{0|1} - \pi_{01}, \\ \Pr(T = 10|Z = 1) &= \Pr(T = 10|Z = 0) = P_{1|1} - P_{1|0} + \pi_{01}, \\ \Pr(T = 01|Z = 1) &= \Pr(T = 01|Z = 0) = \pi_{01}, \end{aligned} \tag{A.8}$$

where $\pi_{01} \in \mathcal{P}^*$. This probability distribution satisfies Assumption 2(ii), as it does not depend on Z .

We will now introduce four different specifications for h_t^z , $t = 11, 10, 01, 00$, $Z = 1, 0$, (henceforth denoted as $\{h_t^z\}$) that are (i) proper probability density functions, (ii) satisfy Assumption 2, (iii) are compatible with the data generating process, and (iv) reach

Case 1 the lower bound on the ATE of $t = 11$ and the upper bound on the ATE of $t = 00$,

Case 2 the upper bound on the ATE of $t = 11$ and the lower bound on the ATE of $t = 00$,

Case 3 the upper bound on the ATEs of $t = 10$ and $t = 01$,

Case 4 the lower bound on the ATEs of $t = 10$ and $t = 01$.

Case 1

Let the function $I\{A\}$ stand for the indicator function of a set A . Consider the following $\{h_t^z\}$:

$$\begin{aligned}
h_{11}^1 &= \begin{cases} I\{y(0) = y^{UB}\} \cdot f_Y(y(1)|D = 1, Z = 1, Y \leq F_{Y_{1,1}}^{-1}(q_{1,1}^{11})) & \text{if } \bar{Y}_{1,1}(\min |q_{1,1}^{11}|) \geq \bar{Y}_{0,1}(\min |q_{0,1}^{11}|) \\ I\{y(0) = y^{UB}\} \cdot \left\{ \alpha_{11}^1 f_Y(y(1)|D = 1, Z = 1, Y \leq F_{Y_{1,1}}^{-1}(q_{1,1}^{11})) \right. & \text{if } \bar{Y}_{1,1}(\min |q_{1,1}^{11}|) < \bar{Y}_{0,1}(\min |q_{0,1}^{11}|) \\ \quad \left. + (1 - \alpha_{11}^1) f_Y(y(1)|D = 1, Z = 1, Y \geq F_{Y_{1,1}}^{-1}(1 - q_{1,1}^{11})) \right\} & \end{cases} , \\
h_{11}^0 &= \begin{cases} I\{y(0) = y^{UB}\} \cdot \left\{ \alpha_{11}^0 f_Y(y(1)|D = 1, Z = 0, Y \leq F_{Y_{0,1}}^{-1}(q_{0,1}^{11})) \right. & \text{if } \bar{Y}_{1,1}(\min |q_{1,1}^{11}|) \geq \bar{Y}_{0,1}(\min |q_{0,1}^{11}|) \\ \quad \left. + (1 - \alpha_{11}^0) f_Y(y(1)|D = 1, Z = 0, Y \geq F_{Y_{0,1}}^{-1}(1 - q_{0,1}^{11})) \right\} & \\ I\{y(0) = y^{UB}\} \cdot f_Y(y(1)|D = 1, Z = 0, Y \leq F_{Y_{0,1}}^{-1}(q_{0,1}^{11})) & \text{if } \bar{Y}_{1,1}(\min |q_{1,1}^{11}|) < \bar{Y}_{0,1}(\min |q_{0,1}^{11}|) \end{cases} , \\
h_{00}^1 &= \begin{cases} I\{y(1) = y^{UB}\} \cdot f_Y(y(0)|D = 0, Z = 1, Y \leq F_{Y_{1,0}}^{-1}(q_{1,0}^{00})) & \text{if } \bar{Y}_{1,0}(\min |q_{1,0}^{00}|) \geq \bar{Y}_{0,0}(\min |q_{0,0}^{00}|) \\ I\{y(1) = y^{UB}\} \cdot \left\{ \alpha_{00}^1 f_Y(y(0)|D = 0, Z = 1, Y \leq F_{Y_{1,0}}^{-1}(q_{1,0}^{00})) \right. & \text{if } \bar{Y}_{1,0}(\min |q_{1,0}^{00}|) < \bar{Y}_{0,0}(\min |q_{0,0}^{00}|) \\ \quad \left. + (1 - \alpha_{00}^1) f_Y(y(0)|D = 0, Z = 1, Y \geq F_{Y_{1,0}}^{-1}(1 - q_{1,0}^{00})) \right\} & \end{cases} , \\
h_{00}^0 &= \begin{cases} I\{y(1) = y^{UB}\} \cdot \left\{ \alpha_{00}^0 f_Y(y(0)|D = 0, Z = 0, Y \leq F_{Y_{0,0}}^{-1}(q_{0,0}^{00})) \right. & \text{if } \bar{Y}_{1,0}(\min |q_{1,0}^{00}|) \geq \bar{Y}_{0,0}(\min |q_{0,0}^{00}|) \\ \quad \left. + (1 - \alpha_{00}^0) f_Y(y(0)|D = 0, Z = 0, Y \geq F_{Y_{0,0}}^{-1}(1 - q_{0,0}^{00})) \right\} & \\ I\{y(1) = y^{UB}\} \cdot f_Y(y(0)|D = 0, Z = 0, Y \leq F_{Y_{0,0}}^{-1}(q_{0,0}^{00})) & \text{if } \bar{Y}_{1,0}(\min |q_{1,0}^{00}|) < \bar{Y}_{0,0}(\min |q_{0,0}^{00}|) \end{cases} , \\
h_{10}^z &= \begin{cases} (P_{1|1} - P_{1|0} + \pi_{01})^{-2} \cdot (P_{1|1} \cdot f_Y(y(1)|D = 1, Z = 1) - (P_{1|0} - \pi_{01}) \cdot \int h_{11}^1 dy(0)) & \text{if } \pi_{01} > P_{1|0} - P_{1|1} \\ \quad \cdot (P_{0|0} \cdot f_Y(y(0)|D = 0, Z = 0) - (P_{0|1} - \pi_{01}) \cdot \int h_{00}^0 dy(1)) & \\ \text{arbitrary probability density function, (because } \pi_{10} = 0) & \text{if } \pi_{01} = P_{1|0} - P_{1|1}. \end{cases} , \\
h_{01}^z &= \begin{cases} \pi_{01}^{-2} \cdot (P_{1|0} \cdot f_Y(y(1)|D = 1, Z = 0) - (P_{1|0} - \pi_{01}) \cdot \int h_{11}^0 dy(0)) & \text{if } \pi_{01} > 0 \\ \quad \cdot (P_{0|1} \cdot f_Y(y(0)|D = 0, Z = 1) - (P_{0|1} - \pi_{01}) \cdot \int h_{00}^1 dy(1)) & \\ \text{arbitrary probability density function, (because } \pi_{01} = 0) & \text{if } \pi_{01} = 0. \end{cases} .
\end{aligned}$$

where we set π_{01} to $\pi_{01}^{\max}$ and the parameters

$$\begin{aligned}
\alpha_{11}^1 &= \frac{\bar{Y}_{1,1}(\max |q_{1,1}^{11}|) - \bar{Y}_{0,1}(\min |q_{0,1}^{11}|)}{\bar{Y}_{1,1}(\max |q_{1,1}^{11}|) - \bar{Y}_{1,1}(\min |q_{1,1}^{11}|)}, \\
\alpha_{11}^0 &= \frac{\bar{Y}_{0,1}(\max |q_{0,1}^{11}|) - \bar{Y}_{1,1}(\min |q_{1,1}^{11}|)}{\bar{Y}_{0,1}(\max |q_{0,1}^{11}|) - \bar{Y}_{0,1}(\min |q_{0,1}^{11}|)}, \\
\alpha_{00}^1 &= \frac{\bar{Y}_{1,0}(\max |q_{1,0}^{00}|) - \bar{Y}_{0,0}(\min |q_{0,0}^{00}|)}{\bar{Y}_{1,0}(\max |q_{1,0}^{00}|) - \bar{Y}_{1,0}(\min |q_{1,0}^{00}|)}, \\
\alpha_{00}^0 &= \frac{\bar{Y}_{0,0}(\max |q_{0,0}^{00}|) - \bar{Y}_{1,0}(\min |q_{1,0}^{00}|)}{\bar{Y}_{0,0}(\max |q_{0,0}^{00}|) - \bar{Y}_{0,0}(\min |q_{0,0}^{00}|)},
\end{aligned}$$

are chosen so that the mean independence within principal strata holds.

(i) Proper probability functions. We have to check that these functions are proper probability density

functions. For h_{11}^z and h_{00}^z it is sufficient to verify that $0 \leq \alpha_t^z \leq 1$ for $T = 11, 00$ and $z = 1, 0$, because then all functions h_{11}^z and h_{00}^z are products of two marginal probability density functions that are convex combinations of two proper marginal probability density functions. We show this for α_{11}^1 and the same argument extends to the other parameters. The parameter α_{11}^1 only occurs in h_{11}^1 if $\bar{Y}_{1,1}(\min |q_{1,1}^{11}|) < \bar{Y}_{0,1}(\min |q_{0,1}^{11}|)$.

Suppose that the denominator of α_{11}^1 is positive so that $\bar{Y}_{1,1}(\max |q_{1,1}^{11}|) > \bar{Y}_{1,1}(\min |q_{1,1}^{11}|)$. As for the non-negativity of α_{11}^1 , it is therefore sufficient to verify that the numerator is nonnegative. Assume the contrary, so that $\bar{Y}_{1,1}(\max |q_{1,1}^{11}|) < \bar{Y}_{0,1}(\min |q_{0,1}^{11}|)$. Now we have that

$$\bar{Y}_{1,1}(\min |q_{1,1}^{11}|) \leq \bar{Y}_{1,1} \leq \bar{Y}_{1,1}(\max |q_{1,1}^{11}|) < \bar{Y}_{0,1}(\min |q_{0,1}^{11}|) \leq \bar{Y}_{0,1} \leq \bar{Y}_{0,1}(\max |q_{0,1}^{11}|), \quad (\text{A.9})$$

where the weak inequalities follow from the definition of $\bar{Y}_{z,d}(\min |q_{z,d}^t|)$ and $\bar{Y}_{z,d}(\max |q_{z,d}^t|)$.

Because $\pi_{01}^{\max}$ is an admissible value for π_{01} , we know that there exists some set of proper probability densities $\{h_t^{*z}\}$ that satisfy Assumptions 1 and 2 and are compatible with the data. Consider the value of $E[Y(1)|T = 11]$ that is implied by $\{h_t^{*z}\}$, so that $E[Y(1)|T = 11] = \iint y(1)h_{11}^{*1} dy(0) dy(1) = \iint y(1)h_{11}^{*0} dy(0) dy(1)$. We have proven the validity of the bounds for $E[Y(1)|T = 11]$, so it must hold that

$$\max \{ \bar{Y}_{1,1}(\min |q_{1,1}^{11}|), \bar{Y}_{0,1}(\min |q_{0,1}^{11}|) \} \leq E[Y(1)|T = 11] \leq \min \{ \bar{Y}_{1,1}(\max |q_{1,1}^{11}|), \bar{Y}_{0,1}(\max |q_{0,1}^{11}|) \}.$$

This results in

$$\bar{Y}_{0,1}(\min |q_{0,1}^{11}|) \leq E[Y(1)|T = 11] \leq \bar{Y}_{1,1}(\max |q_{1,1}^{11}|),$$

which contradicts (A.9). We therefore conclude that $\bar{Y}_{0,1}(\min |q_{0,1}^{11}|) \leq \bar{Y}_{1,1}(\max |q_{1,1}^{11}|)$ and consequently $0 \leq \alpha_{11}^1$. As $\bar{Y}_{1,0}(\min |q_{1,0}^{11}|) > \bar{Y}_{1,1}(\min |q_{1,1}^{11}|)$ we obtain $\alpha_{11}^1 \leq 1$.

In the case that $\bar{Y}_{1,1}(\max |q_{1,1}^{11}|) = \bar{Y}_{1,1}(\min |q_{1,1}^{11}|)$, we immediately get that the upper bound on $E[Y(1)|T = 11]$ is smaller than the lower bound and this contradicts the admissibility of π_{01} .

Let us now consider h_{10}^1 . It is a product of two marginal probability densities. Both the first and the second parenthesis in the expression for h_{10}^1 integrate to $(P_{1|1} - P_{1|0} + \pi_{01})$, so it remains to verify that both marginal probability density functions are positive.

Consider the first parenthesis, that is $(P_{1|1} \cdot f_Y(y(1)|D = 1, Z = 1) - (P_{1|0} - \pi_{01}) \cdot \int h_{11}^1 dy(0))$, the marginal density of h_{10}^1 with respect to $y(1)$. Assume that $\bar{Y}_{1,1}(\min |q_{1,1}^{11}|) \geq \bar{Y}_{0,1}(\min |q_{0,1}^{11}|)$. Because $q_{1,1}^{11} = \frac{P_{1|0} - \pi_{01}}{P_{1|1}}$, it follows that the parenthesis,

$$\left(P_{1|1} \cdot f_Y(y(1)|D = 1, Z = 1) - (P_{1|0} - \pi_{01}) \cdot f_Y(y(1)|D = 1, Z = 1, Y \leq F_{Y_{1,1}}^{-1}(q_{1,1}^{11})) \right), \quad \text{is zero for } y(1) \leq F_{Y_{1,1}}^{-1}(q_{1,1}^{11}) \text{ and equal to } P_{1|1} \cdot f_Y(y(1)|D = 1, Z = 1) \geq 0 \text{ elsewhere.}$$

Now assume $\bar{Y}_{1,1}(\min |q_{1,1}^{11}|) < \bar{Y}_{0,1}(\min |q_{0,1}^{11}|)$.

- If $q_{1,1}^{11} < 0.5$ so that $F_{Y_{1,1}}^{-1}(q_{1,1}^{11}) < F_{Y_{1,1}}^{-1}(1 - q_{1,1}^{11})$, we inspect three cases:
 - (i) $y(1) < F_{Y_{1,1}}^{-1}(q_{1,1}^{11})$: the second component of the marginal distribution of h_{11}^1 is zero and the fact that $0 \leq \alpha_{11}^1 \leq 1$ guarantees the non-negativity of the parenthesis,
 - (ii) $F_{Y_{1,1}}^{-1}(q_{1,1}^{11}) < y(1) < F_{Y_{1,1}}^{-1}(1 - q_{1,1}^{11})$: the marginal density of h_{11}^1 is zero, so the parenthesis is equal to $P_{1|1} f_Y(y(1)|Z=1, D=1)$,
 - (iii) $y(1) > F_{Y_{1,1}}^{-1}(1 - q_{1,1}^{11})$: the first component of the marginal distribution of h_{11}^1 is zero and the fact that $0 \leq \alpha_{11}^1 \leq 1$ guarantees the non-negativity of the parenthesis.
- If $q_{1,1}^{11} > 0.5$ so that $F_{Y_{1,1}}^{-1}(q_{1,1}^{11}) > F_{Y_{1,1}}^{-1}(1 - q_{1,1}^{11})$, we also inspect three cases:
 - (i) $y(1) < F_{Y_{1,1}}^{-1}(1 - q_{1,1}^{11})$: the second component of the marginal distribution of h_{11}^1 is zero and the fact that $0 \leq \alpha_{11}^1 \leq 1$ guarantees the non-negativity of the parenthesis
 - (ii) $F_{Y_{1,1}}^{-1}(1 - q_{1,1}^{11}) < y(1) < F_{Y_{1,1}}^{-1}(q_{1,1}^{11})$: both components of the marginal distribution of h_{11}^1 are equal to $f_Y(y(1)|Z=1, D=1)/q_{1,1}^{11}$ and therefore the whole parenthesis becomes 0.
 - (iii) $y(1) > F_{Y_{1,1}}^{-1}(q_{1,1}^{11})$: the first component of the marginal distribution of h_{11}^1 is zero and the fact that $0 \leq \alpha_{11}^1 \leq 1$ guarantees the non-negativity of the parenthesis.

The non-negativity of the second marginal of h_{10}^1 follows in a similar fashion.

(ii) Mean independence within principal strata

We have to show that for a given type $T = t$, the marginal probability densities of $\{h_t^z\}$ produce the same expected values of the potential outcomes across $z = 1, 0$. Let us consider h_{11}^1 and h_{11}^0 . Both probability density functions lead to $E[Y(0)|T=11] = y^{UB}$. If $\bar{Y}_{1,1}(\min |q_{1,1}^{11}|) \geq \bar{Y}_{0,1}(\min |q_{0,1}^{11}|)$, then

$$\iint y(1) h_{11}^1 dy(1) dy(0) = \bar{Y}_{1,1}(\min |q_{1,1}^{11}|),$$

and

$$\begin{aligned} \iint y(1) h_{11}^0 dy(1) dy(0) &= \iint y(1) \left\{ \alpha_{11}^0 f_Y(y(1)|D=1, Z=0, Y \leq F_{Y_{0,1}}^{-1}(q_{0,1}^{11})) \right. \\ &\quad \left. + (1 - \alpha_{11}^0) f_Y(y(1)|D=1, Z=0, Y \geq F_{Y_{0,1}}^{-1}(1 - q_{0,1}^{11})) dy(1) dy(0) \right\} \\ &= \alpha_{11}^0 \bar{Y}_{0,1}(\min |q_{0,1}^{11}|) + (1 - \alpha_{11}^0) \bar{Y}_{0,1}(\max |q_{0,1}^{11}|) \\ &= \alpha_{11}^0 (\bar{Y}_{0,1}(\min |q_{0,1}^{11}|) - \bar{Y}_{0,1}(\max |q_{0,1}^{11}|)) + \bar{Y}_{0,1}(\max |q_{0,1}^{11}|) \\ &= \bar{Y}_{1,1}(\min |q_{1,1}^{11}|). \end{aligned}$$

Now if $\bar{Y}_{1,1}(\min |q_{1,1}^{11}|) < \bar{Y}_{0,1}(\min |q_{0,1}^{11}|)$, we have that

$$\iint y(1)h_{11}^0 dy(1) dy(0) = \bar{Y}_{0,1}(\min |q_{0,1}^{11}|),$$

and

$$\begin{aligned} \iint y(1)h_{11}^1 dy(1) dy(0) &= \iint y(1) \left\{ \alpha_{11}^1 f_Y(y(1)|D=1, Z=1, Y \leq F_{Y_{1,1}}^{-1}(q_{1,1}^{11})) \right. \\ &\quad \left. + (1 - \alpha_{11}^1) f_Y(y(1)|D=1, Z=1, Y \geq F_{Y_{1,1}}^{-1}(1 - q_{1,1}^{11})) \right\} dy(1) dy(0) \\ &= \alpha_{11}^1 \bar{Y}_{1,1}(\min |q_{1,1}^{11}|) + (1 - \alpha_{11}^1) \bar{Y}_{1,1}(\max |q_{1,1}^{11}|) \\ &= \alpha_{11}^1 (\bar{Y}_{1,1}(\min |q_{1,1}^{11}|) - \bar{Y}_{1,1}(\max |q_{1,1}^{11}|)) + \bar{Y}_{1,1}(\max |q_{1,1}^{11}|) \\ &= \bar{Y}_{0,1}(\min |q_{0,1}^{11}|). \end{aligned}$$

Mean independence of potential outcomes within principal strata for $T = 11, 00$ together with the definition of $\{h_t^z\}$ in (A.9) guarantees that the mean independence within principal strata must also hold for $T = 10, 01$.

(iii) Compatibility with the data generating process. From the definition of $\{h_t^z\}$ (A.9), it follows that $\{h_t^z\}$ are compatible with the data generating process:

$$\begin{aligned} P_{1|1} \cdot f_Y(y(1)|Z=1, D=1) &= (P_{1|0} - \pi_{01}) \int h_{11}^1 dy(0) + (P_{1|1} - P_{1|0} + \pi_{01}) \int h_{10}^1 dy(0), \\ P_{1|0} \cdot f_Y(y(1)|Z=0, D=1) &= (P_{1|0} - \pi_{01}) \int h_{11}^0 dy(0) + \pi_{01} \int h_{01}^0 dy(0), \\ P_{0|1} \cdot f_Y(y(0)|Z=1, D=0) &= \pi_{01} \int h_{01}^1 dy(1) + (P_{0|1} - \pi_{01}) \int h_{00}^1 dy(1), \\ P_{0|0} \cdot f_Y(y(0)|Z=0, D=0) &= (P_{1|1} - P_{1|0} + \pi_{01}) \int h_{10}^0 dy(1) + (P_{0|1} - \pi_{01}) \int h_{00}^0 dy(1). \end{aligned} \tag{A.10}$$

The fact that $\{h_t^z\}$ are proper probability density functions and compatible with the data generating process and that they satisfy Assumption 2 extends to Cases 2 – 4.

(iv) The upper bound on the ATE of $T = 11$ and the lower bound on the ATE of $T = 00$ are attained. We have already shown in part (ii) that

$$\begin{aligned} E(Y(0)|T=11) &= \iint y(0)h_{11}^z dy(1) dy(0) = y^{UB}, \\ E(Y(1)|T=00) &= \iint y(1)h_{00}^z dy(1) dy(0) = y^{UB}, \\ E(Y(1)|T=11) &= \iint y(1)h_{11}^z dy(1) dy(0) = \max(\bar{Y}_{0,1}(\min |q_{0,1}^{11}|), \bar{Y}_{1,1}(\min |q_{1,1}^{11}|)), \end{aligned} \tag{A.11}$$

hold for $z = 1, 0$. We can show in an analogous way that the following equation holds:

$$E(Y(0)|T = 00) = \iint y(0)h_{00}^z dy(1) dy(0) = \max(\bar{Y}_{0,0}(\min |q_{0,0}^{00}), \bar{Y}_{1,0}(\min |q_{1,0}^{00})).$$

The first equality signs in each of the previous four lines in (A.11) and (A.10) follow from (A.9). The respective second equations in the first and second lines follow from the fact that h_{11} is zero at $y(0) \neq y^{UB}$ and that h_{00} is zero at $y(1) \neq y^{UB}$. We recall the definition of $\bar{Y}_{z,d}(\min |q_{z,d}^t) = E(Y|Z = z, D = d, Y \leq F_{Y_{z,d}}^{-1}(q_{z,d}^t))$ that motivates the respective second equations in third and fourth lines. We now inspect the remaining mean potential outcomes within principal strata. Because of mean independence within principal strata we obtain for both $z = 1$ and $z = 0$:

$$\begin{aligned} E(Y(1)|T = 10) &= \iint y(1)h_{10}^z dy(1) dy(0) \\ &= \iint y(1) \frac{1}{(P_{1|1} - P_{1|0} + \pi_{01})^2} \cdot \left(P_{1|1} \cdot f_Y(y(1)|D = 1, Z = 1) - (P_{1|0} - \pi_{01}) \cdot \int h_{11}^z dy(0) \right) \\ &\quad \cdot \left(P_{0|0} \cdot f_Y(y(0)|D = 0, Z = 0) - (P_{0|1} - \pi_{01}) \cdot \int h_{00}^z dy(1) \right) dy(1) dy(0) \\ &= \frac{1}{(P_{1|1} - P_{1|0} + \pi_{01})} \int y(1) \left(P_{1|1} \cdot f_Y(y(1)|D = 1, Z = 1) - (P_{1|0} - \pi_{01}) \cdot \int h_{11}^z dy(0) \right) dy(1) \\ &\quad \cdot \int \frac{(P_{0|0} \cdot f_Y(y(0)|D = 0, Z = 0) - (P_{0|1} - \pi_{01}) \cdot \int h_{00}^z dy(1))}{P_{1|1} - P_{1|0} + \pi_{01}} dy(0) \\ &= \frac{P_{1|1} \cdot \bar{Y}_{1,1} - (P_{1|0} - \pi_{01}) \cdot \max(\bar{Y}_{0,1}(\min |q_{0,1}^{11}), \bar{Y}_{1,1}(\min |q_{1,1}^{11}))}{P_{1|1} - P_{1|0} + \pi_{01}}. \end{aligned}$$

The first equation follows from (A.9). In the second equation we plug in h_{10}^z . In the third equation we use the fact that the part in the first parenthesis in the definition of h_{10}^z does not depend on $y(0)$ and that the second part does not depend on $y(1)$. The last equation follows from (A.11) and from the fact that the second integral is equal to one. We obtain the following equations in an analogous way

$$\begin{aligned} E(Y(0)|T = 10) &= \iint y(0)h_{10}^z dy(1) dy(0) = \frac{P_{0|0} \cdot \bar{Y}_{0,0} - (P_{0|1} - \pi_{01}) \cdot \max(\bar{Y}_{0,0}(\min |q_{0,0}^{00}), \bar{Y}_{1,0}(\min |q_{1,0}^{00}))}{P_{1|1} - P_{1|0} + \pi_{01}}, \\ E(Y(1)|T = 01) &= \iint y(1)h_{01}^z dy(1) dy(0) = \frac{P_{1|0} \cdot \bar{Y}_{0,1} - (P_{1|0} - \pi_{01}) \cdot \max(\bar{Y}_{0,1}(\min |q_{0,1}^{11}), \bar{Y}_{1,1}(\min |q_{1,1}^{11}))}{\pi_{01}}, \\ E(Y(0)|T = 01) &= \iint y(0)h_{01}^z dy(1) dy(0) = \frac{P_{0|1} \cdot \bar{Y}_{1,0} - (P_{0|1} - \pi_{01}) \cdot \max(\bar{Y}_{0,0}(\min |q_{0,0}^{00}), \bar{Y}_{1,0}(\min |q_{1,0}^{00}))}{\pi_{01}}. \end{aligned}$$

This demonstrates that $E(Y(1)|T = t)^{UB}, t = 10, 01, 00, E(Y(0)|T = t)^{UB}, t = 11, 10, 01, E(Y(1)|T = 11)^{LB}$, and $E(Y(0)|T = 00)^{LB}$ are sharp bounds. Therefore, we have found distributions of the potential outcomes $(Y(1), Y(0))$ given T, Z and $\Pr(T = t|Z = z)$, i.e. strata probabilities given Z (which are uniquely determined by the value of $\pi_{01} = \pi_{01}^{\max}$), that achieve the lower bound on the ATE for $t = 11$ and the upper bound on the ATE for $t = 00$. In Appendix A.1.3 we show that if $\mathcal{P}^*$ is non-empty, then $\pi_{01}^{\max} = \min\{P_{1|0}, P_{0|1}\}$ must be admissible.

Sharpness of the bounds in Case 2 can be proven in an analogous way. Cases 3 and 4 follow similarly, but with the important difference that the bounds are suprema (infima) of the bounds conditional on π_{01} . In these cases, we also show that the suprema (infima) of $\mathcal{P}^*$ are attained.

Case 2 By setting π_{01} to $\pi_{01}^{\max}$, the distribution of the types to (A.8), h_t for $t \in \{11, 10, 01, 00\}$ to

$$\begin{aligned}
h_{11}^1 &= \begin{cases} I\{y(0) = y^{LB}\} \cdot f_Y(y(1)|D=1, Z=1, Y \geq F_{Y_{1,1}}^{-1}(1 - q_{1,1}^{11})) & \text{if } \bar{Y}_{1,1}(\max |q_{1,1}^{11}|) \leq \bar{Y}_{0,1}(\max |q_{0,1}^{11}|) \\ I\{y(0) = y^{LB}\} \cdot \left\{ \alpha_{11}^1 f_Y(y(1)|D=1, Z=1, Y \leq F_{Y_{1,1}}^{-1}(q_{1,1}^{11})) \right. & \text{if } \bar{Y}_{1,1}(\max |q_{1,1}^{11}|) > \bar{Y}_{0,1}(\max |q_{0,1}^{11}|) \\ \quad \left. + (1 - \alpha_{11}^1) f_Y(y(1)|D=1, Z=1, Y \geq F_{Y_{1,1}}^{-1}(1 - q_{1,1}^{11})) \right\} & \end{cases} , \\
h_{11}^0 &= \begin{cases} I\{y(0) = y^{LB}\} \cdot \left\{ \alpha_{11}^0 f_Y(y(1)|D=1, Z=0, Y \leq F_{Y_{0,1}}^{-1}(q_{0,1}^{11})) \right. & \text{if } \bar{Y}_{1,1}(\max |q_{1,1}^{11}|) \leq \bar{Y}_{0,1}(\max |q_{0,1}^{11}|) \\ \quad \left. + (1 - \alpha_{11}^0) f_Y(y(1)|D=1, Z=0, Y \geq F_{Y_{0,1}}^{-1}(1 - q_{0,1}^{11})) \right\} & \\ I\{y(0) = y^{LB}\} \cdot f_Y(y(1)|D=1, Z=0, Y \geq F_{Y_{0,1}}^{-1}(1 - q_{0,1}^{11})) & \text{if } \bar{Y}_{1,1}(\max |q_{1,1}^{11}|) > \bar{Y}_{0,1}(\max |q_{0,1}^{11}|) \end{cases} , \\
h_{00}^1 &= \begin{cases} I\{y(1) = y^{LB}\} \cdot f_Y(y(0)|D=0, Z=1, Y \geq F_{Y_{1,0}}^{-1}(1 - q_{1,0}^{00})) & \text{if } \bar{Y}_{1,0}(\max |q_{1,0}^{00}|) \leq \bar{Y}_{0,0}(\max |q_{0,0}^{00}|) \\ I\{y(1) = y^{LB}\} \cdot \left\{ \alpha_{00}^1 f_Y(y(0)|D=0, Z=1, Y \leq F_{Y_{1,0}}^{-1}(q_{1,0}^{00})) \right. & \text{if } \bar{Y}_{1,0}(\max |q_{1,0}^{00}|) > \bar{Y}_{0,0}(\max |q_{0,0}^{00}|) \\ \quad \left. + (1 - \alpha_{00}^1) f_Y(y(0)|D=0, Z=1, Y \geq F_{Y_{1,0}}^{-1}(1 - q_{1,0}^{00})) \right\} & \end{cases} , \\
h_{00}^0 &= \begin{cases} I\{y(1) = y^{LB}\} \cdot \left\{ \alpha_{00}^0 f_Y(y(0)|D=0, Z=0, Y \leq F_{Y_{0,0}}^{-1}(q_{0,0}^{00})) \right. & \text{if } \bar{Y}_{1,0}(\max |q_{1,0}^{00}|) \leq \bar{Y}_{0,0}(\max |q_{0,0}^{00}|) \\ \quad \left. + (1 - \alpha_{00}^0) f_Y(y(0)|D=0, Z=0, Y \geq F_{Y_{0,0}}^{-1}(1 - q_{0,0}^{00})) \right\} & \\ I\{y(1) = y^{LB}\} \cdot f_Y(y(0)|D=0, Z=0, Y \geq F_{Y_{0,0}}^{-1}(1 - q_{0,0}^{00})) & \text{if } \bar{Y}_{1,0}(\max |q_{1,0}^{00}|) > \bar{Y}_{0,0}(\max |q_{0,0}^{00}|) \end{cases} ,
\end{aligned}$$

and h_t^z for $t = 10, 01$ and $z = 1, 0$ to values as defined in (A.9), where

$$\begin{aligned}
\alpha_{11}^1 &= \frac{\bar{Y}_{1,1}(\max |q_{1,1}^{11}|) - \bar{Y}_{0,1}(\max |q_{0,1}^{11}|)}{\bar{Y}_{1,1}(\max |q_{1,1}^{11}|) - \bar{Y}_{1,1}(\min |q_{1,1}^{11}|)} , \\
\alpha_{11}^0 &= \frac{\bar{Y}_{0,1}(\max |q_{0,1}^{11}|) - \bar{Y}_{1,1}(\max |q_{1,1}^{11}|)}{\bar{Y}_{0,1}(\max |q_{0,1}^{11}|) - \bar{Y}_{0,1}(\min |q_{0,1}^{11}|)} , \\
\alpha_{00}^1 &= \frac{\bar{Y}_{1,0}(\max |q_{1,0}^{00}|) - \bar{Y}_{0,0}(\max |q_{0,0}^{00}|)}{\bar{Y}_{1,0}(\max |q_{1,0}^{00}|) - \bar{Y}_{1,0}(\min |q_{1,0}^{00}|)} , \\
\alpha_{00}^0 &= \frac{\bar{Y}_{0,0}(\max |q_{0,0}^{00}|) - \bar{Y}_{1,0}(\max |q_{1,0}^{00}|)}{\bar{Y}_{0,0}(\max |q_{0,0}^{00}|) - \bar{Y}_{0,0}(\min |q_{0,0}^{00}|)} ,
\end{aligned}$$

one obtains the lower bound on the ATE for $t = 11$ and the upper bound on the ATE for $t = 00$.

Case 3

For a given value of π_{01} , h_t for $t = 11, 10, 01, 00$ is set to

$$\begin{aligned}
h_{11}^1 &= \begin{cases} I\{y(0) = y^{LB}\} \cdot f_Y(y(1)|D=1, Z=1, Y \geq F_{Y_{1,1}}^{-1}(1 - q_{1,1}^{11})) & \text{if } \bar{Y}_{1,1}(\max |q_{1,1}^{11}|) \leq \bar{Y}_{0,1}(\max |q_{0,1}^{11}|) \\ I\{y(0) = y^{LB}\} \cdot \left\{ \alpha_{11}^1 f_Y(y(1)|D=1, Z=1, Y \leq F_{Y_{1,1}}^{-1}(q_{1,1}^{11})) \right. & \text{if } \bar{Y}_{1,1}(\max |q_{1,1}^{11}|) > \bar{Y}_{0,1}(\max |q_{0,1}^{11}|) \\ \quad \left. + (1 - \alpha_{11}^1) f_Y(y(1)|D=1, Z=1, Y \geq F_{Y_{1,1}}^{-1}(1 - q_{1,1}^{11})) \right\} & \end{cases} , \\
h_{11}^0 &= \begin{cases} I\{y(0) = y^{LB}\} \cdot \left\{ \alpha_{11}^0 f_Y(y(1)|D=1, Z=0, Y \leq F_{Y_{0,1}}^{-1}(q_{0,1}^{11})) \right. & \text{if } \bar{Y}_{1,1}(\max |q_{1,1}^{11}|) \leq \bar{Y}_{0,1}(\max |q_{0,1}^{11}|) \\ \quad \left. + (1 - \alpha_{11}^0) f_Y(y(1)|D=1, Z=0, Y \geq F_{Y_{0,1}}^{-1}(1 - q_{0,1}^{11})) \right\} & \\ I\{y(0) = y^{LB}\} \cdot f_Y(y(1)|D=1, Z=0, Y \geq F_{Y_{0,1}}^{-1}(1 - q_{0,1}^{11})) & \text{if } \bar{Y}_{1,1}(\max |q_{1,1}^{11}|) > \bar{Y}_{0,1}(\max |q_{0,1}^{11}|) \end{cases} , \\
h_{00}^1 &= \begin{cases} I\{y(1) = y^{UB}\} \cdot f_Y(y(0)|D=0, Z=1, Y \leq F_{Y_{1,0}}^{-1}(q_{1,0}^{00})) & \text{if } \bar{Y}_{1,0}(\min |q_{1,0}^{00}|) \geq \bar{Y}_{0,0}(\min |q_{0,0}^{00}|) \\ I\{y(1) = y^{UB}\} \cdot \left\{ \alpha_{00}^1 f_Y(y(0)|D=0, Z=1, Y \leq F_{Y_{1,0}}^{-1}(q_{1,0}^{00})) \right. & \text{if } \bar{Y}_{1,0}(\min |q_{1,0}^{00}|) < \bar{Y}_{0,0}(\min |q_{0,0}^{00}|) \\ \quad \left. + (1 - \alpha_{00}^1) f_Y(y(0)|D=0, Z=1, Y \geq F_{Y_{1,0}}^{-1}(1 - q_{1,0}^{00})) \right\} & \end{cases} , \\
h_{00}^0 &= \begin{cases} I\{y(1) = y^{UB}\} \cdot \left\{ \alpha_{00}^0 f_Y(y(0)|D=0, Z=0, Y \leq F_{Y_{0,0}}^{-1}(q_{0,0}^{00})) \right. & \text{if } \bar{Y}_{1,0}(\min |q_{1,0}^{00}|) \geq \bar{Y}_{0,0}(\min |q_{0,0}^{00}|) \\ \quad \left. + (1 - \alpha_{00}^0) f_Y(y(0)|D=0, Z=0, Y \geq F_{Y_{0,0}}^{-1}(1 - q_{0,0}^{00})) \right\} & \\ I\{y(1) = y^{UB}\} \cdot f_Y(y(0)|D=0, Z=0, Y \leq F_{Y_{0,0}}^{-1}(q_{0,0}^{00})) & \text{if } \bar{Y}_{1,0}(\min |q_{1,0}^{00}|) < \bar{Y}_{0,0}(\min |q_{0,0}^{00}|) \end{cases} , \\
h_{10}^z &= \begin{cases} (P_{1|1} - P_{1|0} + \pi_{01})^{-2} \cdot (P_{1|1} \cdot f_Y(y(1)|D=1, Z=1) - (P_{1|0} - \pi_{01}) \cdot \int h_{11}^1 dy(0)) & \text{if } \pi_{01} > P_{1|0} - P_{1|1} \\ \quad \cdot (P_{0|0} \cdot f_Y(y(0)|D=0, Z=0) - (P_{0|1} - \pi_{01}) \cdot \int h_{00}^0 dy(1)) & \\ I\{y(0) = y^{LB}\} \cdot I\{y(1) = y^{UB}\} & \text{if } \pi_{01} = P_{1|0} - P_{1|1} . \end{cases} , \\
h_{01}^z &= \begin{cases} \pi_{01}^{-2} \cdot (P_{1|0} \cdot f_Y(y(1)|D=1, Z=0) - (P_{1|0} - \pi_{01}) \cdot \int h_{11}^0 dy(0)) & \text{if } \pi_{01} > 0 \\ \quad \cdot (P_{0|1} \cdot f_Y(y(0)|D=0, Z=1) - (P_{0|1} - \pi_{01}) \cdot \int h_{00}^1 dy(1)) & \\ I\{y(0) = y^{LB}\} \cdot I\{y(1) = y^{UB}\} & \text{if } \pi_{01} = 0 . \end{cases} ,
\end{aligned}$$

where

$$\begin{aligned}
\alpha_{11}^1 &= \frac{\bar{Y}_{1,1}(\max |q_{1,1}^{11}|) - \bar{Y}_{0,1}(\max |q_{0,1}^{11}|)}{\bar{Y}_{1,1}(\max |q_{1,1}^{11}|) - \bar{Y}_{1,1}(\min |q_{1,1}^{11}|)} , \\
\alpha_{11}^0 &= \frac{\bar{Y}_{0,1}(\max |q_{0,1}^{11}|) - \bar{Y}_{1,1}(\max |q_{1,1}^{11}|)}{\bar{Y}_{0,1}(\max |q_{0,1}^{11}|) - \bar{Y}_{0,1}(\min |q_{0,1}^{11}|)} , \\
\alpha_{00}^1 &= \frac{\bar{Y}_{1,0}(\max |q_{1,0}^{00}|) - \bar{Y}_{0,0}(\min |q_{0,0}^{00}|)}{\bar{Y}_{1,0}(\max |q_{1,0}^{00}|) - \bar{Y}_{1,0}(\min |q_{1,0}^{00}|)} , \\
\alpha_{00}^0 &= \frac{\bar{Y}_{0,0}(\max |q_{0,0}^{00}|) - \bar{Y}_{1,0}(\min |q_{1,0}^{00}|)}{\bar{Y}_{0,0}(\max |q_{0,0}^{00}|) - \bar{Y}_{0,0}(\min |q_{0,0}^{00}|)} ,
\end{aligned}$$

entails the upper bounds on the ATE for $t = 10$ if π_{01} maximizes (7) and the upper bound on the ATE for $t = 01$ if π_{01} maximizes (8). Note that because the bounds conditional on π_{01} are continuous in π_{01} (as shown in Appendix A.1.4) and $\mathcal{P}^*$ is an interval (as shown in Lemma 1), we get that the maxima of $\mathcal{P}^*$ are attained by the extreme value theorem.

Case 4

Consider the following conditional densities

$$\begin{aligned}
h_{11}^1 &= \begin{cases} I\{y(0) = y^{UB}\} \cdot f_Y(y(1)|D=1, Z=1, Y \leq F_{Y_{1,1}}^{-1}(q_{1,1}^{11})) & \text{if } \bar{Y}_{1,1}(\min |q_{1,1}^{11}|) \geq \bar{Y}_{0,1}(\min |q_{0,1}^{11}|) \\ I\{y(0) = y^{UB}\} \cdot \left\{ \alpha_{11}^1 f_Y(y(1)|D=1, Z=1, Y \leq F_{Y_{1,1}}^{-1}(q_{1,1}^{11})) \right. & \text{if } \bar{Y}_{1,1}(\min |q_{1,1}^{11}|) < \bar{Y}_{0,1}(\min |q_{0,1}^{11}|) \\ \quad \left. + (1 - \alpha_{11}^1) f_Y(y(1)|D=1, Z=1, Y \geq F_{Y_{1,1}}^{-1}(1 - q_{1,1}^{11})) \right\} & \end{cases} , \\
h_{11}^0 &= \begin{cases} I\{y(0) = y^{UB}\} \cdot \left\{ \alpha_{11}^0 f_Y(y(1)|D=1, Z=0, Y \leq F_{Y_{0,1}}^{-1}(q_{0,1}^{11})) \right. & \text{if } \bar{Y}_{1,1}(\min |q_{1,1}^{11}|) \geq \bar{Y}_{0,1}(\min |q_{0,1}^{11}|) \\ \quad \left. + (1 - \alpha_{11}^0) f_Y(y(1)|D=1, Z=0, Y \geq F_{Y_{0,1}}^{-1}(1 - q_{0,1}^{11})) \right\} & \\ I\{y(0) = y^{UB}\} \cdot f_Y(y(1)|D=1, Z=0, Y \leq F_{Y_{0,1}}^{-1}(q_{0,1}^{11})) & \text{if } \bar{Y}_{1,1}(\min |q_{1,1}^{11}|) < \bar{Y}_{0,1}(\min |q_{0,1}^{11}|) \end{cases} , \\
h_{00}^1 &= \begin{cases} I\{y(1) = y^{LB}\} \cdot f_Y(y(0)|D=0, Z=1, Y \geq F_{Y_{1,0}}^{-1}(1 - q_{1,0}^{00})) & \text{if } \bar{Y}_{1,0}(\max |q_{1,0}^{00}|) \leq \bar{Y}_{0,0}(\max |q_{0,0}^{00}|) \\ I\{y(1) = y^{LB}\} \cdot \left\{ \alpha_{00}^1 f_Y(y(0)|D=0, Z=1, Y \leq F_{Y_{1,0}}^{-1}(q_{1,0}^{00})) \right. & \text{if } \bar{Y}_{1,0}(\max |q_{1,0}^{00}|) > \bar{Y}_{0,0}(\max |q_{0,0}^{00}|) \\ \quad \left. + (1 - \alpha_{00}^1) f_Y(y(0)|D=0, Z=1, Y \geq F_{Y_{1,0}}^{-1}(1 - q_{1,0}^{00})) \right\} & \end{cases} , \\
h_{00}^0 &= \begin{cases} I\{y(1) = y^{LB}\} \cdot \left\{ \alpha_{00}^0 f_Y(y(0)|D=0, Z=0, Y \leq F_{Y_{0,0}}^{-1}(q_{0,0}^{00})) \right. & \text{if } \bar{Y}_{1,0}(\max |q_{1,0}^{00}|) \leq \bar{Y}_{0,0}(\max |q_{0,0}^{00}|) \\ \quad \left. + (1 - \alpha_{00}^0) f_Y(y(0)|D=0, Z=0, Y \geq F_{Y_{0,0}}^{-1}(1 - q_{0,0}^{00})) \right\} & \\ I\{y(1) = y^{LB}\} \cdot f_Y(y(0)|D=0, Z=0, Y \geq F_{Y_{0,0}}^{-1}(1 - q_{0,0}^{00})) & \text{if } \bar{Y}_{1,0}(\max |q_{1,0}^{00}|) > \bar{Y}_{0,0}(\max |q_{0,0}^{00}|) \end{cases} , \\
h_{10}^z &= \begin{cases} (P_{1|1} - P_{1|0} + \pi_{01})^{-2} \cdot (P_{1|1} \cdot f_Y(y(1)|D=1, Z=1) - (P_{1|0} - \pi_{01}) \cdot \int h_{11}^1 dy(0)) & \text{if } \pi_{01} > P_{1|0} - P_{1|1} \\ \quad \cdot (P_{0|0} \cdot f_Y(y(0)|D=0, Z=0) - (P_{0|1} - \pi_{01}) \cdot \int h_{00}^0 dy(1)) & \\ I\{y(0) = y^{UB}\} \cdot I\{y(1) = y^{LB}\} & \text{if } \pi_{01} = P_{1|0} - P_{1|1} . \end{cases} , \\
h_{01}^z &= \begin{cases} \pi_{01}^{-2} \cdot (P_{1|0} \cdot f_Y(y(1)|D=1, Z=0) - (P_{1|0} - \pi_{01}) \cdot \int h_{11}^0 dy(0)) & \text{if } \pi_{01} > 0 \\ \quad \cdot (P_{0|1} \cdot f_Y(y(0)|D=0, Z=1) - (P_{0|1} - \pi_{01}) \cdot \int h_{00}^1 dy(1)) & \\ I\{y(0) = y^{UB}\} \cdot I\{y(1) = y^{LB}\} & \text{if } \pi_{01} = 0, \end{cases} ,
\end{aligned}$$

where

$$\begin{aligned}
\alpha_{11}^1 &= \frac{\bar{Y}_{1,1}(\max |q_{1,1}^{11}|) - \bar{Y}_{1,0}(\min |q_{1,0}^{11}|)}{\bar{Y}_{1,1}(\max |q_{1,1}^{11}|) - \bar{Y}_{1,1}(\min |q_{1,1}^{11}|)} , \\
\alpha_{11}^0 &= \frac{\bar{Y}_{0,1}(\max |q_{0,1}^{11}|) - \bar{Y}_{1,1}(\min |q_{1,1}^{11}|)}{\bar{Y}_{0,1}(\max |q_{0,1}^{11}|) - \bar{Y}_{0,1}(\min |q_{0,1}^{11}|)} , \\
\alpha_{00}^1 &= \frac{\bar{Y}_{1,0}(\max |q_{1,0}^{00}|) - \bar{Y}_{0,0}(\max |q_{0,0}^{00}|)}{\bar{Y}_{1,0}(\max |q_{1,0}^{00}|) - \bar{Y}_{1,0}(\min |q_{1,0}^{00}|)} , \\
\alpha_{00}^0 &= \frac{\bar{Y}_{0,0}(\max |q_{0,0}^{00}|) - \bar{Y}_{1,0}(\max |q_{1,0}^{00}|)}{\bar{Y}_{0,0}(\max |q_{0,0}^{00}|) - \bar{Y}_{0,0}(\min |q_{0,0}^{00}|)} ,
\end{aligned}$$

entails the lower bound on the ATE for $t = 10$ if π_{01} minimizes (7) and the lower bound on the ATE for $t = 01$ if π_{01} minimizes (8). The bounds conditional on π_{01} are continuous in π_{01} (as shown in Appendix A.1.4) and $\mathcal{P}^*$ is an interval (as shown in Lemma 1), so that the minima of $\mathcal{P}^*$ are attained by the extreme value theorem. This ends the proof. ■

A.1.3 Identified set for the proportion of defiers based on moment inequalities

From the construction of $\{h_t^z\}$ in Appendix A.1.2 we note that, if $\mathcal{P}^*$ is non-empty, so that there exists at least one value of π_{01} that is admissible, then $\pi_{01}^{\max} = \min\{P_{1|0}, P_{0|1}\}$ must also be admissible. The reason is that $\bar{Y}_{0,1}(\min |q_{0,1}^{11}|), \bar{Y}_{0,1}(\min |q_{1,1}^{11}|), \bar{Y}_{1,0}(\min |q_{1,0}^{00}|)$ and $\bar{Y}_{0,0}(\min |q_{0,0}^{00}|)$ are non-increasing in π_{01} and $\bar{Y}_{1,1}(\max |q_{1,1}^{11}|), \bar{Y}_{0,1}(\max |q_{0,1}^{11}|), \bar{Y}_{0,0}(\max |q_{0,0}^{00}|)$ and $\bar{Y}_{1,0}(\max |q_{1,0}^{00}|)$ are non-decreasing in π_{01} . Therefore if the following four inequalities hold for a particular value of π_{01}^* ,

$$\begin{aligned} \bar{Y}_{0,1}(\min |q_{0,1}^{11}|) &\leq \bar{Y}_{1,1}(\max |q_{1,1}^{11}|) \\ \bar{Y}_{1,1}(\min |q_{1,1}^{11}|) &\leq \bar{Y}_{0,1}(\max |q_{0,1}^{11}|) \\ \bar{Y}_{1,0}(\min |q_{1,0}^{00}|) &\leq \bar{Y}_{0,0}(\max |q_{0,0}^{00}|) \\ \bar{Y}_{0,0}(\min |q_{0,0}^{00}|) &\leq \bar{Y}_{1,0}(\max |q_{1,0}^{00}|) \end{aligned} \tag{A.12}$$

then they must also hold for any $\pi_{01} > \pi_{01}^*$.

The inequalities (A.12) guarantee that $0 \leq \alpha_t^z \leq 1$ for $t = 11, 00$ and $Z = 0, 1$ and hence the possibility of constructing $\{h_t^z\}$ as outlined in (A.9). This makes any $\pi_{01} \leq \pi_{01}^{\max}$ admissible and therefore (A.12) provides a **sufficient condition** for $\pi_{01} \in \mathcal{P}^*$. But at the same time, for an admissible value of π_{01} , inequalities (A.12) must hold so that the upper bounds of $E[Y(1)|T = 11]$ and $E[Y(0)|T = 00]$ are larger than the lower bounds, such that (A.12) is a **necessary condition** for $\pi_{01} \in \mathcal{P}^*$. This shows that, given that $\pi_{01} \in \mathcal{P}$, (A.12) is a necessary and sufficient condition for the admissibility of π_{01} under Assumptions 1 and 2, that is $\pi_{01} \in \mathcal{P}^*$.

A.1.4 Continuity of bounds on the ATE for compliers and defiers in the share of defiers

Here we prove the continuity of the upper bound on the ATE for compliers conditional on π_{01} . The continuity of the other bounds on the ATE for other principal strata follows similarly. We make the dependence of quantiles on the value of π_{01} explicit. The quantiles $q_{z,d}^t$ are continuous functions in π_{01} , the trimmed means are continuous in quantiles and therefore it is sufficient to verify cases when the denominator in the analytic expression $E[Y(1)|T = 10]$ for is zero. Therefore we have to show that

$$\lim_{\pi_{01} \rightarrow (P_{1|0} - P_{1|1})^+} E[Y(1)|T = 10](\pi_{01}) = y^{UB},$$

for any admissible value of π_{01} .

Note that $q_{1,1}^{11}(P_{1|0} - P_{1|1}) = 1$ and $q_{0,1}^{11}(P_{1|0} - P_{1|1}) = \frac{P_{1|1}}{P_{1|0}}$. If $\bar{Y}_{1,1}(\min|1) > \bar{Y}_{0,1}(\min|\frac{P_{1|1}}{P_{1|0}})$

$$\begin{aligned}
& \lim_{\pi_{01} \rightarrow (P_{1|0} - P_{1|1})^+} \frac{P_{1|1} \cdot \bar{Y}_{1,1} - (P_{1|0} - \pi_{01}) \cdot \max(\bar{Y}_{1,1}(\min|q_{1,1}^{11}(\pi_{01})), \bar{Y}_{0,1}(\min|q_{0,1}^{11}(\pi_{01})))}{P_{1|1} - P_{1|0} + \pi_{01}} \\
&= \lim_{\pi_{01} \rightarrow (P_{1|0} - P_{1|1})^+} \frac{\bar{Y}_{1,1}(\min|q_{1,1}^{11}(\pi_{01})) - (P_{1|0} - \pi_{01}) \frac{\partial \bar{Y}_{1,1}(\min|q_{1,1}^{11})}{\partial q_{1,1}^{11}} \frac{\partial q_{1,1}^{11}}{\partial \pi_{01}}}{1} \\
&= \bar{Y}_{1,1} - (P_{1|1}) \left(\bar{Y}_{1,1} - y^{UB} \right) \left(-\frac{1}{P_{1|1}} \right) = y^{UB},
\end{aligned}$$

where the first equation follows from the l'Hospital rule and in the second we used the definition of the trimmed mean. If $\bar{Y}_{1,1}(\min|1) < \bar{Y}_{0,1}(\min|\frac{P_{1|1}}{P_{1|0}})$, then we get that $\bar{Y}_{1,1}(\min|1) = \bar{Y}_{1,1}(\max|1) < \bar{Y}_{0,1}(\min|\frac{P_{1|1}}{P_{1|0}})$, which contradicts the admissibility of $\pi_{01} = P_{1|0} - P_{1|1}$, because then the upper bound for $E[Y(1)|T = 11]$ is smaller than its' lower bound.

Using the same argument we get that

$$\lim_{\pi_{01} \rightarrow (P_{1|0} - P_{1|1})^+} E[Y(0)|T = 10](\pi_{01}) = y^{LB},$$

and this leads to

$$\lim_{\pi_{01} \rightarrow (P_{1|0} - P_{1|1})^+} \Delta_{10}^{UB}(\pi_{01}) = y^{UB} - y^{LB}.$$

The intuitive argument is that we could have chosen the parametrization differently. Instead of expressing the bounds in π_{01} , we could have expressed the bounds in π_{11} or π_{00} and in such case, we would not get zero in the denominator for the analytic expression for the bounds on compliers but for always takers or never takers.

A.1.5 Proof of the validity and the sharpness of the bounds on the mean potential outcomes $E(Y(1))$, $E(Y(0))$

We only demonstrate sharpness of the upper bounds on $E(Y(1))$, $E(Y(0))$, as the proof for the lower bounds is symmetric and therefore omitted. The proof consists of three steps. In the first step, we show that for π_{01} fixed, the upper bounds on $E(Y(1))^{UB}$, $E(Y(0))^{UB}$ in expressions (9) and (10) in the main text are valid in the sense that they are weakly greater than $E(Y(1))$ and $E(Y(0))$, respectively, with probability one. In the second step, we prove that these upper bounds are sharp conditional on π_{01} by showing that there exists a distribution of T given Z and of $(Y(1), Y(0))$ given T and Z (for $T = 11, 10, 01, 00$ and $Z = 1, 0$) that is compatible with a data generating process satisfying Assumption 2 and at the same time yielding $E(Y(1)) = E(Y(1))^{UB}$ and $E(Y(0)) = E(Y(0))^{UB}$. Thirdly, since π_{01} is unknown, the sharp upper bounds over all admissible π_{01} are (by the virtue of the extreme value theorem and the continuity of the bounds in π_{01} as shown in Appendix A.1.4) obtained by taking the supremum of the conditional upper bounds w.r.t. π_{01} , which is shown to be achieved at $\pi_{01} = \pi_{01}^{\min}$.

We start the proof by showing that for π_{01} given,

$$\begin{aligned}
E(Y(1)) \leq E(Y(1))^{UB}(\pi_{01}) &= (P_{0|1} - \pi_{01}) \cdot y^{UB} - (P_{1|0} - \pi_{01}) \cdot \max(\bar{Y}_{1,1}(\min |q_{1,1}^{11}), \bar{Y}_{0,1}(\min |q_{0,1}^{11})) \\
&+ P_{1|0} \cdot \bar{Y}_{0,1} + P_{1|1} \cdot \bar{Y}_{1,1}, \\
E(Y(0)) \leq E(Y(0))^{UB}(\pi_{01}) &= (P_{1|0} - \pi_{01}) \cdot y^{UB} - (P_{0|1} - \pi_{01}) \cdot \max(\bar{Y}_{0,0}(\min |q_{0,0}^{00}), \bar{Y}_{1,0}(\min |q_{1,0}^{00})) \\
&+ P_{0|0} \cdot \bar{Y}_{0,0} + P_{0|1} \cdot \bar{Y}_{1,0},
\end{aligned}$$

so that the upper bounds are valid. For the sake of notational convenience we omit (π_{01}) in the subsequent discussion. Note that by the law of total probability,

$$E(Y(0)) = \pi_{11} \cdot E(Y(0)|T = 11) + \pi_{10} \cdot E(Y(0)|T = 10) + \pi_{01} \cdot E(Y(0)|T = 01) + \pi_{00} \cdot E(Y(0)|T = 00),$$

which, using equations (3) and (4), can be written as

$$\begin{aligned}
E(Y(0)) &= \pi_{11} \cdot E(Y(0)|T = 11) + P_{0|0} \cdot \bar{Y}_{0,0} - \pi_{00} \cdot E(Y(0)|T = 00) \\
&+ P_{0|1} \cdot \bar{Y}_{1,0} - \pi_{00} \cdot E(Y(0)|T = 00) + \pi_{00} \cdot E(Y(0)|T = 00) \\
&= \pi_{11} \cdot E(Y(0)|T = 11) - \pi_{00} \cdot E(Y(0)|T = 00) + P_{0|0} \cdot \bar{Y}_{0,0} + P_{0|1} \cdot \bar{Y}_{1,0} \\
&= (P_{1|0} - \pi_{01}) \cdot E(Y(0)|T = 11) - (P_{0|1} - \pi_{01}) \cdot E(Y(0)|T = 00) \\
&+ P_{0|0} \cdot \bar{Y}_{0,0} + P_{0|1} \cdot \bar{Y}_{1,0}.
\end{aligned} \tag{A.13}$$

Similarly,

$$\begin{aligned}
E(Y(1)) &= \pi_{11} \cdot E(Y(1)|T = 11) + \pi_{10} \cdot E(Y(1)|T = 10) \\
&+ \pi_{01} \cdot E(Y(1)|T = 01) + \pi_{00} \cdot E(Y(1)|T = 00) \\
&= (P_{0|1} - \pi_{01}) \cdot E(Y(1)|T = 00) - (P_{1|0} - \pi_{11}) \cdot E(Y(1)|T = 11) \\
&+ P_{1|1} \cdot \bar{Y}_{1,1} + P_{1|0} \cdot \bar{Y}_{0,1},
\end{aligned} \tag{A.14}$$

where we use equations (1) and (2) to go from the first to the second line.

It is easy to see that $E(Y(0))^{UB}$ is a valid upper bound for $E(Y(0))$:

$$\begin{aligned}
E(Y(0))^{UB} - E(Y(0)) &= (P_{1|0} - \pi_{01}) \cdot (y^{UB} - E(Y(0)|T = 11)) \\
&+ (P_{0|1} - \pi_{01}) \cdot (E(Y(0)|T = 00) - \max(\bar{Y}_{0,0}(\min |q_{0,0}^{00}), \bar{Y}_{1,0}(\min |q_{1,0}^{00}))) \geq 0,
\end{aligned}$$

because $(P_{1|0} - \pi_{01}) \geq 0$, $(P_{0|1} - \pi_{01}) \geq 0$, $y^{UB} \geq E(Y(0)|T = 11)$, and, as already shown in Section A.1.2,

$E(Y(0)|T = 00) \geq E(Y(0)|T = 00)^{LB} = \max(\bar{Y}_{0,0}(\min |q_{0,0}^{00}), \bar{Y}_{1,0}(\min |q_{1,0}^{00}))$. In a similar way one can prove the validity of $E(Y(1))^{UB}$.

To demonstrate sharpness we need to show that for each $E(Y(d)) \in [E(Y(d))^{LB}, E(Y(d))^{UB}]$, $d = 1, 0$ there exist principal strata proportions $\Pr(T = t|Z = z) : t = 11, 10, 01, 00, z = 1, 0$ and potential outcome distributions $f(y(1), y(0)|t, z) : T = 11, 10, 01, 00, Z = 1, 0$ that are compatible with a data generating process that satisfies Assumptions 2. As $E(Y(d))^{LB}, E(Y(d))^{UB}$, $d = 1, 0$ are the smallest and largest values of the interval $[E(Y(d))^{LB}, E(Y(d))^{UB}]$, it is sufficient to prove the existence at those two extremes. The reason is that the values of $E(Y(d)|T = t)$ inside this interval can be achieved as convex combinations of the potential outcome distributions $f(y(1), y(0)|t, z) : t = 11, 10, 01, 00, z = 1, 0$ that generate $E(Y(d)|T = t)^{LB}$ and $E(Y(d)|T = t)^{UB}$. For the upper bound, one therefore has to show that such distributions exist so that $E(Y(d)) = E(Y(d))^{UB}$.

To this end, reconsider (A.8) and (A.9), which have been shown to be consistent with the data generating process (see Section A.1.2), satisfy Assumption 2, and give

$$\begin{aligned} \int y(0)h_{11}^z dy(1) &= E(Y(0)|T = 11) = y^{UB}, \\ \int y(1)h_{00}^z dy(0) &= E(Y(1)|T = 00) = y^{UB}, \\ \int y(1)h_{11}^z dy(0) &= E(Y(1)|T = 11) = \max(\bar{Y}_{0,1}(\min |q_{0,1}^{11}), \bar{Y}_{1,1}(\min |q_{1,1}^{11})), \end{aligned}$$

and

$$\int y(0)h_{00}^z dy(1) = E(Y(0)|T = 00) = \max(\bar{Y}_{0,0}(\min |q_{0,0}^{00}), \bar{Y}_{1,0}(\min |q_{1,0}^{00})).$$

Therefore, (A.8) and (A.9) imply $E(Y(1)) = E(Y(1))^{UB}$ and $E(Y(0)) = E(Y(0))^{UB}$, which proves that they are sharp for a given value of π_{01} .

Finally, to obtain the upper bound on $E(Y(0))$ over all admissible values of π_{01} , we show that $E(Y(0))^{UB}(\pi_{01})$, where we now make conditioning on π_{01} explicit again, is a decreasing function of π_{01} so that $E(Y(0))^{UB} = \sup_{\pi_{01} \in \mathcal{P}^*} [E(Y(0))^{UB}(\pi_{01})] = E(Y(0))^{UB}(\pi_{01}^{\min})$. In an analogous way one can demonstrate that $E(Y(1))^{UB} = \sup_{\pi_{01} \in \mathcal{P}^*} [E(Y(1))^{UB}(\pi_{01})] = E(Y(1))^{UB}(\pi_{01}^{\min})$, which is omitted for the sake of brevity. We first rewrite $E(Y(0))^{UB}(\pi_{01})$ as

$$\begin{aligned} E(Y(0))^{UB}(\pi_{01}) &= \min \left\{ \begin{aligned} &(P_{1|0} - \pi_{01}) \cdot y^{UB} - (P_{0|1} - \pi_{01}) \cdot \bar{Y}_{0,0}(\min |q_{0,0}^{00}) + P_{0|0} \cdot \bar{Y}_{0,0} + P_{0|1} \cdot \bar{Y}_{1,0} \\ &(P_{1|0} - \pi_{01}) \cdot y^{UB} - (P_{0|1} - \pi_{01}) \cdot \bar{Y}_{1,0}(\min |q_{1,0}^{00}) + P_{0|0} \cdot \bar{Y}_{0,0} + P_{0|1} \cdot \bar{Y}_{1,0} \end{aligned} \right\} \\ &= \min \left\{ \begin{aligned} &E(Y(0))_1^{UB}(\pi_{01}) \\ &E(Y(0))_2^{UB}(\pi_{01}) \end{aligned} \right\}. \end{aligned}$$

Note that if Y is a random variable with a continuous probability density function, the trimmed means

$\bar{Y}_{z,0}(\min |q_{z,0}^{00})$ are differentiable in quantile $q_{z,0}^{00}$, because the quantile is differentiable in π_{01} . We can therefore verify that both $E(Y(0))_1^{UB}(\pi_{01})$ and $E(Y(0))_2^{UB}(\pi_{01})$ are decreasing in π_{01} .

$$\frac{\partial E(Y(0))_1^{UB}}{\partial \pi_{01}} = (\bar{Y}_{0,0}(\min |q_{0,0}^{00}) - y^{UB}) - (P_{0|1} - \pi_{01}) \cdot \frac{\partial \bar{Y}_{0,0}(\min |q_{0,0}^{00})}{\partial \pi_{01}} < 0$$

and

$$\frac{\partial E(Y(0))_2^{UB}}{\partial \pi_{01}} = (\bar{Y}_{1,0}(\min |q_{1,0}^{00}) - y^{UB}) - (P_{0|1} - \pi_{01}) \cdot \frac{\partial \bar{Y}_{1,0}(\min |q_{1,0}^{00})}{\partial \pi_{01}} < 0.$$

The two inequalities are always satisfied, because $0 \leq (P_{0|1} - \pi_{01}) \leq 1$ and $\frac{\partial \bar{Y}_{1,0}(\min |q_{1,0}^{00})}{\partial \pi_{01}}$ (the marginal decrease in $\bar{Y}_{z,0}(\min |q_{z,0}^{00})$ due to a marginal increase in π_{01}), albeit always negative, cannot be larger in absolute terms than the difference of $\bar{Y}_{z,0}(\min |q_{z,0}^{00})$ and the largest possible value y^{UB} . Since the minimum of two monotonically decreasing functions is itself a monotonically decreasing function we conclude that $\sup_{\pi_{01} \in \mathcal{P}^*} [E(Y(0))^{UB}(\pi_{01})] = E(Y(0))^{UB}(\pi_{01}^{\min})$, because $\mathcal{P}^*$ is a compact set. This ends the proof. ■

A.1.6 Comparing our bounds to Manski (1990)

This section compares our bounds under mean independence within strata to those of Manski (1990). First, we write Manski's upper bound, denoted by Δ_{Ma}^{UB} , in our notation as

$$\begin{aligned} \Delta_{Ma}^{UB} &= \min(P_{1|1} \cdot \bar{Y}_{1,1} + P_{0|1} \cdot y^{UB}, P_{1|0} \cdot \bar{Y}_{0,1} + P_{0|0} \cdot y^{UB}) \\ &\quad - \max(P_{0|0} \cdot \bar{Y}_{0,0} + P_{1|0} \cdot y^{LB}, P_{0|1} \cdot \bar{Y}_{1,0} + P_{1|1} \cdot y^{LB}). \end{aligned}$$

In order to see the differences between Manski's bounds and ours, consider the case where positive monotonicity is consistent with the data ($P_{1|1} > P_{1|0}$, which implies $P_{0|0} > P_{0|1}$). We have that¹

$$\begin{aligned} P_{1|1} \cdot \bar{Y}_{1,1} + P_{0|1} \cdot y^{UB} &\leq P_{1|0} \cdot \bar{Y}_{0,1} + P_{0|0} \cdot y^{UB} \\ (P_{1|0} - P_{1|1}) \cdot y^{UB} &\leq (P_{1|0} - P_{1|1}) \cdot \left(\frac{P_{1|0}}{P_{1|0} - P_{1|1}} \cdot \bar{Y}_{0,1} - \frac{P_{1|1}}{P_{1|0} - P_{1|1}} \cdot \bar{Y}_{1,1} \right) \\ y^{UB} &\geq \frac{P_{1|0}}{P_{1|0} - P_{1|1}} \cdot \bar{Y}_{0,1} - \frac{P_{1|1}}{P_{1|0} - P_{1|1}} \cdot \bar{Y}_{1,1}. \end{aligned} \tag{A.15}$$

This is always satisfied since $\frac{P_{1|0}}{P_{1|0} - P_{1|1}} \cdot \bar{Y}_{0,1} - \frac{P_{1|1}}{P_{1|0} - P_{1|1}} \cdot \bar{Y}_{1,1}$, which is a weighted difference of two means, cannot be higher than y^{UB} , which is the largest value that Y can theoretically take. A symmetric argument can be used to show that $\max(P_{0|0} \cdot \bar{Y}_{0,0} + P_{1|0} \cdot y^{LB}, P_{0|1} \cdot \bar{Y}_{1,0} + P_{1|1} \cdot y^{LB}) = P_{0|0} \cdot \bar{Y}_{0,0} + P_{1|0} \cdot y^{LB}$.

¹Note that the sign of the inequality changes from the second to the third line because $P_{1|0} - P_{1|1}$ is negative.

Therefore, Δ_{Ma}^{UB} simplifies to

$$\Delta_{Ma}^{UB} = \left(P_{1|1} \cdot \bar{Y}_{1,1} + P_{0|1} \cdot y^{UB} \right) - \left(P_{0|0} \cdot \bar{Y}_{0,0} + P_{1|0} \cdot y^{LB} \right).$$

On the other hand, since $\bar{Y}_{0,1}(\min |q_{0,1}^{11,0}|) = \bar{Y}_{0,1}$ and $\bar{Y}_{1,0}(\min |q_{1,0}^{00,0}|) = \bar{Y}_{1,0}$, our upper bound becomes

$$\begin{aligned} \Delta^{UB} &= \left(P_{1|1} \cdot \bar{Y}_{1,1} + P_{1|0} \cdot \bar{Y}_{0,1} - P_{1|0} \cdot \max(\bar{Y}_{1,1}(\min |q_{1,1}^{11,0}|), \bar{Y}_{0,1}) + P_{0|1} \cdot y^{UB} \right) \\ &- \left(P_{0|0} \cdot \bar{Y}_{0,0} + P_{0|1} \cdot \bar{Y}_{1,0} - P_{0|1} \cdot \min(\bar{Y}_{0,0}(\max |q_{0,0}^{00,0}|), \bar{Y}_{1,0}) + P_{1|1} \cdot y^{LB} \right). \end{aligned}$$

This implies that

$$\begin{aligned} \Delta^{UB} - \Delta_{Ma}^{UB} &= P_{1|0} \cdot (\max(\bar{Y}_{1,1}(\min |q_{1,1}^{11,0}|), \bar{Y}_{0,1}) - \bar{Y}_{0,1}) \\ &- P_{0|1} \cdot (\min(\bar{Y}_{0,0}(\max |q_{0,0}^{00,0}|), \bar{Y}_{1,0}) - \bar{Y}_{1,0}) \leq 0 \end{aligned} \tag{A.16}$$

It is easy to see that $\Delta^{UB} = \Delta_{Ma}^{UB}$, unless either $\bar{Y}_{1,1}(\min |q_{1,1}^{11,0}|) > \bar{Y}_{0,1}$ or $\bar{Y}_{0,0}(\max |q_{0,0}^{00,0}|) < \bar{Y}_{1,0}$ (or both) is satisfied. As discussed in Section 3.2, any of these inequalities can be interpreted as a violation of Assumptions 2 and 3. This shows that Manski's and our bounds are the same unless we detect a violation of the LATE assumptions.

A.1.7 Comparing our bounds to Kitagawa (2009)

To compare our bounds to those proposed in Kitagawa (2009), we need to introduce some extra notation. Let

$$\begin{aligned} p_1(y) &= f(y, D=1|Z=1), \quad p_0(y) = f(y, D=0|Z=1), \\ q_1(y) &= f(y, D=1|Z=0), \quad q_0(y) = f(y, D=0|Z=0), \\ \delta_1 &= \int \max(p_1(y), q_1(y)) dy, \quad \delta_0 = \int \max(p_0(y), q_0(y)) dy, \\ \lambda_1 &= \int \min(p_1(y), q_1(y)) dy, \quad \lambda_0 = \int \min(p_0(y), q_0(y)) dy. \end{aligned}$$

Moreover, Kitagawa (2009) defines the α -th left- and right- trimming of a nonnegative integrable function $g : \mathcal{Y} \mapsto \mathbb{R}$. Let $q_\alpha^{lt} = \inf \left\{ t : \int_{[y^{LB}, t]} g(y) dy \geq \alpha \right\}$ and $q_\alpha^{rt} = \inf \left\{ t : \int_{[t, y^{UB}]} g(y) dy \geq \alpha \right\}$. The α -th left- and right- trimming functions are

$$\begin{aligned} (g)_\alpha^{lt}(y) &= g(y) \cdot I\{y > q_\alpha^{lt}\} + \left(\int_{[y^{LB}, q_\alpha^{lt}]} g(y) dy - \alpha \right) \cdot I\{y = q_\alpha^{lt}\}, \\ (g)_\alpha^{rt}(y) &= g(y) \cdot I\{y < q_\alpha^{rt}\} + \left(\int_{[q_\alpha^{rt}, y^{UB}]} g(y) dy - \alpha \right) \cdot I\{y = q_\alpha^{rt}\}. \end{aligned}$$

Notice that those functions degenerate to zero for $\alpha \geq \int g(y) dy$, sum up to $g(y)$ and integrate to $\int g(y) dy - \alpha$.

Given this notation Kitagawa's bounds on Δ can be written in a compact way as

$$\begin{aligned}
\Delta_{Ki}^{UB} &= \min(1 - \delta_1, \lambda_0) \cdot y^{UB} + \int y_1 \cdot (\max(p_1(y), q_1(y)) + (\min(p_1(y), q_1(y)))_{\min(1-\delta_1, \lambda_0)}^{lt}) dy \\
&- \min(1 - \delta_0, \lambda_1) \cdot y^{LB} - \int y_0 \cdot (\max(p_0(y), q_0(y)) + (\min(p_0(y), q_0(y)))_{\min(1-\delta_0, \lambda_1)}^{rt}) dy, \\
\Delta_{Ki}^{LB} &= \min(1 - \delta_1, \lambda_0) \cdot y^{LB} + \int y_1 \cdot (\max(p_1(y), q_1(y)) + (\min(p_1(y), q_1(y)))_{\min(1-\delta_1, \lambda_0)}^{rt}) dy \\
&- \min(1 - \delta_0, \lambda_1) \cdot y^{UB} - \int y_0 \cdot (\max(p_0(y), q_0(y)) + (\min(p_0(y), q_0(y)))_{\min(1-\delta_0, \lambda_1)}^{lt}) dy.
\end{aligned}$$

For the comparison we will only consider the upper bound, a symmetric argument holds for the lower bound.

For the same reason we only consider the case of $P_{1|1} > P_{1|0}$. Our upper bound on Δ can be rewritten as

$$\begin{aligned}
\Delta^{UB} &= P_{0|1} \cdot y^{UB} + \int y_1 \cdot (\max(p_1(y), q_1(y)) + \min(p_1(y), q_1(y))) dy \\
&- P_{1|0} \cdot \max(\bar{Y}_{1,1}(\min |q_{1,1}^{11,0}|), \bar{Y}_{0,1}) \\
&- P_{1|0} \cdot y^{LB} - \int y_0 \cdot (\max(p_0(y), q_0(y)) + \min(p_0(y), q_0(y))) dy \\
&+ P_{0|1} \cdot \min(\bar{Y}_{0,0}(\max |q_{0,0}^{00,0}|), \bar{Y}_{1,0}).
\end{aligned}$$

Therefore, the difference between Kitagawa's upper bound and ours is

$$\begin{aligned}
\Delta^{UB} - \Delta_{Ki}^{UB} &= (P_{0|1} - \min(1 - \delta_1, \lambda_0)) \cdot y^{UB} + \int y_1 \cdot \min(p_1(y), q_1(y)) dy \\
&- \int y_1 \cdot (\min(p_1(y), q_1(y)))_{\min(1-\delta_1, \lambda_0)}^{lt} dy - P_{1|0} \cdot \max(\bar{Y}_{1,1}(\min |q_{1,1}^{11,0}|), \bar{Y}_{0,1}) \\
&- (P_{1|0} - \min(1 - \delta_0, \lambda_1)) \cdot y^{LB} - \int y_0 \cdot \min(p_0(y), q_0(y)) dy \\
&+ \int y_0 \cdot (\min(p_0(y), q_0(y)))_{\min(1-\delta_0, \lambda_1)}^{rt} dy + P_{0|1} \cdot \min(\bar{Y}_{0,0}(\max |q_{0,0}^{00,0}|), \bar{Y}_{1,0}) \geq 0
\end{aligned} \tag{A.17}$$

We now show that this inequality is always satisfied. First of all, we demonstrate that

$$(P_{0|1} - \min(1 - \delta_1, \lambda_0)) \cdot y^{UB} - (P_{1|0} - \min(1 - \delta_0, \lambda_1)) \cdot y^{LB} \geq 0$$

In order to do so, it is sufficient to show that $P_{1|0} - \min(1 - \delta_0, \lambda_1) = P_{0|1} - \min(1 - \delta_1, \lambda_0) \geq 0$ Kitagawa (2009) proves that $P_{1|1} + P_{1|0} = \delta_1 + \lambda_1$ and similarly, one can show that $P_{0|0} + P_{0|1} = \delta_0 + \lambda_0$. Thus, $P_{1|0} - \lambda_1 = P_{0|1} - (1 - \delta_1)$ and $P_{0|1} - \lambda_0 = P_{1|0} - (1 - \delta_0)$. Moreover, from $\delta_1 + \lambda_1 + \delta_0 + \lambda_0 = 2$, we have $(1 - \delta_1) - \lambda_0 = \lambda_1 - (1 + \delta_0)$, which implies $(1 - \delta_1) > \lambda_0 \Rightarrow \lambda_1 > (1 - \delta_0)$ and $(1 - \delta_1) < \lambda_0 \Rightarrow \lambda_1 < (1 - \delta_0)$. Therefore, $P_{1|0} - \min(1 - \delta_0, \lambda_1) = P_{0|1} - \min(1 - \delta_1, \lambda_0)$. Finally $P_{1|0} \geq \lambda_1$ by construction and $P_{1|0} \geq (1 - \delta_0)$, since $P_{0|1} \geq \lambda_0$ by construction and $P_{0|1} - \lambda_0 = P_{1|0} - (1 - \delta_0)$.

To end the proof, we need to show that

$$\left(\int y_1 \cdot \min(p_1(y), q_1(y)) dy - \int y_1 \cdot (\min(p_1(y), q_1(y)))_{\min(1-\delta_1, \lambda_0)}^{lt} dy \right) - P_{1|0} \cdot \max(\bar{Y}_{1,1}(\min |q_{1,1}^{11,0}|), \bar{Y}_{0,1}) \geq 0,$$

$$P_{0|1} \cdot \min(\bar{Y}_{0,0}(\max |q_{0,0}^{00,0}|), \bar{Y}_{1,0}) - \left(\int y_0 \cdot \min(p_0(y), q_0(y)) dy - \int y_0 \cdot (\min(p_0(y), q_0(y)))_{\min(1-\delta_0, \lambda_1)}^{rt} dy \right) \geq 0.$$

From the results of Kitagawa it follows that $\int y_1 \cdot \min(p_1(y), q_1(y)) dy - \int y_1 \cdot (\min(p_1(y), q_1(y)))_{\min(1-\delta_1, \lambda_0)}^{lt} dy$ is the upper bound for $E(Y, D = 1 | T = 11)$ while from our results it follows that $P_{1|0} \cdot \max(\bar{Y}_{1,1}(\min |q_{1,1}^{11,0}|), \bar{Y}_{0,1})$ is the lower bound on the same parameter, therefore the first inequality is always satisfied. Similarly, $\int y_0 \cdot \min(p_0(y), q_0(y)) dy - \int y_0 \cdot (\min(p_0(y), q_0(y)))_{\min(1-\delta_0, \lambda_1)}^{rt} dy$ is the lower bound for $E(Y, D = 0 | T = 00)$, while $P_{0|1} \cdot \min(\bar{Y}_{0,0}(\max |q_{0,0}^{00,0}|), \bar{Y}_{1,0})$ is the upper bound on the same parameter. This ends the proof.

An important result of Kitagawa (2009) is that under full independence of the instrument and the potential outcomes/treatment states, monotonicity does not hold if one or both of the following inequalities are violated for some $y \in \mathcal{Y}$:

$$p_1(y) \geq q_1(y), \quad q_0(y) \geq p_0(y) \quad \forall y \in \mathcal{Y}. \quad (\text{A.18})$$

However, under the satisfaction of (A.18), Kitagawa's bounds are equivalent to Manski's (1990), which are in turn equivalent to ours, because (A.18) implies the satisfaction of Assumptions 2 and 3 (mean independence within strata and monotonicity). Therefore, only if (A.18) is violated, the inequality in (A.17) becomes strict (and turns into an equality otherwise).

A.1.8 Order of the bounds on the treated, non-treated and the entire population w.r.t. their tightness

First of all, note that $\Delta_{D=1}^{UB} > \Delta_{D=0}^{UB}$ immediately implies that $\Delta_{D=1}^{LB} < \Delta_{D=0}^{LB}$ since the bounds are symmetric. Moreover,

$$\begin{aligned} \Delta_{D=1}^{UB} - \Delta^{UB} &= \Delta_{D=1}^{UB} - \Pr(D=1) \cdot \Delta_{D=1}^{UB} - (1 - \Pr(D=1)) \cdot \Delta_{D=0}^{UB}, \\ &= \Pr(D=0) \cdot (\Delta_{D=1}^{UB} - \Delta_{D=0}^{UB}), \end{aligned}$$

and

$$\begin{aligned} \Delta_{D=0}^{UB} - \Delta^{UB} &= \Delta_{D=0}^{UB} - \Pr(D=0) \cdot \Delta_{D=0}^{UB} - (1 - \Pr(D=0)) \cdot \Delta_{D=1}^{UB}, \\ &= \Pr(D=1) \cdot (\Delta_{D=0}^{UB} - \Delta_{D=1}^{UB}). \end{aligned}$$

Therefore, $\Delta_{D=1}^{UB} < \Delta_{D=0}^{UB}$ implies $\Delta_{D=1}^{UB} < \Delta^{UB} < \Delta_{D=0}^{UB}$ and $\Delta_{D=0}^{LB} < \Delta^{LB} < \Delta_{D=1}^{LB}$. The converse is true if $\Delta_{D=1}^{UB} > \Delta_{D=0}^{UB}$. This ends the proof.

A.2 Monotonicity

A.2.1 Proof of the sharpness of the bounds for the always takers

Under monotonicity and Assumption 2, $E(Y(1)|T = 11)$ is identified by $\bar{Y}_{0,1}$. Since monotonicity does not impose any restrictions on the distribution of $Y(0)|T = 11$, $E(Y(1)|T = 11)^{UB} = y^{UB}$ and $E(Y(1)|T = 11)^{LB} = y^{LB}$. This implies that Δ_{11}^{UB} and Δ_{11}^{LB} are the sharp upper and lower bounds of Δ_{11} .

A.2.2 Proof of the sharpness of the bounds for the never takers

Under monotonicity and Assumption 2, $E(Y(0)|T = 00)$ is identified by $\bar{Y}_{1,0}$. Since monotonicity does not impose any restrictions on the distribution of $Y(1)|T = 00$, $E(Y(0)|T = 11)^{UB} = y^{UB}$ and $E(Y(0)|T = 11)^{LB} = y^{LB}$. This implies that Δ_{00}^{UB} and Δ_{00}^{LB} are the sharp upper and lower bounds of Δ_{00} .

A.2.3 Proof of the sharpness of the bounds on the mean potential outcomes $E(Y(1))$, $E(Y(0))$

Under monotonicity, $\pi_{11} = P_{1|0}$, $\pi_{00} = P_{0|1}$, $E(Y(1)|T = 11) = \bar{Y}_{0,1}$ and $E(Y(0)|T = 00) = \bar{Y}_{1,0}$, so that equations (A.13) and (A.14) simplify to

$$\begin{aligned} E(Y(0)) &= P_{1|0} \cdot E(Y(0)|T = 11) + P_{0|0} \cdot \bar{Y}_{0,0}, \\ E(Y(1)) &= P_{0|1} \cdot E(Y(1)|T = 00) + P_{1|1} \cdot \bar{Y}_{1,1}. \end{aligned}$$

The only unidentified element of $E(Y(0))$ is $E(Y(0)|T = 11)$. Therefore, its sharp upper and lower bounds are obtained by substituting $E(Y(0)|T = 11)$ with $E(Y(0)|T = 11)^{UB} = y^{UB}$ and $E(Y(0)|T = 11)^{LB} = y^{LB}$, respectively. Similarly, the sharp bounds upper and lower bounds on $E(Y(1))$ are obtained by substituting $E(Y(1)|T = 11)$ with $E(Y(1)|T = 11)^{UB} = y^{UB}$ and $E(Y(1)|T = 11)^{LB} = y^{LB}$, respectively.

A.3 Mean dominance

A.3.1 Identified set of the proportion of defiers under mean dominance based on linear programming

Assumption 4 together with Assumption 2 (i) can be rewritten as

$$E[Y(d)|T = 10, Z = 0] \geq E[Y(d)|T = t, Z = 0] \quad \forall d \in \{0, 1\}, \quad t \in \{11, 00\},$$

and the corresponding additional restrictions on h (recall that $h_z^t(y_i, y_j) = \Pr(Y(1) = y_i, Y(0) = y_j | T = t, Z = z)$) are

$$\begin{aligned} \sum_{i=1}^k \sum_{j=1}^k y_i h_0^{10}(y_i, y_j) &\geq \sum_{i=1}^k \sum_{j=1}^k y_i h_0^{11}(y_i, y_j), \\ \sum_{i=1}^k \sum_{j=1}^k y_i h_0^{10}(y_i, y_j) &\geq \sum_{i=1}^k \sum_{j=1}^k y_i h_0^{00}(y_i, y_j), \\ \sum_{i=1}^k \sum_{j=1}^k y_j h_0^{10}(y_i, y_j) &\geq \sum_{i=1}^k \sum_{j=1}^k y_j h_0^{11}(y_i, y_j), \\ \sum_{i=1}^k \sum_{j=1}^k y_j h_0^{10}(y_i, y_j) &\geq \sum_{i=1}^k \sum_{j=1}^k y_j h_0^{00}(y_i, y_j). \end{aligned} \tag{A.19}$$

If there exists a solution to (A.4) – (A.6) and (A.19) for a given value of the defier share π_{01} , then such a π_{01} is admissible. Note that the sharp identified set for the share of defiers under Assumptions 1, 2, and 4 is an interval because the mechanism of the proof of Lemma 1 applies.

A.3.2 Identified set of the proportion of defiers under mean dominance based on moment inequalities

The inequalities are similar to (A.12), with the only difference that $\bar{Y}_{1,1}(\max |q_{1,1}^{11}|)$ is replaced by $\bar{Y}_{1,1}$ and $\bar{Y}_{0,0}(\max |q_{0,0}^{00}|)$ is replaced by $\bar{Y}_{0,0}$.

$$\begin{aligned} \bar{Y}_{0,1}(\min |q_{0,1}^{11}|) &\leq \bar{Y}_{1,1}, \\ \bar{Y}_{1,1}(\min |q_{1,1}^{11}|) &\leq \bar{Y}_{0,1}(\max |q_{0,1}^{11}|), \\ \bar{Y}_{1,0}(\min |q_{1,0}^{00}|) &\leq \bar{Y}_{0,0}, \\ \bar{Y}_{0,0}(\min |q_{0,0}^{00}|) &\leq \bar{Y}_{1,0}(\max |q_{1,0}^{00}|). \end{aligned} \tag{A.20}$$

The proof for necessity and sufficiency of (A.20) for the share of defiers π_{01} to be included in $\mathcal{P}^{**}$ is analogous to the one presented in Appendix A.1.3. A similar argument as in Appendix A.1.3 also applies to prove that $\pi_{01}^{\max} = \min\{P_{1|0}, P_{0|1}\} \in \mathcal{P}^{**}$.

A.3.3 Bounds on the ATEs under mean dominance only

We begin with a lemma that we will make use of in the proof.

Lemma 2 *Let W be a random variable that is distributed as a two components mixture:*

$$f(w) = p \cdot f_1(w) + (1 - p) \cdot f_2(w) \quad p \in [0, 1],$$

so that $W = AW_1 + (1 - A)W_2$, where f_1 , f_2 , and f are probability density functions of W_1 , W_2 , and W , respectively, and A denotes a random variable such that $A \in \{0, 1\}$, $A \perp (W_1, W_2)$, $A \sim \text{Bernoulli}(p)$. Denote by $E(W_1)$ and $E(W_2)$ the expected values of the first and second component, respectively, and assume that $E(W_2) \geq E(W_1)$. Then,

1. *$E(W)$ is the sharp upper bound for $E(W_1)$,*
2. *$E(W)$ is the sharp lower bound for $E(W_2)$.*

Proof of Lemma 2. First of all, we need to show that $E(W_1) \leq E(W) \leq E(W_2)$. To see this, note that

$$E(W) = p \cdot E(W_1) + (1 - p) \cdot E(W_2). \tag{A.21}$$

This implies

$$E(W_2) = E(W) + (p \cdot E(W_2) - p \cdot E(W_1)).$$

Since $E(W_2) \geq E(W_1)$ by assumption, $p \cdot E(W_2) - p \cdot E(W_1)$ cannot be negative, which implies that $E(W) \leq E(W_2)$. In a symmetric manner one can show that $E(W_1) \leq E(W)$. Finally, we need to show that $E(W)$ is the respective sharp lower bound for $E(W_2)$ and the sharp upper bound for $E(W_1)$. We only demonstrate the latter part as the proof for the former is symmetric. Let ψ_2 be a generic member of the identification region of $E(W_2)$, denoted by $\Psi_2 : \psi_2 \geq E(W) \geq E(W_1)$. Clearly, $E(W) \in \Psi_2$. From (A.21) we have

$$E(W_1) \leq \frac{E(W) - (1 - p) \cdot \psi_2}{p}.$$

Suppose there exists an element of Ψ_2 , denoted by ψ_2^* , such that

$$E(W_1) \leq \frac{E(W) - (1 - p) \cdot \psi_2^*}{p} \leq E(W). \tag{A.22}$$

From (A.22) it must hold that

$$\psi_2^* \leq E(W). \quad (\text{A.23})$$

Since we have already shown that $E(W_2) \geq E(W)$, the only admissible element of Ψ_2 that satisfies (A.23) is $E(W)$, which shows that the latter is the sharp upper bound of $E(W_1)$. ■

Under mean dominance, the ATE on the compliers is bounded by

$$\begin{aligned} \Delta_{10}^{UB} &= \sup_{\pi_{01} \in \mathcal{P}^{**}} \left[\frac{P_{1|1} \cdot \bar{Y}_{1,1} - (P_{1|0} - \pi_{01}) \cdot \max(\bar{Y}_{1,1}(\min |q_{1,1}^{11}), \bar{Y}_{0,1}(\min |q_{0,1}^{11}))}{P_{1|1} - P_{1|0} + \pi_{01}} \right. \\ &\quad \left. - \frac{P_{0|0} \cdot \bar{Y}_{0,0} - (P_{0|1} - \pi_{01}) \cdot \min(\bar{Y}_{0,0}, \bar{Y}_{1,0}(\max |q_{1,0}^{00}))}{P_{1|1} - P_{1|0} + \pi_{01}} \right], \\ \Delta_{10}^{LB} &= \inf_{\pi_{01} \in \mathcal{P}^{**}} \left[\frac{P_{1|1} \cdot \bar{Y}_{1,1} - (P_{1|0} - \pi_{01}) \cdot \min(\bar{Y}_{1,1}, \bar{Y}_{0,1}(\max |q_{0,1}^{11}))}{P_{1|1} - P_{1|0} + \pi_{01}} \right. \\ &\quad \left. - \frac{P_{0|0} \cdot \bar{Y}_{0,0} - (P_{0|1} - \pi_{01}) \cdot \max(\bar{Y}_{0,0}(\min |q_{0,0}^{00}), \bar{Y}_{1,0}(\min |q_{1,0}^{00}))}{P_{1|1} - P_{1|0} + \pi_{01}} \right]. \end{aligned}$$

The intuition of this result is that the compliers' mean potential outcome under treatment cannot be lower than that of the always takers, while under non-treatment it cannot be lower than the one of the never takers. Therefore, we now have the minimization problems $\min(\bar{Y}_{0,0}, \bar{Y}_{1,0}(\max |q_{1,0}^{00}))$ and $\min(\bar{Y}_{1,1}, \bar{Y}_{0,1}(\max |q_{0,1}^{11}))$, as $\bar{Y}_{0,0}$ is the lower bound on the compliers' mean potential outcome in the mixed population with the defiers and $\bar{Y}_{1,1}$ in the mixed group with the always takers. The sharpness of these bounds and all other bounds under mean dominance proposed below follows from the fact that they are special cases of the bounds derived in Section 3.1 and that we can apply Lemma 2 in Appendix A.3 to formally prove their sharpness.

The bounds for the defiers are the same as in Section 3.1, since we do not impose any mean dominance assumption w.r.t. the potential outcomes of this population. The bounds for the ATEs on the always takers and never takers are, respectively,

$$\begin{aligned} \Delta_{11}^{UB} &= \min(\bar{Y}_{1,1}, \bar{Y}_{0,1}(\max |q_{0,1}^{11, \pi_{01}^{\max}})) - y^{LB}, \\ \Delta_{11}^{LB} &= \inf_{\pi_{01} \in \mathcal{P}^{**}} \left[\max(\bar{Y}_{1,1}(\min |q_{1,1}^{11}), \bar{Y}_{0,1}(\min |q_{0,1}^{11})) \right. \\ &\quad \left. - \frac{P_{0|0} \cdot \bar{Y}_{0,0} - (P_{0|1} - \pi_{01}) \cdot \max(\bar{Y}_{0,0}(\min |q_{0,0}^{00}), \bar{Y}_{1,0}(\min |q_{1,0}^{00}))}{P_{1|1} - P_{1|0} + \pi_{01}} \right], \end{aligned}$$

and

$$\begin{aligned}
\Delta_{00}^{UB} &= \sup_{\pi_{01} \in \mathcal{P}^{**}} \left[\frac{P_{1|1} \cdot \bar{Y}_{1,1} - (P_{1|0} - \pi_{01}) \cdot \max(\bar{Y}_{1,1}(\min |q_{1,1}^{11}), \bar{Y}_{0,1}(\min |q_{0,1}^{11}))}{P_{1|1} - P_{1|0} + \pi_{01}} \right. \\
&\quad \left. - \max(\bar{Y}_{1,0}(\min |q_{1,0}^{00}), \bar{Y}_{0,0}(\min |q_{0,0}^{00})) \right], \\
\Delta_{00}^{LB} &= y^{LB} - \min(\bar{Y}_{1,0}(\max |q_{1,0}^{00, \pi_{01}^{\max}}), \bar{Y}_{0,0}).
\end{aligned}$$

$\bar{Y}_{1,1}$, $\bar{Y}_{0,0}$ are now the upper bounds on the mean potential outcome under treatment and the mean potential outcome under non-treatment for the always takers and the never takers, respectively. Moreover, mean dominance implies that the always takers' upper bound under non-treatment cannot be higher than the compliers' upper bound under non-treatment and that the never takers' upper bound under treatment cannot be higher than the compliers' upper bound under treatment. This is a considerable improvement over the bounds only invoking mean independence within strata, as it allows us to replace y^{UB} by observed quantities. It, however, requires optimization over all possible values of π_{01} .

Under mean dominance the sharp bounds on $E(Y(1))$ and $E(Y(0))$ become

$$\begin{aligned}
E(Y(0))^{UB} &= \sup_{\pi_{01} \in \mathcal{P}^{**}} \left[(P_{1|0} - \pi_{01}) \cdot \frac{P_{1|1} \cdot \bar{Y}_{1,1} - (P_{1|0} - \pi_{01}) \cdot \max(\bar{Y}_{1,1}(\min |q_{1,1}^{11}), \bar{Y}_{0,1}(\min |q_{0,1}^{11}))}{P_{1|1} - P_{1|0} + \pi_{01}} \right. \\
&\quad \left. - (P_{0|1} - \pi_{01}) \cdot \max(\bar{Y}_{0,0}(\min |q_{0,0}^{00}), \bar{Y}_{1,0}(\min |q_{1,0}^{00})) + P_{0|0} \cdot \bar{Y}_{0,0} + P_{0|1} \cdot \bar{Y}_{1,0} \right], \\
E(Y(0))^{LB} &= \inf_{\pi_{01} \in \mathcal{P}^{**}} \left[(P_{1|0} - \pi_{01}) \cdot y^{LB} - (P_{0|1} - \pi_{01}) \cdot \min(\bar{Y}_{0,0}, \bar{Y}_{1,0}(\max |q_{1,0}^{00})) \right. \\
&\quad \left. + P_{0|0} \cdot \bar{Y}_{0,0} + P_{0|1} \cdot \bar{Y}_{1,0} \right],
\end{aligned}$$

and

$$\begin{aligned}
E(Y(1))^{UB} &= \sup_{\pi_{01} \in \mathcal{P}^{**}} \left[(P_{0|1} - \pi_{01}) \cdot \frac{P_{0|0} \cdot \bar{Y}_{0,0} - (P_{0|1} - \pi_{01}) \cdot \max(\bar{Y}_{0,0}(\min |q_{0,0}^{00}), \bar{Y}_{1,0}(\min |q_{1,0}^{00}))}{P_{1|1} - P_{1|0} + \pi_{01}} \right. \\
&\quad \left. - (P_{1|0} - \pi_{01}) \cdot \max(\bar{Y}_{1,1}(\min |q_{1,1}^{11}), \bar{Y}_{0,1}(\min |q_{0,1}^{11})) + P_{1|0} \cdot \bar{Y}_{0,1} + P_{1|1} \cdot \bar{Y}_{1,1} \right], \\
E(Y(1))^{LB} &= \inf_{\pi_{01} \in \mathcal{P}^{**}} \left[(P_{0|1} - \pi_{01}) \cdot y^{LB} - (P_{1|0} - \pi_{01}) \cdot \min(\bar{Y}_{1,1}, \bar{Y}_{0,1}(\max |q_{0,1}^{11})) \right. \\
&\quad \left. + P_{1|0} \cdot \bar{Y}_{0,1} + P_{1|1} \cdot \bar{Y}_{1,1} \right].
\end{aligned}$$

Therefore, the bounds on the ATEs on the treated, non-treated, and the entire population correspond to

$$\begin{aligned}
\Delta_{D=1}^{UB} &= E(Y|D=1) - \inf_{\pi_{01} \in \mathcal{P}^{**}} \left[\frac{(P_{1|0} - \pi_{01}) \cdot y^{LB} + \Pr(Z=1) \cdot P_{0|0} \cdot \bar{Y}_{0,0} + \Pr(Z=0) \cdot P_{0|1} \cdot \bar{Y}_{1,0}}{\Pr(D=1)} \right. \\
&\quad \left. - \frac{(P_{0|1} - \pi_{01}) \cdot \min(\bar{Y}_{0,0}, \bar{Y}_{1,0}(\max |q_{1,0}^{00}))}{\Pr(D=1)} \right], \\
\Delta_{D=1}^{LB} &= E(Y|D=1) - \sup_{\pi_{01} \in \mathcal{P}^{**}} \left[\frac{(P_{1|0} - \pi_{01}) \cdot \frac{P_{0|0} \cdot \bar{Y}_{0,0} - (P_{0|1} - \pi_{01}) \cdot \max(\bar{Y}_{0,0}(\min |q_{0,0}^{00}), \bar{Y}_{1,0}(\min |q_{1,0}^{00}))}{P_{1|1} - P_{1|0} + \pi_{01}}}{\Pr(D=1)} \right. \\
&\quad \left. + \frac{\Pr(Z=1) \cdot P_{0|0} \cdot \bar{Y}_{0,0} + \Pr(Z=0) \cdot P_{0|1} \cdot \bar{Y}_{1,0}}{\Pr(D=1)} - \frac{(P_{0|1} - \pi_{01}) \cdot \max(\bar{Y}_{0,0}(\min |q_{0,0}^{00}), \bar{Y}_{1,0}(\min |q_{1,0}^{00}))}{\Pr(D=1)} \right], \\
\Delta_{D=0}^{UB} &= \sup_{\pi_{01} \in \mathcal{P}^{**}} \left[\frac{(P_{0|1} - \pi_{01}) \cdot \frac{P_{1|1} \cdot \bar{Y}_{1,1} - (P_{1|0} - \pi_{01}) \cdot \max(\bar{Y}_{1,1}(\min |q_{1,1}^{11}), \bar{Y}_{0,1}(\min |q_{0,1}^{11}))}{P_{1|1} - P_{1|0} + \pi_{01}}}{\Pr(D=0)} \right. \\
&\quad \left. + \frac{\Pr(Z=1) \cdot P_{1|0} \cdot \bar{Y}_{1,0} + \Pr(Z=0) \cdot P_{1|1} \cdot \bar{Y}_{1,1}}{\Pr(D=0)} - \frac{(P_{1|0} - \pi_{01}) \cdot \max(\bar{Y}_{1,1}(\min |q_{1,1}^{11}), \bar{Y}_{0,1}(\min |q_{0,1}^{11}))}{\Pr(D=0)} \right] \\
&\quad - E(Y|D=0), \\
\Delta_{D=0}^{LB} &= \inf_{\pi_{01} \in \mathcal{P}^{**}} \left[\frac{(P_{0|1} - \pi_{01}) \cdot y^{LB} + \Pr(Z=1) \cdot P_{1|0} \cdot \bar{Y}_{1,0} + \Pr(Z=0) \cdot P_{1|1} \cdot \bar{Y}_{1,1}}{\Pr(D=0)} \right. \\
&\quad \left. - \frac{(P_{1|0} - \pi_{01}) \cdot \min(\bar{Y}_{1,1}, \bar{Y}_{0,1}(\max |q_{0,1}^{11}))}{\Pr(D=0)} \right] - E(Y|D=0),
\end{aligned}$$

and

$$\begin{aligned}
\Delta^{UB} &= \sup_{\pi_{01} \in \mathcal{P}^{**}} \left[P_{1|0} \cdot \bar{Y}_{0,1} - (P_{1|0} - \pi_{01}) \cdot y^{LB} + P_{1|1} \cdot \bar{Y}_{1,1} - (P_{1|0} - \pi_{01}) \cdot \max(\bar{Y}_{1,1}(\min |q_{1,1}^{11}), \bar{Y}_{0,1}(\min |q_{0,1}^{11})) \right. \\
&\quad + (P_{0|1} - \pi_{01}) \cdot \min(\bar{Y}_{0,0}, \bar{Y}_{1,0}(\max |q_{1,0}^{00})) - P_{0|1} \cdot \bar{Y}_{1,0} \\
&\quad \left. + (P_{0|1} - \pi_{01}) \cdot \frac{P_{1|1} \cdot \bar{Y}_{1,1} - (P_{1|0} - \pi_{01}) \cdot \max(\bar{Y}_{1,1}(\min |q_{1,1}^{11}), \bar{Y}_{0,1}(\min |q_{0,1}^{11}))}{P_{1|1} - P_{1|0} + \pi_{01}} - P_{0|0} \cdot \bar{Y}_{0,0} \right], \\
\Delta^{LB} &= \inf_{\pi_{01} \in \mathcal{P}^{**}} \left[P_{1|0} \cdot \bar{Y}_{0,1} + P_{1|1} \cdot \bar{Y}_{1,1} \right. \\
&\quad - (P_{1|0} - \pi_{01}) \cdot \frac{P_{0|0} \cdot \bar{Y}_{0,0} - (P_{0|1} - \pi_{01}) \cdot \max(\bar{Y}_{0,0}(\min |q_{0,0}^{00}), \bar{Y}_{1,0}(\min |q_{1,0}^{00}))}{P_{1|1} - P_{1|0} + \pi_{01}} \\
&\quad - (P_{1|0} - \pi_{01}) \cdot \min(\bar{Y}_{1,1}, \bar{Y}_{0,1}(\max |q_{0,1}^{11})) + (P_{0|1} - \pi_{01}) \cdot \max(\bar{Y}_{0,0}(\min |q_{0,0}^{00}), \bar{Y}_{1,0}(\min |q_{1,0}^{00})) \\
&\quad \left. - P_{0|1} \cdot \bar{Y}_{1,0} + (P_{0|1} - \pi_{01}) \cdot y^{LB} - P_{0|0} \cdot \bar{Y}_{0,0} \right].
\end{aligned}$$

Note that in contrast to the bounds of Section 3.1, mean dominance requires optimizing w.r.t. π_{01} , because any of the previously used y^{UB} has been substituted by the upper bounds on the mean potential outcomes of the compliers.

Lemma 2 shows that under mean dominance, the upper bounds of $E(Y(1)|Z=1, T=11)$ and $E(Y(0)|Z=0, T=00)$ are $\bar{Y}_{1,1}$ and $\bar{Y}_{0,0}$, respectively. Moreover, mean dominance implies that $E(Y(0)|T=11) \leq E(Y(0)|T=$

10) and $E(Y(1)|T = 00) \leq E(Y(1)|T = 10)$, thus $E(Y(0)|T = 11)^{UB} \leq E(Y(0)|T = 10)^{UB}$ and $E(Y(1)|T = 00)^{UB} \leq E(Y(1)|T = 10)^{UB}$. The bounds on the ATEs within the four principal strata are special cases of the bounds derived in Section 3.1 under the restrictions just mentioned. Therefore, all of them are sharp.

A.3.4 Proof of the sharpness of the bounds on the mean potential outcomes $E(Y(1))$, $E(Y(0))$

The proofs for the lower bounds are very similar to the ones provided in Appendix A.1.5 (with the only difference that $\bar{Y}_{1,1}(\max |q_{1,1}^{11})$ is replaced by $\bar{Y}_{1,1}$ and $\bar{Y}_{0,0}(\max |q_{0,0}^{00})$ is replaced by $\bar{Y}_{0,0}$ in the definitions of densities $\{h_t\}$ and coefficients $\{\alpha\}$) and therefore omitted. Hence, we only consider the upper bounds of $E(Y(1))$, $E(Y(0))$ for a given π_{01} :

$$\begin{aligned} E(Y(0))^{UB}(\pi_{01}) &= (P_{1|0} - \pi_{01}) \cdot \frac{P_{1|1} \cdot \bar{Y}_{1,1} - (P_{1|0} - \pi_{01}) \cdot \max(\bar{Y}_{1,1}(\min |q_{1,1}^{11}), \bar{Y}_{0,1}(\min |q_{0,1}^{11}))}{P_{1|1} - P_{1|0} + \pi_{01}} \\ &\quad - (P_{0|1} - \pi_{01}) \cdot \max(\bar{Y}_{0,0}(\min |q_{0,0}^{00}), \bar{Y}_{1,0}(\min |q_{1,0}^{00})) + P_{0|0} \cdot \bar{Y}_{0,0} + P_{0|1} \cdot \bar{Y}_{1,0}, \end{aligned}$$

and

$$\begin{aligned} E(Y(1))^{UB}(\pi_{01}) &= (P_{0|0} - \pi_{01}) \cdot \frac{P_{0|0} \cdot \bar{Y}_{0,0} - (P_{0|1} - \pi_{01}) \cdot \max(\bar{Y}_{0,0}(\min |q_{0,0}^{00}), \bar{Y}_{1,0}(\min |q_{1,0}^{00}))}{P_{1|1} - P_{1|0} + \pi_{01}} \\ &\quad - (P_{1|0} - \pi_{01}) \cdot \max(\bar{Y}_{1,1}(\min |q_{1,1}^{11}), \bar{Y}_{0,1}(\min |q_{0,1}^{11})) + P_{1|0} \cdot \bar{Y}_{0,1} + P_{1|1} \cdot \bar{Y}_{1,1}. \end{aligned}$$

By assumption, $E(Y(0)|T = 11)^{UB} \leq E(Y(0)|T = 10)^{UB} = \frac{P_{0|0} \cdot \bar{Y}_{0,0} - (P_{0|1} - \pi_{01}) \cdot \max(\bar{Y}_{0,0}(\min |q_{0,0}^{00}), \bar{Y}_{1,0}(\min |q_{1,0}^{00}))}{P_{1|1} - P_{1|0} + \pi_{01}}$ and $E(Y(1)|T = 00)^{UB} \leq E(Y(1)|T = 10)^{UB} = \frac{P_{0|0} \cdot \bar{Y}_{0,0} - (P_{0|1} - \pi_{01}) \cdot \max(\bar{Y}_{0,0}(\min |q_{0,0}^{00}), \bar{Y}_{1,0}(\min |q_{1,0}^{00}))}{P_{1|1} - P_{1|0} + \pi_{01}}$. Therefore, the bounds are valid.

To show that they are sharp, we need to show that for $T = 11, 10, 01, 00$ and $Z = 1, 0$, there exist distributions of T given Z and of $(Y(1), Y(0))$ given T and Z that are compatible with a data generating process satisfying Assumptions 2 and 4 and implying $E(Y(0)) = E(Y(0))^{UB}$ and $E(Y(1)) = E(Y(1))^{UB}$.

We reconsider (A.8) and (A.9) and change the marginal distribution of h_{11}^1 and h_{11}^0 with respect to $y(0)$ by replacing $I\{y(0) = y^{UB}\}$ with the marginal distributions of h_{10}^z with respect to $y(0)$. We also modify the marginal distribution of h_{00}^1 and h_{00}^0 with respect to $y(1)$ by replacing $I\{y(1) = y^{UB}\}$ by the marginal distributions of h_{10}^z with respect to $y(1)$.

This choice of $\Pr(T|Z)$ and $\{h_t^z\}$ satisfies Assumption 2 by construction. Note that $\iint y(0)h_{11} dy(1) dy(0) = \iint y(0)h_{10} dy(1) dy(0)$ and $\iint y(1)h_{00} dy(1) dy(0) = \iint y(1)h_{10} dy(1) dy(0)$ imply $E(Y(0)|T = 11) = E(Y(0)|T = 10)$ and $E(Y(1)|T = 00) = E(Y(1)|T = 10)$. Therefore, Assumption 4 is also satisfied. Since all equations in

(A.10) are still valid because the marginal distributions that we replaced do not affect restriction (A.10), this distributional choice is compatible with the data generating process. In an analogous way as in A.1.5, it is easy to see that the distributions imply $E(Y(0)) = E(Y(0))^{UB}$ and $E(Y(1)) = E(Y(1))^{UB}$. This shows that the provided bounds are sharp for a given value of π_{01} . The sharp upper bounds over all admissible π_{01} are obtained by taking the supremum w.r.t. to π_{01} . Because of the continuity of the bounds in π_{01} (see Appendix A.1.4) for a compact set $\mathcal{P}^{**}$, the maximum is attained by the extreme value theorem.

A.4 Monotonicity and mean dominance

A.4.1 Proof of the sharpness of the bounds for always takers

Mean dominance implies that $E(Y(0)|T = 11) \leq E(Y(0)|T = 10) = \frac{P_{0|0} \cdot \bar{Y}_{0,0} - P_{0|1} \cdot \bar{Y}_{1,0}}{P_{1|1} - P_{1|0}}$. Thus, $E(Y(0)|T = 11)^{LB} = \frac{P_{0|0} \cdot \bar{Y}_{0,0} - P_{0|1} \cdot \bar{Y}_{1,0}}{P_{1|1} - P_{1|0}}$ and Δ_{11}^{LB} is the sharp lower bound of Δ_{11} .

A.4.2 Proof of the sharpness of the bounds for never takers

Mean dominance implies that $E(Y(1)|T = 00) \leq E(Y(1)|T = 10) = \frac{P_{1|1} \cdot \bar{Y}_{1,1} - P_{1|0} \cdot \bar{Y}_{0,1}}{P_{1|1} - P_{1|0}}$. Thus, $E(Y(1)|T = 00)^{LB} = \frac{P_{1|1} \cdot \bar{Y}_{1,1} - P_{1|0} \cdot \bar{Y}_{0,1}}{P_{1|1} - P_{1|0}}$ and Δ_{00}^{UB} is the sharp upper bound of Δ_{00} .

A.4.3 Proof of the sharpness of the bounds on the mean potential outcomes $E(Y(1))$, $E(Y(0))$

Mean dominance implies $E(Y(0)|T = 11)^{LB} = \frac{P_{0|0} \cdot \bar{Y}_{0,0} - P_{0|1} \cdot \bar{Y}_{1,0}}{P_{1|1} - P_{1|0}}$, $E(Y(1)|T = 00)^{LB} = \frac{P_{1|1} \cdot \bar{Y}_{1,1} - P_{1|0} \cdot \bar{Y}_{0,1}}{P_{1|1} - P_{1|0}}$. Monotonicity implies

$$\begin{aligned} E(Y(0)) &= P_{1|0} \cdot E(Y(0)|T = 11) + P_{0|0} \cdot \bar{Y}_{0,0}, \\ E(Y(1)) &= P_{0|1} \cdot E(Y(1)|T = 00) + P_{1|1} \cdot \bar{Y}_{1,1}. \end{aligned}$$

Therefore,

$$\begin{aligned} E(Y(0))^{LB} &= P_{1|0} \cdot \frac{P_{0|0} \cdot \bar{Y}_{0,0} - P_{0|1} \cdot \bar{Y}_{1,0}}{P_{1|1} - P_{1|0}} + P_{0|0} \cdot \bar{Y}_{0,0}, \\ E(Y(1))^{LB} &= P_{0|1} \cdot \frac{P_{1|1} \cdot \bar{Y}_{1,1} - P_{1|0} \cdot \bar{Y}_{0,1}}{P_{1|1} - P_{1|0}} + P_{1|1} \cdot \bar{Y}_{1,1}, \end{aligned}$$

are sharp.

A.5 Discrete outcomes

A.5.1 Identification

If the outcome Y is discrete, the bounds based on Proposition 4 in Horowitz and Manski (1995) are generally not valid. This is due to the presence of ties in the outcome, i.e. the occurrence of mass points with equal outcome values, which entails a non-unique quantile function such that a particular outcome value is observed at several ranks. The quantile function is required to construct (i) a distribution which is stochastically dominated by any feasible distribution that is consistent with the identification region of some mixture component (i.e., stratum) of interest to determine the lower bound of its mean outcome and (ii) a distribution which stochastically dominates any distribution consistent with the identification to determine the upper bound, respectively. In the presence of discrete outcomes we have to replace the non-unique quantile function, which gives equal ranks to all ties, by a modified version which accounts for ties.

To this end, we denote by $Y_{z,d}$ the outcome variable in the respective observed group and introduce the following trimming functions which are similar to the ones proposed by Kitagawa (2009)

$$\begin{aligned} Trim_{z,d}^{\min,t} &= \frac{I\{Y_{z,d} < F^{-1}(q_{z,d}^t)\} \cdot \Pr(Y_{z,d} < F^{-1}(q_{z,d}^t))}{1 - q_{z,d}^t} \\ &+ \frac{I\{Y_{z,d} = F^{-1}(q_{z,d}^t)\} \cdot (\Pr(Y_{z,d} \geq F^{-1}(q_{z,d}^t)) - q_{z,d}^t)}{1 - q_{z,d}^t} \\ Trim_{z,d}^{\max,t} &= \frac{I\{Y_{z,d} > F^{-1}(1 - q_{z,d}^t)\} \cdot \Pr(Y_{z,d} > F^{-1}(1 - q_{z,d}^t))}{q_{z,d}^t} \\ &+ \frac{I\{Y_{z,d} = F^{-1}(1 - q_{z,d}^t)\} \cdot (\Pr(Y_{z,d} \leq F^{-1}(1 - q_{z,d}^t)) - (1 - q_{z,d}^t))}{q_{z,d}^t} \end{aligned}$$

Let $Y_{z,d}^{\min,t} = Y_{z,d} \cdot Trim_{z,d}^{\min,t}$ and $Y_{z,d}^{\max,t} = Y_{z,d} \cdot Trim_{z,d}^{\max,t}$. Since $F_{Y_{z,d}^{\min,t}}(y)$ is stochastically dominated by any feasible distribution consistent with the identification region and $F_{Y_{z,d}^{\max,t}}(y)$ stochastically dominates any feasible distribution consistent with the identification region, we can define the lower and upper bounds of $E(Y(d)|Z = z, T = t)$ as $\bar{Y}_{z,d}(\min | q_{z,d}^t) = E(Y_{z,d}^{\min,t})$ and $\bar{Y}_{z,d}(\max | q_{z,d}^t) = E(Y_{z,d}^{\max,t})$, respectively.

A.6 Estimation

The trimming function can be easily estimated by its sample analog. A more intuitive and asymptotically equivalent estimator can be obtained as follows. Denote by $n_{z,d}$ the number of observations with $Z = z$ and $D = d$. Let $Y_{z,d}^{(1)}, \dots, Y_{z,d}^{(n_{z,d})}$ be the order statistic of $Y_{z,d}$ which assigns increasing ranks at the ties. E.g., if there are two observations i and $j \in \{1, \dots, n_{z,d}\}$ with the same outcome values y , the rank of the first one, denoted by the function $rank(\cdot)$, will be some integer i while the rank of the second observation will be $i + 1$. Define $\widetilde{q_{z,d}^t} \cdot n_{z,d}$ and

$n_{z,d} - \widetilde{q_{z,d}^t} \cdot n_{z,d}$ to be the integer part of $q_{z,d}^t \cdot n_{z,d}$ and $n_{z,d} - q_{z,d}^t \cdot n_{z,d}$, respectively, if $q_{z,d}^t \cdot n_{z,d}$ and $n_{z,d} - q_{z,d}^t \cdot n_{z,d}$ are larger than 1, while they are 1 otherwise.

The only difference with Section 4 is that the trimmed means are now estimated by

$$\begin{aligned}\hat{Y}_{z,d}(\max | q_{z,d}^t) &= \frac{\sum_{i=1}^n Y_i \cdot I\{D_i = d\} \cdot I\{Z_i = z\} \cdot I\{\text{rank}(Y_i) \geq n_{z,d} - \widetilde{q_{z,d}^t} \cdot n_{z,d}\}}{\sum_{i=1}^n I\{D_i = d\} \cdot I\{Z_i = z\} \cdot I\{\text{rank}(Y_i) \geq n_{z,d} - \widetilde{q_{z,d}^t} \cdot n_{z,d}\}}, \\ \hat{Y}_{z,d}(\min | q_{z,d}^t) &= \frac{\sum_{i=1}^n Y_i \cdot I\{D_i = d\} \cdot I\{Z_i = z\} \cdot I\{\text{rank}(Y_i) \leq \widetilde{q_{z,d}^t} \cdot n_{z,d}\}}{\sum_{i=1}^n I\{D_i = d\} \cdot I\{Z_i = z\} \cdot I\{\text{rank}(Y_i) \leq \widetilde{q_{z,d}^t} \cdot n_{z,d}\}}.\end{aligned}$$

Since we just replace the quantile function by the modified rank function, this only results in using a different indicator function when computing the trimmed means. Note that if Y is continuous, defining the bounds in terms of the modified rank function is equivalent to defining them in terms of the regular quantile function, as each outcome value has a unique rank in this case. Therefore, the two approaches converge to each other as the number of support points of Y goes to infinity.

A.7 Inference based on Chernozhukov, Lee, and Rosen (2009)

Recall from Section 4 that $\Delta_t^{LB}(\pi_{01}, v)$, $\Delta_t^{UB}(\pi_{01}, v)$ are the conditional bounds on the ATE in some subpopulation t given π_{01} and v^2 and that the identification region of Δ_t (the unconditional ATE in the subpopulation) is obtained by optimizing over admissible values of $\pi_{01} \in \mathcal{P}$ and $v \in V = \{1, 2, 3, 4\}$:

$$\inf_{\pi_{01} \in \mathcal{P}} \{ \max_{v \in V} [\Delta_t^{LB}(\pi_{01}, v)] \} \leq \Delta_t \leq \sup_{\pi_{01} \in \mathcal{P}} \{ \min_{v \in V} [\Delta_t^{UB}(\pi_{01}, v)] \},$$

We subsequently discuss the estimation of the upper bound along with its confidence region (the proceeding for the lower bound is analogous) under Assumptions 1 and 2 or 1, 2, and 4, where non-differentiability complicates inference. We use the procedure of Chernozhukov, Lee and Rosen (2013) to obtain a half-median-unbiased estimator of $\min_{v \in V} [\Delta_t^{UB}(\pi_{01}, v)]$ conditional on π_{01} , see also the application of this method in Chen and Flores (2012).

The main idea is that instead of taking the minimum of the estimated upper bounds $\hat{\Delta}_t^{UB}(\pi_{01}, v)$ directly, one uses the following precision adjusted version, denoted by $\tilde{\Delta}_t^{UB}(\pi_{01}, p)$, which consists of the initial estimate plus $s(v)$, a measure of the precision of $\hat{\Delta}_t^{UB}(\pi_{01}, v)$, times an appropriate critical value $k(p)$:

$$\tilde{\Delta}_t^{UB}(\pi_{01}, p) = \min_{v \in V} [\hat{\Delta}_t^{UB}(\pi_{01}, v) + k(p) \cdot s(v)].$$

$$s_v = \begin{cases} 1 & \text{if } z = 1, z' = 1 \\ 2 & \text{if } z = 1, z' = 0 \\ 3 & \text{if } z = 0, z' = 1 \\ 4 & \text{if } z = 0, z' = 0 \end{cases}.$$

Define $\Delta_t^{UB}(\pi_{01})$ to be the four dimensional vector with elements $\Delta_t^{UB}(\pi_{01}, v)$, $v = 1, 2, 3, 4$. As outlined below, $k(p)$ is a function of the sample size and the estimated variance-covariance matrix of $\sqrt{n}(\hat{\Delta}_t^{UB}(\pi_{01}) - \Delta_t^{UB}(\pi_{01}))$, denoted by $\hat{\Omega}$. For $p = \frac{1}{2}$, the estimator $\tilde{\Delta}_t^{UB}(\pi_{01}, p)$ is half-median-unbiased, which implies that the estimate of the upper bound exceeds its true value with probability at least one half asymptotically. Finally, by taking the supremum of $\tilde{\Delta}_t^{UB}(\pi_{01}, p)$ over π_{01} , we obtain a conservative estimate for Δ_t^{UB} .

The following algorithm briefly sketches the estimation of Δ_t^{UB} along with its upper confidence band based on the precision adjustment.

1. Estimate the vector $\hat{\Delta}_t^{UB}(\pi_{01})$ by its sample analog. Estimate the variance-covariance matrix $\hat{\Omega}$ by bootstrapping B times.
2. Denoting by $\hat{g}(v)^\top$ the v -th row of $\hat{\Omega}^{\frac{1}{2}}$, estimate $\hat{s}(v) = \frac{\|\hat{g}(v)\|}{\sqrt{n}}$, where $\|\cdot\|$ is the Euclidean norm.
3. Simulate R draws, $H_1, \dots, H_R$ from a $\mathcal{N}(\mathbf{0}, I_4)$, where $\mathbf{0}$ and I_4 are the null vector and the identity matrix of dimension 4, respectively.
4. Let $H_r^*(v) = \hat{g}(v)^\top Z_r / \|\hat{g}(v)\|$ for $r = 1, \dots, R$.
5. Let $\tilde{k}(c)$ be the c -th quantile of $\min_{v \in V} H_r^*(v)$, $r = 1, \dots, R$, where $c = 1 - \frac{0.1}{\log(n)}$.
6. Compute the set estimator $\hat{V} = \{v \in V : \hat{\Delta}_t^{UB}(\pi_{01}, v) \leq \min_{v' \in V} \{[\hat{\Delta}_t^{UB}(\pi_{01}, v') + \tilde{k}(c) \cdot \hat{s}(v')] + 2 \cdot \tilde{k}(c) \cdot \hat{s}(v')\}\}$.
7. Estimate the critical value $\hat{k}(p)$ by the p -th quantile of $\min_{v \in \hat{V}} H_r^*(v)$, $r = 1, \dots, R$.
8. For half-median-unbiasedness, set $p = \frac{1}{2}$ and compute $\tilde{\Delta}_t^{UB}(\pi_{01}, \frac{1}{2}) = \min_{v \in V} [\hat{\Delta}_t^{UB}(\pi_{01}, v) + \hat{k}(\frac{1}{2}) \cdot \hat{s}(v)]$.
9. Estimate the upper bound Δ_t^{UB} by $\hat{\Delta}_t^{UB} = \sup_{\pi_{01} \in \mathcal{P}} [\tilde{\Delta}_t^{UB}(\pi_{01}, \frac{1}{2})]$.
10. To obtain the upper confidence band, estimate the half-median-unbiased lower bound given π_{01} , $\tilde{\Delta}_t^{LB}(\pi_{01}, \frac{1}{2})$.
11. Let $\Gamma = \max(0, \tilde{\Delta}_t^{UB}(\pi_{01}, \frac{1}{2}) - \tilde{\Delta}_t^{LB}(\pi_{01}, \frac{1}{2}))$, $\rho = \max(\tilde{\Delta}_t^{UB}(\pi_{01}, \frac{3}{4}) - \tilde{\Delta}_t^{UB}(\pi_{01}, \frac{1}{4}), \tilde{\Delta}_t^{LB}(\pi_{01}, \frac{3}{4}) - \tilde{\Delta}_t^{LB}(\pi_{01}, \frac{1}{4}))$ and $\tau = (\rho \cdot \log(n))^{-1}$. Compute $\hat{a} = 1 - \Phi(\tau \cdot \Gamma) \cdot \alpha$, where α is the chosen confidence level.
12. The upper confidence band for the estimate of Δ_t^{UB} is obtained by $\sup_{\pi_{01} \in \mathcal{P}} [\tilde{\Delta}_t^{UB}(\pi_{01}, \hat{a})]$.

A.8 Bounding the ATEs on the treated with $Z = 1$ and $Z = 0$

In the subsequent sections³, we derive bounds on $\Delta_{D=1, Z=1} = E(Y(1) - Y(0)|D = 1, Z = 1)$, the ATE among those treated receiving the instrument (consisting of always takers and compliers), and on $\Delta_{D=1, Z=0} = E(Y(1) - Y(0)|D = 1, Z = 0)$, the ATE among those treated not receiving the instrument (consisting of defiers and never takers) under various assumptions. These parameters appear interesting in the context of experimentally evaluated encouragement designs aiming at increasing treatment participation (e.g. by advertisement). $\Delta_{D=1, Z=1}$ then gives

³We would like to thank an anonymous referee for suggesting those parameters.

the effect on the treated that have been encouraged ($Z = 1$) to take the treatment, whereas $\Delta_{D=1, Z=0}$ is the impact on those who are treated despite of not being encouraged ($Z = 0$). This allows assessing whether/how the ATE on the treated differs with and without encouragement (however, bearing in mind that the treated are not identical under $Z = 1$ and $Z = 0$), which may be relevant for deciding whether an encouragement should be provided to everybody or nobody in some population of interest.

A.8.1 Mean independence within principal strata without further assumptions

To derive bounds on $\Delta_{D=1, Z=1} = E(Y(1) - Y(0)|D = 1, Z = 1)$, note that $E(Y(1)|D = 1, Z = 1) = E(Y|D = 1, Z = 1) = \bar{Y}_{1,1}$. Therefore, we only need to bound

$$\begin{aligned} E(Y(0)|D = 1, Z = 1) &= \frac{\pi_{11}}{P_{1|1}} \cdot E(Y(0)|T = 11) + \frac{\pi_{10}}{P_{1|0}} \cdot E(Y(0)|T = 10) \\ &= \frac{(P_{1|0} - \pi_{01}) \cdot E(Y(0)|T = 11) + P_{0|0} \cdot \bar{Y}_{0,0} - (P_{0|1} - \pi_{01}) \cdot E(Y(0)|T = 00)}{P_{1|1}}. \end{aligned}$$

The sharp bounds on $\Delta_{D=1, Z=1}$ are given by

$$\begin{aligned} \Delta_{D=1, Z=1}^{UB} &= \bar{Y}_{1,1} - \frac{(P_{1|0} - \pi_{01}^{\min}) \cdot y^{LB} + P_{0|0} \cdot \bar{Y}_{0,0}}{P_{1|1}} \\ &\quad + \frac{(P_{0|1} - \pi_{01}^{\min}) \cdot \min\left(\bar{Y}_{0,0}(\max |q_{0,0}^{00, \pi_{01}^{\min}}), \bar{Y}_{1,0}(\max |q_{1,0}^{00, \pi_{01}^{\min}})\right)}{P_{1|1}}, \\ \Delta_{D=1, Z=1}^{LB} &= \bar{Y}_{1,1} - \frac{(P_{1|0} - \pi_{01}^{\min}) \cdot y^{UB} + P_{0|0} \cdot \bar{Y}_{0,0}}{P_{1|1}} \\ &\quad + \frac{(P_{0|1} - \pi_{01}^{\min}) \cdot \max\left(\bar{Y}_{0,0}(\min |q_{0,0}^{00, \pi_{01}^{\min}}), \bar{Y}_{1,0}(\min |q_{1,0}^{00, \pi_{01}^{\min}})\right)}{P_{1|1}}. \end{aligned}$$

To see this, note that (A.8) and (A.9) imply $\Delta_{D=1, Z=1} = \Delta_{D=1, Z=1}^{UB}(\pi_{01})$. In an analogous manner as in Appendix A.1.5, one can show that $\sup_{\pi_{01} \in \mathcal{P}} \Delta_{D=1, Z=1}^{UB}(\pi_{01}) = \Delta_{D=1, Z=1}^{UB}(\pi_{01}^{\min})$.

Concerning $\Delta_{D=1, Z=0} = E(Y(1) - Y(0)|D = 1, Z = 0)$, note that $E(Y(1)|D = 1, Z = 0) = E(Y|D = 1, Z = 0) = \bar{Y}_{0,1}$, so that we only need to bound

$$\begin{aligned} E(Y(0)|D = 1, Z = 0) &= \frac{\pi_{11}}{P_{1|0}} \cdot E(Y(0)|T = 11) + \frac{\pi_{01}}{P_{1|0}} \cdot E(Y(0)|T = 01) \\ &= \frac{(P_{1|0} - \pi_{01}) \cdot E(Y(0)|T = 11) + P_{0|1} \cdot \bar{Y}_{1,0} - (P_{0|1} - \pi_{01}) \cdot E(Y(0)|T = 00)}{P_{1|0}}. \end{aligned}$$

In the same way as for $\Delta_{D=1,Z=1}^{UB}$, one can show that the sharp bounds on $\Delta_{D=1,Z=1}$ are given by

$$\begin{aligned}\Delta_{D=1,Z=0}^{UB} &= \bar{Y}_{0,1} - \frac{(P_{1|0} - \pi_{01}^{\min}) \cdot y^{LB} + P_{0|1} \cdot \bar{Y}_{0,1}}{P_{1|0}} \\ &\quad + \frac{(P_{0|1} - \pi_{01}^{\min}) \cdot \min\left(\bar{Y}_{0,0}(\max |q_{0,0}^{00, \pi_{01}^{\min}}), \bar{Y}_{1,0}(\max |q_{1,0}^{00, \pi_{01}^{\min}})\right)}{P_{1|0}}, \\ \Delta_{D=1,Z=0}^{LB} &= \bar{Y}_{0,1} - \frac{(P_{1|0} - \pi_{01}^{\min}) \cdot y^{UB} + P_{0|1} \cdot \bar{Y}_{1,0}}{P_{1|0}} \\ &\quad + \frac{(P_{0|1} - \pi_{01}^{\min}) \cdot \max\left(\bar{Y}_{0,0}(\min |q_{0,0}^{00, \pi_{01}^{\min}}), \bar{Y}_{1,0}(\min |q_{1,0}^{00, \pi_{01}^{\min}})\right)}{P_{1|0}}.\end{aligned}$$

A.8.2 Monotonicity

Under monotonicity, $\Delta_{D=1,Z=0} = \Delta_{11}$ because defiers do not exist. Concerning $\Delta_{D=1,Z=1}$, $\pi_{01} = 0$ and $E(Y(0)|T = 00) = \bar{Y}_{1,0}$ imply that the bounds simplify to

$$\begin{aligned}\Delta_{D=1,Z=1}^{UB} &= \frac{P_{1|1} \cdot \bar{Y}_{1,1} + P_{0|1} \cdot \bar{Y}_{1,0} - P_{1|0} \cdot y^{LB} - P_{0|0} \cdot \bar{Y}_{0,0}}{P_{1|1}}, \\ \Delta_{D=1,Z=1}^{LB} &= \frac{P_{1|1} \cdot \bar{Y}_{1,1} + P_{0|1} \cdot \bar{Y}_{1,0} - P_{1|0} \cdot y^{UB} - P_{0|0} \cdot \bar{Y}_{0,0}}{P_{1|1}}.\end{aligned}$$

Sharpness follows immediately.

A.8.3 Mean dominance

Using a similar argument as the one used for the bounds under Assumption 2 alone, one can show that the following bounds are sharp.

$$\begin{aligned}\Delta_{D=1,Z=1}^{UB} &= \sup_{\pi_{01} \in \mathcal{P}} \left[\bar{Y}_{1,1} - \frac{(P_{1|0} - \pi_{01}^{\min}) \cdot y^{LB} + P_{0|0} \cdot \bar{Y}_{0,0}}{P_{1|1}} \right. \\ &\quad \left. + \frac{(P_{0|1} - \pi_{01}^{\min}) \cdot \min\left(\bar{Y}_{0,0}(\max |q_{0,0}^{00, \pi_{01}^{\min}}), \bar{Y}_{1,0}(\max |q_{1,0}^{00, \pi_{01}^{\min}})\right)}{P_{1|1}} \right], \\ \Delta_{D=1,Z=1}^{LB} &= \inf_{\pi_{01} \in \mathcal{P}} \left[\bar{Y}_{1,1} - \frac{(P_{1|0} - \pi_{01}^{\min}) \cdot y^{UB} + P_{0|0} \cdot \bar{Y}_{0,0}}{P_{1|1}} \right. \\ &\quad \left. + \frac{(P_{0|1} - \pi_{01}^{\min}) \cdot \max\left(\bar{Y}_{0,0}(\min |q_{0,0}^{00, \pi_{01}^{\min}}), \bar{Y}_{1,0}(\min |q_{1,0}^{00, \pi_{01}^{\min}})\right)}{P_{1|1}} \right].\end{aligned}$$

$$\begin{aligned}
\Delta_{D=1,Z=0}^{UB} &= \sup_{\pi_{01} \in \mathcal{P}} \left[\bar{Y}_{0,1} - \frac{(P_{1|0} - \pi_{01}^{\min}) \cdot y^{LB} + P_{0|1} \cdot \bar{Y}_{0,1}}{P_{1|0}} \right. \\
&\quad \left. + \frac{(P_{0|1} - \pi_{01}^{\min}) \cdot \min \left(\bar{Y}_{0,0}(\max |q_{0,0}^{00, \pi_{01}^{\min}}), \bar{Y}_{1,0}(\max |q_{1,0}^{00, \pi_{01}^{\min}}) \right)}{P_{1|0}} \right], \\
\Delta_{D=1,Z=0}^{LB} &= \inf_{\pi_{01} \in \mathcal{P}} \left[\bar{Y}_{0,1} - \frac{(P_{1|0} - \pi_{01}^{\min}) \cdot y^{UB} + P_{0|1} \cdot \bar{Y}_{1,0}}{P_{1|0}} \right. \\
&\quad \left. + \frac{(P_{0|1} - \pi_{01}^{\min}) \cdot \max \left(\bar{Y}_{0,0}(\min |q_{0,0}^{00, \pi_{01}^{\min}}), \bar{Y}_{1,0}(\min |q_{1,0}^{00, \pi_{01}^{\min}}) \right)}{P_{1|0}} \right].
\end{aligned}$$

A.8.4 Monotonicity and mean dominance

Under both monotonicity and mean dominance, $\Delta_{D=1,Z=0} = \Delta_{11}$ and $\Delta_{D=1,Z=1}^{UB}$ are the same as under monotonicity alone. Concerning $\Delta_{D=1,Z=1}^{LB}$, note that $E(Y(0)|T=11)^{UB} \leq E(Y(0)|T=10) = \frac{P_{0|0} \cdot \bar{Y}_{0,0} - P_{0|1} \cdot \bar{Y}_{1,0}}{P_{1|1} - P_{1|0}}$. Therefore,

$$\Delta_{D=1,Z=1}^{LB} = \frac{P_{1|1} \cdot \bar{Y}_{1,1} + P_{0|1} \cdot \bar{Y}_{1,0} - P_{1|0} \cdot \frac{P_{0|0} \cdot \bar{Y}_{0,0} - P_{0|1} \cdot \bar{Y}_{1,0}}{P_{1|1} - P_{1|0}} - P_{0|0} \cdot \bar{Y}_{0,0}}{P_{1|1}}.$$

Sharpness follows immediately.

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