Should I stay or should I go? Multiple option valuation based on LSMC for life insurance contracts

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Abstract

Life insurance contracts typically possess various embedded options. We focus on common options with early exercise features such as paid-up options, resumption options, surrender options and combinations among them. Contrary to the existing literature, showing these options value little under a deterministic-term structure, we demonstrate the situation changes dramatically whenever stochastic interest rates are introduced. With two stochastic sources (asset and interest rates), deriving optimal stopping strategies for multiple options becomes complex. Therefore, in this paper, an extension of the least squares Monte Carlo method (LSMC) is developed to allow valuation of these multi-feature options.

Keywords: Participating life insurance contracts; Embedded option pricing; Stochastic interest rate; Insurance management; Risk-neutral valuation; Least-squares Monte Carlo

JEL classification: E17; G13; G22

1 Introduction

Life insurance contracts are typically offered with various embedded options. In this paper, we focus on premium payment options with early exercise features, which can be found in essentially all life insurance contract. As insurance contracts usually go with multiple premium payments, embedded premium options can be offered with different varieties: A paid-up option allows policyholders to stop premium payments while the main contract continues with adjusted benefits. A resumption option allows policyholders to resume payments after the paid-up option has been exercised (again, benefits will be adjusted accordingly). With a surrender option, policyholders can terminate their contracts and receive a surrender amount. With a combined paid-up and surrender option, policyholders may surrender their policy with or without previously exercising the paid-up option. The combined of all these three options allows policyholders to exercise paid-up, resumption, surrender options during the contract period.

In the current low-interest rate environment, insurance companies are particularly struggling with the high long-term interest guarantees which they previously provided to their policyholders. The situation for the insurer can be even more problematic, as policyholders tend to exercise their surrender or paid-up options once the interest rate rebounds (cf. Feodoria and Förstemann (2015)). Therefore, if options are not priced adequately and hence no proper risk management has taken place, insurance companies may encounter severe difficulties (cf. the cases of Equitable Life in 2000 or the Hartford in 2009). In addition, current solvency regulation schemes such as Solvency II or the Swiss Solvency Test require insurers to consider lapse risk and provide proper risk management and equity capital for options provided to their customers. Proper models for the valuation of premium payment options and the related risk assessment are thus essential for life insurance companies and should be conducted with care.

Most of literature dealing with premium payment options centers on a single surrender option with the assumption that one single premium payment is paid upfront. Thus, life insurance policies without surrender option are as European option while the insurance policies with surrender option embedded are as a Bermudan or American options (cf. Courtadon (1982), Grosen and Jørgensen (1997)). The surrender option can therefore be valued as the difference between the American option and the European option. To value these options, Bacinello (2003a), Bacinello (2003b), and Bacinello (2005) apply the recursive binomial-tree approach discussed in Cox et al. (1979). First suggested by Longstaff and Schwartz (2001) and further discussed and compared in Ibanez and Zapatero (2004), least squares Monte Carlo method (LSMC) is another approach to value Amer-

ican options. This method has then been applied in the life insurance case by Bacinello (2008) and by Bacinello et al. (2009). Andreatta and Corradin (2003) compare these two approaches and conclude that these two methods are similarly accurate, while LSMC can be better applied for high-dimensional derivative valuation. Bauer et al. (2010) build a general model and compare these numerical valuation approaches. Again, LSMC was found to be superior because of its efficiency.

Generally, life insurance contracts include multiple premium payments during the contract period. In addition, premium options include not only surrender options but paid-up and resumption options. The latter two optios do not exist under the single premium payment assumption. With the multiple-premium-payment contract model, Kling et al. (2006), Gatzert and Schmeiser (2008), and Schmeiser and Wagner (2011) value both single and combined premium payment options. With geometric Brownian motion for assets and a deterministic interest rate, the studies base on a fair pricing concept (net present value (NPV) of zero for insurance contracts without premium payment options). Option values are then derived as NPV of the contract with premium payment options. In particular, if the options are exercised at their maximum level, the provider may face severe risk (cf. Gatzert and Schmeiser (2008)). However, as Kling et al. (2006), Gatzert and Schmeiser (2008), and Reuß et al. (2016) point out, this strategy is not feasible from the policyholders' point of view. Schmeiser and Wagner (2011) value premium payment options with the optimal stopping strategy proposed by Andersen (1999). In the case of participating life insurance with cliquet-style option, Schmeiser and Wagner (2011) show that the value of premium payment options is fairly small. Moreover, for a combined option, it is not possible to exercise each option at its optimal level. Thus, a combined option values only little more than a single option but far less than the sum of the individual. MacKay et al. (2017) moreover design a marketable contract with a state-dependent fee structure, with which it is never optimal to exercise the premium payment options. This fee structure depends on the asset-development only while interest rates are kept constant.

The existing literature regarding combined option valuation assumes deterministic interest rate. However, as Chung (2002) contends, the constant-interest-rate assumption is not realistic if the main contracts are relatively long. Life insurance has typically long maturity and the empirical findings confirm the strong influence of interest rate on policyholders' option exercising behaviors (cf. Russell et al. (2013) and Kuo et al. (2003) in USA market, Kiesenbauer (2012) in German market and Kim (2005) in Korean market). This paper therefore values single and combined premium payment options with two stochastic risk sources (assets and interest rates). With multiple-premium-payment contract, we show the influence of stochastic interest rates on not only the surrender option but the general premium payment is enormous. This aspect must be considered to

prevent underpricing of life insurance contracts. In addition, only risk-adequate pricing allows insurers to provide adequate risk management measures such as equity capital to always ensure the payments to policyholders (cf. Grosen and Jørgensen (2002) and Jørgensen (2001)).

This paper introduces an adjusted LSMC method. Comparing LSMC and the optimal stopping strategy, LSMC is slightly biased downwards (cf. Douady (2002)). However, the optimal stopping strategy is only feasible when considering single risk source. In addtion, the LSMC method is much effiecient, solving the technical challenges Schmeiser and Wagner (2011) encounter. Thus, the combinations of three or more options are feasible to calculate with this method. Except for the surrender option, which most of literature focuses on, the pay off of the premium options such as paid-up and resumption option involves a series of uncertain future cash flow. This payoff scheme and multiple exercise points for combined options make the option valuation challenging and multifactor model with closed-form solution such as Chung (2002) and Peterson et al. (2003) hard to apply. The original LSMC method aims to find one optimal exercising point for the option triggering a known cash flow, such as American, Bermudan options, and surrender option in insurance life contract. To calculate other single and combined premium options, we adjust the LSMC method, with which policyholders make exercise decisions based on two or more conditional expected values. This adjusted LSMC method allows more general features and produces multiple optimal exercise decision points. In our case, the surrender value under the adjusted LSMC method is slightly better than the value under the original LSMC method.[1]

For the remainder of this paper: Section II introduces the general contract framework. Section III analyzes option values based both on a rational and feasible exercise strategy (adjusted LSMC) and on the options' maximal value. Section IV provides numerical results. Section V discusses the economic implications of our findings and concludes the paper.

2 The Model Framework

The basic contract with cliquet-style option includes two standard features: a guaranteed yearly interest rate (g) and a surplus participation (with participation rate α in Grosen and Jørgensen (2000)). We then extend the basic contract with different premium payment options (cf. Schmeiser and Wagner (2011) and Gatzert and Schmeiser (2008)). These premium payment options are assumed to only be exercised at the end of each year, given that the main basic contract is still in

[1] Whether this adjusted LSMC method provides a better result and improves the downward bias in general may be a subject for future research.

force (i.e., at the end of year t, policyholders must still be alive and the relevant options have not yet been exercised). We assume the insurer faces no default risk and hence legitimate payments to the policyholders can always be achieved by the insurer. In other words, the insurer can hedge out all the risk from both the basic contract options and premium payment options.

2.1 The Basic Contract ($\Pi^{\emptyset}$)

We start with a basic life insurance endowment contract with a duration of T years and time index $t = 1...T$. Let ${}_tp_x$ be the probability that a policyholder aged x years survives the next t years, while q_x=$1 - {}_1p_x$ represents the probability of death over the next year. Following general actuarial practice, we assume mortality risk is uncorrelated to financial risk sources and diversifiable (cf. Biffis et al. (2010)).

Annual premium payments, B_t, are paid by the policyholder at the beginning of t if the policyholder is alive at the end of $t-1$. Without premium options, premium payments are constant in time, i.e., $B_t = B$. Present value (PV) of premium payments can be written as $B\sum_{t=0}^{T-1} {}_tp_x(1+r)^{-t}$, where r is the interest rate, served as the technical discount rate.

Benefit payments provided to the policyholder include death benefits and survival benefits. If the policyholder dies during year t, death benefits γ are payable at the end of year t. The death benefits are constant with PV formalized as $\gamma\sum_{t=0}^{T-1} {}_tp_xq_{x+t}(1+r)^{-(t+1)}$. Survival benefits are paid out at T if the policyholder survives when the contract matures. The minimum amount of the survival benefits (guaranteed survival benefit) provided to the policyholder equals to death benefit amount, γ. The PV of the guaranteed survival benefit can be written as $\gamma {}_Tp_x(1+r)^{-T}$.

According to the actuarial equivalence principle, the PV of the premium payments and that of the death and survival benefits should be identical. Hence, γ can be derived with a fixed premium payment amount B. In order to be on the safe side, we discount the premium and benefit payments using the interest guarantee rate, g (cf. Linnemann (2003)). The relationship between the premium payments and the benefits is shown via the following equation:

$$B\sum_{t=0}^{T-1} {}_tp_x(1+g)^{-t} = \gamma(\sum_{t=0}^{T-1} {}_tp_xq_{x+t}(1+g)^{-(t+1)} + {}_T p_x(1+g)^{-T}) \tag{1}$$

Hence, γ is given by

$$\gamma = \frac{B\sum_{t=0}^{T-1} {}_tp_x(1+g)^{-t}}{\sum_{t=0}^{T-1} {}_tp_x q_{x+t}(1+g)^{-(t+1)} + {}_Tp_x(1+g)^{-T}} \tag{2}$$

Note that for the survival benefit, γ only represents the minimum benefit given by the guaranteed interest rate. The actual amount of the survival benefit depends on the policy's accumulated assets, A_T, including both the guarantee option and a surplus participation. To calculate this policy's accumulated asset, A_t, we separate the annual premium payment, B, into two parts, denoted by B_t^R and B_t^A. B_t^R as $q_{x+t-1}\max(\gamma - A_{t-1}, 0)$ is used to pay the difference between the death benefits and the policy's asset accumulated by the end of the previous year (A_{t-1}). The remainder, B_t^A serves as the savings premium and becomes part of the policy's accumulated asset account for the coming year, t:

$$\begin{aligned} B &= B_t^R + B_t^A \\ B_t^A &= B - q_{x+t-1}\max(\gamma - A_{t-1}, 0) \end{aligned} \tag{3}$$

At the beginning of t, the policy's accumulated asset contains two parts: the accumulated amount at the end of the previous year, A_{t-1}, and the annual savings premium, B_t^A, collected at the beginning of t. With both the guarantee and surplus, the accumulated assets earn an annual return at the guaranteed interest rate or an annual surplus rate, whichever is greater. The annual surplus rate is a fraction, α, of the annual insurer's investment result at t, i.e., $S_t/S_{t-1} - 1$. Hence, α serves as a participation rate. The development of the policy's assets over time can be formally written as

$$A_t = (A_{t-1} + {}_{t-1}p_x B_t^A)(\max(g, \alpha(S_t/S_{t-1} - 1)) + 1) \tag{4}$$

with $A_0 = 0$

The policy's asset is subject to investment risk, which includes two risk sources, the interest rate risk and the asset risk. The interest rate, r, evolves according to the one-factor Vasicek model (cf. Vasicek (1977)):

$$dr_t = \kappa(\theta - r_t)dt + \sigma^I dZ^{\mathbb{P}} \tag{5}$$

Here, $Z^{\mathbb{P}}$ is a Wiener process on a probability space $(\Omega, \phi, \mathbb{P})$. To capture the interest rate risk, σ^I determines how much randomness of $Z^{\mathbb{P}}$ is acquired in the model. κ and θ are positive constants representing the speed of reversion and the long-term mean, respectively. A constant market price of risk, λ, is introduced to transfer the model into the risk-neutral probability space. If the market participants are risk averse, we have $\lambda < 0$. Under the risk-neutral measure, $\mathbb{Q}$, the interest spot

rate process given in equation (5) changes to

$$dr_t = \kappa(\theta - \frac{\sigma^I \lambda}{\kappa} - r_t)dt + \sigma^I dZ_t^{\mathbb{Q}} \tag{6}$$

where $Z_t^{\mathbb{Q}}$ denotes the Wiener process under the risk-neutral measure, $\mathbb{Q}$. The solution of the Vasicek model for one period return can be derived as

$$r_t = e^{(-\kappa)} + (\theta - \frac{\sigma^I \lambda}{\kappa})(1 - e^{-\kappa}) + \frac{\sigma^I}{\sqrt{2\kappa}}\sqrt{1 - e^{-2\kappa}}(Z_t^{\mathbb{Q}} - Z_{t-1}^{\mathbb{Q}}) \tag{7}$$

For asset risk, we assume the policy's asset follows a geometric Brownian motion (μ and σ^s) with stochastic interest rate derived via equation (7). That is, though insurers usually invest in a bundle of assets, the whole portfolio contains one asset risk. For the geometric Brownian motion, we have:

$$S_t = S_0 e^{(\mu - \sigma^{s2}/2)dt + \sigma^s dW}$$

Under the risk-neutral measure, $\mathbb{Q}$, and combined with a stochastic interest rate, the deterministic drift for the asset risk changes into the stochastic spot rate. One-period return can be derived as:

$$\ln \frac{S_t}{S_{t-1}} = r_t - \sigma^{s2}/2 + \sigma^s(\rho(Z_t^{\mathbb{Q}} - Z_{t-1}^{\mathbb{Q}}) + \sqrt{1-\rho^2}(W_t^{\mathbb{Q}} - W_{t-1}^{\mathbb{Q}})) \tag{8}$$

In this context, $W_t^{\mathbb{Q}}$ represents a second Wiener process under the risk-neutral measure $\mathbb{Q}$. σ^s captures the investment risk, which relates to both the asset risk and the interest risk. ρ indicates the correlation coefficient between these two risks.

NPV of the contract, denoted by Π, is the PV difference between two cash flows: the benefit paid to the policyholder and the premium paid by the policyholder to the insurer. For basic contract, $\Pi^{\emptyset}$ can be formalized as:

$$\Pi^{\emptyset} = E^{\mathbb{Q}}[(\gamma \sum_{t=0}^{T-1} {}_t p_x q_{x+t} \delta_{t+1} + A_T \delta_T - B \sum_{t=0}^{T-1} {}_t p_x \delta_t)] \tag{9}$$

where $\delta_t = \prod_{i=0}^{t}((1 + r_i))^{(-1)}$.

We call a contract fair whenever its NPV is zero ($\Pi^{\emptyset} = 0$).

For different parameters, we aim to derive their respective participation rate, α (with $0 \leq \alpha \leq 1$),

that leads to a fair condition for policyholders and equity holders.

In what follows, the value of premium payment options ϑ is derived as the NPV difference between the basic contract $\Pi^{\emptyset}$ and the contract with the options Π^{ϑ}, or the NPV of ϑ (as $\Pi^{\emptyset} = 0$ in equation (9)). Policyholders are assumed to pay ϑ at $t = 0$ so that the contract with the option is fair. The premium payment options considered in this paper are: paid-up option, resumption, surrender option, and the combinations among them.

g	guaranteed interest rate
r_t	stochastic annual spot rate for year t
δ_t	stochastic discount factor for year t back to year 0
α	participation rate ($0 \leq \alpha \leq 1$)
$\Pi^{\emptyset}$	NPV of the basic contract
B_t	constant premium payment, paid at the beginning of year t ($B_t = B = B_t^A + B_t^R$)
B_t^R	term life premium
B_t^A	savings premium
γ	constant death benefit paid at the end of the year
A_t	policy's accumulated asset at the end of the year t
ϑ	present value of a premium payment option

Table 1: Summary of basic contract notation

2.2 Contract with a Option ϑ

Policyholders stop premium payments while the contract continues with adjusted benefits when the paid-up option is exercised. The resumption option allows policyholders to resume premium payments after exercising the paid-up option. Again, the benefit will then be adjuted. The surrender option moreover allows the policyholder to terminate the policy and receive a surrender value before the agreed end of maturity. All these options allow the policyholder to change their premium payments in different ways. Thus when a premium option is exercised, B_t' is no longer constant. Here, we assume there is no fee applied when the options are exercised. The adjusted benefit with options can be derived as following:

$$\gamma' = \frac{\sum_{t=0}^{T-1} B'_{t\,t} p_x (1+g)^{-t}}{\sum_{t=1}^{T} {}_t p_x q_{x+t} (1+g)^{-(t+1)} + {}_T p_x (1+g)^{-T}}$$

The adjusted contract account is demonstrated as:

$$A'_t = (A'_{t-1} +_{t-1} p_x B_t^{'A})(\max(g, \alpha(S_t/S_{t*1} - 1)) + 1$$

with $A'_0 = 0$ and $B_t^{'A} = B'_t - q_{x+t}\max(\gamma' - A'_{t-1}, 0)$

This setup generalizes all the premium option schemes. When none of the options has been exercised (or $t < \theta$), $B'_t = B$, $A'_t = A_t$ and $\gamma' = \gamma$. For example, with the paid-up option exercised at θ_1 and resumption exercised at θ_2, $B'_t = B$ for $0 < t < \theta_1$ and $\theta_2 \leq t < T$ and $B'_t = 0$ for $\theta_1 \leq t < \theta_2$.

With the surrender option, the contract stops whenever this option is exercised and the policyholder receives the surrender amount equals to the accumulated account value. The NPV of the contract becomes:

$$\vartheta_\theta = \Pi^\vartheta = E^{\mathbb{Q}}[(\sum_{t=0}^{\theta} (\gamma' - \gamma)_t p_x q_{x+t} \delta_{t+1} + (A'_\theta - A_\theta)\delta_T - \sum_{t=0}^{\theta-1} (B'_t - B)_t p_x \delta_t)] \tag{10}$$

NPV of the contract with the options other then surrender options can be demonstrated as:

$$\vartheta_\theta = \Pi^\vartheta = E^{\mathbb{Q}}[(\sum_{t=0}^{T-1} (\gamma'_\theta - \gamma)_t p_x q_{x+t} \delta_{t+1} + (A'_T - A_T)\delta_T - \sum_{t=0}^{T-1} (B'_t - B)_t p_x \delta_t)] \tag{11}$$

3 Valuation of Premium Payment Options

The assumed policyholder's exercise strategy is central when valuing embedded premium payment options in life insurance contracts. We begin by calculating the upper limit of the premium payment options for any possible exercise strategy. This method indicates the options' value range but not the option's accessible value. Since such a method uses information which is not available in a neoclassical finance setting and hence not a feasible strategy for policyholders, we use the adjusted LSMC (least-squares Monte Carlo) strategy as an approximation of an optimal exercise approach and as the basis for the value of the premium payment options.

3.1 Deriving an Upper Limit ($^{UP}\vartheta$)

Kling et al. (2006), Gatzert and Schmeiser (2008), and Schmeiser and Wagner (2011) discuss calculating an upper limit for premium payment options and its economical interpretation in detail.

Assuming policyholders know the future developments, they would exercise the premium payment option when it is in the money (ITM when $\vartheta > 0$) and at its maximum value for the whole contract period. Using Monte Carlo simulation with $n = 1...N$ paths, we have:

$$^{UP}\vartheta = \frac{1}{N}\sum_{n=1}^{N}(\max(^{n}\vartheta_t, 0)) \qquad t = 1...T-1, \tag{12}$$

where $^{n}\vartheta_t$ denotes the different option values if exercised at t for the n^{th} simulation path. Options are non-negative as policyholders do not exercise these options if their value is negative or out of the money (OTM) for the whole contract period. The upper limit can also be referred to as the PV of the option given perfect information. Although perfect information is clearly not possible in practice, the concept still provides useful insight as it shows the upper bound of the option – or maximum loss from the insurer's viewpoint – under any conceivable exercise strategy if no parameter or model risk can take place.

3.2 Option Valuation via the Least-squares Monte Carlo Strategy ($^{LSMC}\vartheta$)

The LSMC method was first presented by Longstaff and Schwartz (2001) to price American options. It has been used to value the single surrender option in life insurance contracts (cf. Andreatta and Corradin (2003), Nordahl (2008), and summarized by Bauer et al. (2010)) assuming single premium payment is made in the beginning of the contract. The LSMC approach aims to find an optimal exercise point t^* using only accessible information. For different points in time t, the method compares between two values: the exercise and continuation values. The exercise value is the value if the option is exercised at t, while the continuation value is the value if the policyholder does not exercise the option at t and the contract goes forward assuming optimal exercise point $t^* > t$. Following this strategy, policyholders exercise an option if its exercise value is larger than the continuation value and $t^* = t$.

The original strategy determines the exercise value as a defined and deterministic cash flow when exercising the option. The continuation value is the PV of the future cash flows if the options are not exercised immediately at t. However, except for the surrender option, exercising premium payment options does not always cause a defined immediate cash flow as the main contract continues. We adjust the original approach and define both the exercise value and the continuous value as the PV of future cash flows conditioned that the premium option is exercised or not exercised. For the

special case of surrender options, we compare both the adjusted LSMC and the original method proposed by Longstaff and Schwartz (2001) in Appendix. Our numerical examples suggest the optimal option value with the adjusted LSMC strategy is slightly better than that of the original LSMC. In the following, unless stated otherwise, LSMC refers to adjusted LSMC.

The original algorithm contains two approximations to converge the optimal option value (cf. Clément et al. (2002)). First, the continuation value at t denoted by ${}_tC(\vartheta)$ is approximated by the combination of finite functions with accessible relevant information as parameters. The second approximation determines the value functions via a least squares regression. We further add two approximations for the exercise value, ϑ_t, the option value when the option is immediately exercised at t. Note that all the values are discounted for convenient comparison back to the beginning of the contract.

${}_tC(\vartheta)$ is approximated by ${}_tC(\vartheta) \cong E^{\mathbb{Q}}[{}_tC(\vartheta)|\mathscr{F}_t] \cong f(x_t^1 ... x_t^J)$, the expected value under the risk-neutral distribution at year t. $x_t^1 ... x_t^J$ are J relevant variables (all possible information accessible at t). In our model, x^j with $j = 1...3$ includes the interest rate, r_t, the investment rate of return, S_t/S_{t-1}, and the adjusted benefit, γ'.

The second approximation includes K sets of basis functions, υ^k, to approximate $f(x_t^1 ... x_t^J)$ with α^k as constant coefficients. In our model, υ^k is a set of Laguerre polynomials. We set $K = 4$.[2]

$$ {}_tC(\vartheta) \cong f(x_t^1 ... x_t^J) \cong \sum_{k=0}^{K} \alpha_t^k \upsilon^k(x_t^1 ... x_t^J) \tag{13} $$

The coefficients α^k are unknown so far. With Monte Carlo simulation paths $n = 1...N$, we estimate α_t^k via least squares linear regression. In Longstaff and Schwartz (2001), these estimators are based solely on in-the-money paths to reduce computation effort. However, in our case, all paths should be considered since the option we focus is not standard (cf. Andreatta and Corradin (2003)). The estimator for α_t^k is provided by

$$ \hat{\alpha}_t^k = arg\min\{\sum_{n=1}^{N} [{}_t^nC(\vartheta) - \sum_{k=0}^{K} \alpha_t^k \upsilon^k({}^nx_t^1 ... {}^nx_t^J]\} \tag{14} $$

[2] When taking different values for K, our numerical results stabilize after $K = 4$.

With $\hat{\alpha}_t^k$, we can calculate:

$$ {}_t^n\hat{C}(\vartheta) = \sum_{k=0}^{K} \hat{\alpha}_t^k \upsilon^k({}^n x_t^1 ... {}^n x_t^J) \tag{15} $$

As explained above, the option value at the time when the option is exercised is unknown until maturity ($t = T$). Therefore, another two approximations are required to calculate ϑ_t via $E^{\mathbb{Q}}[\vartheta_t|\mathscr{F}_t]$:

$$ \begin{aligned} \vartheta_t &= E^{\mathbb{Q}}[\vartheta_t|\mathscr{F}_t] \cong s(x_t^1 ... x_t^J) \cong \sum_{k=0}^{K} \alpha_t'^k \upsilon^k(x_t^1 ... x_t^J) \\ \hat{\alpha}_t'^k &= arg\min\{\sum_{n=1}^{N} [{}^n\vartheta_t - \sum_{k=0}^{K} \alpha_t'^k \upsilon^k({}^n x_t^1 ... {}^n x_t^J]\} \\ {}^n\hat{\vartheta}_t &= \sum_{k=0}^{K} \hat{\alpha}_t'^k \upsilon^k({}^n x_t^1 ... {}^n x_t^J) \end{aligned} \tag{16} $$

Following we exam five different cases as described in Schmeiser and Wagner (2011):

1. For paid-up only option, ϑ_τ^P exercised at τ with $\tau = 1...T-1$, $B_t' = B$ for $t = 1...\tau-1$ and $B_t' = 0$ for $t = \tau...T-1$. For $\tau = T$, the option is expired and $\vartheta_T^P = \Pi^\emptyset = 0$

2. With the surrender-only option, ϑ_θ^S can be exercised and the contract discontinued at θ with $\theta = 1...T$. For $\theta = T$, the option is expired and $\vartheta_T^S = \Pi^\emptyset = 0$.

3. For the combined option of paid-up on and resumption, $\vartheta_{\tau,\nu}^{PR}$ at τ and ν separately with $\tau = 0...T-1$ and $\nu = \tau+1...T$, $B_t' = B$ for $t = 1...\tau-1$ and $t = \nu...T$ while $B_t' = 0$ for $t = \tau...\nu-1$. For $\tau = 0$ and $\nu = T$, both options are expired and $\vartheta_{0,0}^{PR} = \Pi^\emptyset = 0$. For $\nu = T$ and $\tau > 0$, the resumption option is expired and $\vartheta_{\tau,T}^{PR} = \vartheta_\tau^P$.

4. For the combined option of paid-up and surrender, $\vartheta_{\tau,\theta}^{PS}$ at τ and θ, $B_t' = B$ for $t = 0...\tau$ and $B_t' = 0$ thereafter.

5. For the combined option of paid-up resumption and surrender, $\vartheta_{\tau,\nu,\theta}^{PRS}$ at τ, ν and θ, $B_t' = B$ for $t = 0...\tau-1$ and $t = \nu...\min(\theta, T-1)$ and $B_t' = 0$ for $t = \tau...\min(\nu-1, \theta)$.

3.2.1 Premium Payment Option Case (Single Option Procedure)

For single options, ϑ^P and ϑ^S, we aim to find one optimal exercise point, ${}^n t^*$, that maximizes the option value in each path, n, by using accessible information (x_t^j). At the end of each year, policyholders decide whether to exercise the option or not. The option should be exercised if the exercise value exceeds the continuation value.

For the Monte Carlo path, n, with $n = 1...N$ and m iteration with $m = T...1$, the single-option procedure can be formally described as follows.

Step 1: Initialization: Start with $m = T$ and set all ${}^n t^* = m = T$:
The option value is zero as the contract matures without exercising the option. The optimal option value is given by ${}^n \vartheta_{{}^n t^*} = {}^n \vartheta_T = 0$.

Step 2: One-Year Backward:
One year backward at $m = T - 1$, the continuation value is set zero as ${}^n_{T-1}C(\vartheta) = {}^n \vartheta_{{}^n t^*} = \vartheta_T = 0$. Policyholders decide whether to exercise the option. If ${}^n \hat{\vartheta}_{T-1}$ is positive (and hence exceeds the continuation value, which is zero) the option should be exercised and the optimal exercise point becomes ${}^n t^* = m$. Otherwise, the contract continues and ${}^n t^*$ remains unchanged. In formal terms, we have:

$$ {}^n t^* = T - 1, \text{ if } {}^n \hat{\vartheta}_{T-1} > {}^n_{T-1}C(\vartheta) \text{ (where } {}^n_{T-1}C(\vartheta) = 0). \text{ Otherwise, } {}^n t^* \text{ remains } T. $$

where ${}^n \hat{\vartheta}_{T-1}$ can be derived with the equation (16)

Step 3. Backward Iteration: for $m = T - 2...1$:
(1) ${}^n_m C(\vartheta) = {}^n \vartheta_{{}^n t^*}$

(2) Approximate the continuation value and the exercise value with the equation (15) and (16):
${}^n_m \hat{C}(\vartheta) = \sum_{k=0}^{K} \hat{\alpha}_m^k \upsilon^k({}^n x_m^1 ... {}^n x_m^J)$, with $\hat{\alpha}_m^k = arg\min\{\sum_{n=1}^{N}[{}^n_t C(\vartheta) - \sum_{k=0}^{K} \alpha_m^k \upsilon^k({}^n x_m^1 ... {}^n x_m^J)]\}$

${}^n \hat{\vartheta}_m = \sum_{k=0}^{K} \hat{\alpha}_m'^k \upsilon^k({}^n x_m^1 ... {}^n x_m^J)$, with $\hat{\alpha}_m'^k = arg\min\{\sum_{n=1}^{N}[{}^n \vartheta_m - \sum_{k=0}^{K} \alpha_m'^k \upsilon^k({}^n x_m^1 ... {}^n x_m^J)]\}$

(3) If ${}^n \hat{\vartheta}_m >^n_m \hat{C}(\vartheta)$, ${}^n t^* = m$. Otherwise, ${}^n t^*$ remains unchanged.

With the algorithm above, the optimal option value equals the average of each path option value

exercised at its respective point, ${}^{n}t^{*}$:

$$^{LSMC}\vartheta = \frac{1}{N}\sum_{n=1}^{N}({}^{n}\vartheta_{{}^{n}t^{*}})$$

3.2.2 Double Premium Payment Option Case

For the two option combination, we exam the option of paid-up and resumption $\vartheta^{PR}_{\tau,\nu}$, and the option of paid-up and surrender $\vartheta^{PS}_{\tau,\theta}$. The main difference between these two options is that the resumption option can only exercised when the paid-up option has been exercised, i.e., $\tau < \nu$. However, policyholders can exercise the surrender option independently, even if the paid-up option has not yet been exercised.

The Combined Paid-up and Resumption Option ϑ^{PR}

The two-option combination contains two optimal exercise points. In addition, the value of the combined option cannot be estimated when deciding whether to exercise the first option as the second exercise point has not been determined yet. However, for ϑ^{PR}, while the first paid-up option can be seen as a put option, the resumption option has call option feature. Therefore, at time t, if it is optimal to exercise the paid-up option, the intrinsic value for the resumption option is 0. We can hence first consider the paid-up option only. Conditioned that the paid-up option is exercised, we then determine the second optimal exercise point for the resumption option.

First we determined ${}^{n}\tau^{*}$ with the single option process described in the previous section. If ${}^{n}\tau^{*} = T$, the optimal strategy is not to exercise the first option. In this case, we set ${}^{n}\tau^{*} = 0$. Conditioned that ϑ^{P} being exercised (or $0 <^{n} \tau^{*} < T$), we find the second optimal exercise point ${}^{n}\nu^{*}$ to maximize the second option. The second option value is as the remaining value ϑ'. When exercised at s, the remaining resumption value is derived by $\vartheta'_{s} = \vartheta^{PR}_{t^{*},s} - \vartheta^{P}_{t^{*}}$.

For ${}^{n}\nu^{*}$, the second optimal exercise point at s, we run the 2nd iteration $m' = T...{}^{n}\tau^{*} + 1$ for the single option ϑ'.

The optimal value can be derived by $^{LSMC}\vartheta^{PR} = \frac{1}{N}\sum_{n=1}^{N}({}^{n}\vartheta^{PR}_{{}^{n}\tau^{*},{}^{n}\nu^{*}})$.

The Combined Surrender and Paid-up Option ϑ^{PS}

In the case of paid-up and surrender option, both options are considered as put option and can

be exercised independently. More specifically, the surrender option as the second option can be exercised even if the first option has not yet been exercised. In formal terms, we have:

$$\vartheta_{t,s}^{PS} = \vartheta_s^S \cdot \mathbb{1}_{t=0} + (\vartheta_t^P + \vartheta_{t,s}^{\prime S}) \cdot \mathbb{1}_{T \geq s > t > 0} \tag{17}$$

with $\vartheta_{t,s}^{\prime S} = \vartheta_{t,s}^{PS} - \vartheta_t^P$ for $T \geq s > t > 0$.

Thus, at the end of every year t, two put options are compared so to determine whether to exercise one of the two options. We begin by finding the optimal exercise point for one of these two options ($^n\tau^*$ for ϑ^P and $^n\theta^*$ for ϑ^P) with the independent option procedure described as following:

With m iteration for $m = T...1$:

Step 1. Initialization:
For $m = T$, with $^n\tau^* = T$ and $^n\theta^* = T$, $^n\vartheta^P_{^n\tau^*} = 0$ and $^n\vartheta^S_{^n\theta^*} = 0$.

Step 2. One-Year Backward Comparison:
At $m = T-1$, The option value is $^n\hat{\vartheta}^{PS}_{^n\tau^*,^n\theta^*} = \max(^n_{T-1}C(\vartheta), ^n\hat{\vartheta}^P_{T-1}, ^n\hat{\vartheta}^S_{T-1})$. $^n\hat{\vartheta}^P_{T-1}$ and $^n\hat{\vartheta}^S_{T-1}$ are two estimators of $^n\vartheta^P_{T-1}$ and $^n\vartheta^S_{T-1}$. The continuation value $^n_{T-1}C(\vartheta)$ equals zero.

Three scenarios are considered:

(1) $^n\tau^* = T-1$, $^n\theta^* = T$ if $\max(^n_{T-1}C(\vartheta), ^n\hat{\vartheta}^P_{T-1}, ^n\hat{\vartheta}^S_{T-1}) = {}^n\hat{\vartheta}^P_{T-1}$. The best strategy is to exercise the paid-up option.

(2) $^n\tau^* = 0$, $^n\theta^* = T-1$ if $\max(^n_{T-1}C(\vartheta), ^n\hat{\vartheta}^P_{T-1}, ^n\hat{\vartheta}^S_{T-1}) = {}^n\hat{\vartheta}^S_{T-1}$. Hence, the best strategy is to exercise the surrender option.

(3) $^n\tau^* = 0$, $^n\theta^* = T$ if $\max(^n_{T-1}C(\vartheta), ^n\hat{\vartheta}^P_{T-1}, ^n\hat{\vartheta}^S_{T-1}) = {}^n_{T-1}C(\vartheta) = 0$. In this case, policyholders should keep the two options.

Step 3. Backward Iteration:
For the backward iteration with $m = T-2...1$, we have:

(1) $^n_mC(\vartheta) = {}^n\vartheta_{^n\tau^*,^n\theta^*}$

(2) ${}^n\tau^* = m$, ${}^n\theta^* = T$, if $\max({}^n\hat{\vartheta}^P_m, {}^n\hat{\vartheta}^S_{m,m}\hat{C}(\vartheta)) = {}^n\hat{\vartheta}^P_m$.

(3) ${}^n\tau^* = 0$, ${}^n\theta^* = m$, if $\max({}^n\hat{\vartheta}^P_m, {}^n\hat{\vartheta}^S_{m,m}\hat{C}(\vartheta)) = {}^n\hat{\vartheta}^S_m$.

(4) ${}^n\tau^*$, ${}^n\theta^*$ remain unchanged, if $\max({}^n\hat{\vartheta}^P_m, {}^n\hat{\vartheta}^S_{m,m}\hat{C}(\vartheta)) = {}^n_m\hat{C}(\vartheta)$.

${}^n\hat{\vartheta}^P_m, {}^n\hat{\vartheta}^S_{m,m}\hat{C}(\vartheta)$ are estimators for ${}^n\vartheta^P_m, {}^n\vartheta^S_{m,m}C(\vartheta)$.

After the independent option procedure, we find the two optimal exercise points ${}^n\tau^*$ and ${}^n\theta^*$. ${}^n\tau^* = 0$ means the best strategy is either to exercise the second option only (${}^n\theta^* < T$) or not to exercise any of options (${}^n\theta^* = T$). For the paths that ${}^n\tau^* > 0$, the optimal strategy is to exercise the first option at ${}^n\tau^*$. Given ${}^n\tau^* > 0$, we then derive the second optimal exercise point, ${}^n\theta^*$, to maximize the remaining surrender option value, ${}^n\vartheta'^S_{{}^n\tau^*,s} = {}^n\vartheta^{PS}_{{}^n\tau^*,s} - {}^n\vartheta^P_{{}^n\tau^*}$. For this new option ${}^n\vartheta'^S_{{}^n\tau^*,{}^ns^*}$, ${}^ns^*$ can be determined based as the single option exercise for the 2nd iteration $m_2 = T...{}^n\tau^* + 1$.

The PV of the simulated combined option is determined as ${}^{LSMC}\vartheta^{PS} = \frac{1}{N}\sum_{n=1}^{N}({}^n\vartheta^{PS}_{{}^n\tau^*,{}^n\theta^*})$.

3.2.3 Multiple Premium Payment Option Case ϑ^{PRS}

To extend the algorithm with the combined option of paid-up, resumption, and surrender, the decision process becomes however complex. The paid-up and surrender options are both put options and for ϑ^{PS}, which option to exercise depends on whichever values larger. For ϑ^{PRS}, the surrender option is compared with the combination option of paid-up and resumption. Although when it is optimal to exercise either surrender or paid-up option, the intrinsic value of the resumption option is zero. The time value of resumption option is non-zero if exercising paid-up option. With resumption option, when policyholders exercise the paid-up option, they can still use the resumption option to enter the contract once the interest falls. On the contrary, if surrender option is exercised, the time value of all other options are zero. Policyholders get the surrender amount and the contract is due. They are no more protected by the life contract. Therefore, for ϑ^{PRS}, when we first decide whether to exercise paid-up option or surrender option, resumption option's value attached to paid-up option should be considered.

Step 1: The first iteration for $m = T...1$:
Determine whether to exercise surrender or paid-up option based on the comparison between sur-

render option ϑ_m^S and ϑ_m^{PR}, where ϑ_m^{PR} is the value of combined option of paid-up and resumption with the paid-up option exercised at m. This option value is determined by another LSMC algorithm with $m_1 = T...m+1$.

With the first iteration, we can determine the first two optimal exercise points for ${}^n\tau^*$ and ${}^n\theta^*$.

Step 2: The second iteration for $m_2 = T...{}^n\tau^*+1$ for the path that $0 <^n \tau^* < T$:
For ${}^n\tau^* = T$, reset ${}^n\tau^* = 0$. Conditioned that paid-up option is exercised ($0 <^n \tau^* < T$), we decide either to exercise resumption option or surrender option based on the two remaining option values:

$$ {}^n\vartheta_t'^S =^n \vartheta^{PS}_{{}^n\tau^*,t} -^n \vartheta^P_{{}^n\tau^*} \text{ and } {}^n\vartheta_t'^R =^n \vartheta^{PR}_{{}^n\tau^*,t} -^n \vartheta^P_{{}^n\tau^*} $$

Step 3: The third iteration for $m_3 = T...{}^n\nu^*+1$ for the path ${}^n\nu^* < T$:
Conditioned that it is optimal to exercise the resumption option, we run another iteration $m_3 = T...{}^n\nu^*+1$ to determine whether to exercise the remaining surrender option with value ${}^n\vartheta_t''^S =^n \vartheta^{PRS}_{{}^n\tau^*,{}^n\nu^*,t} -^n \vartheta^{PR}_{{}^n\tau^*,{}^n\nu^*}$.

The PV of the simulated combined option is determined as ${}^{LSMC}\vartheta^{PRS} = \frac{1}{N}\sum_{n=1}^N({}^n\vartheta^{PRS}_{{}^n\tau^*,{}^n\nu^*,{}^n\theta^*})$.

4 Numerical Results

This section presents key results of our numerical analysis for discussion. In particular, we focus on the influence of the interest rate volatility, σ^I. For the easy comparison purpose, the parameters are chosen based on the parameters in Schmeiser and Wagner (2011). Due to the current low interest rate environment, we further run the sensitivity test for the low interest rate. Unless stated otherwise, the numerical results are gathered using a Monte Carlo simulation with $N = 10^4$. In the Appendix, we show that the results stabilize when N reaches 10^4.

4.1 Basic Contract

We consider a basic contract with the following parameters: A 30-year-old female policyholder enters into a 10-year life insurance contract.[3] The premium per annum is 1,200 currency units, with the yearly interest rate guarantee set to 3%. The investment return rate, $S_t/S_{t-1}-1$, combines both the spot interest and the asset process laid down in equation (8) using the risk-neutral measure,

[3]Mortality probabilities are for a 30-year-old US woman in 1994 based on the data from HMD (the Human Mortality Database, 2016).

$\mathbb{Q}$. The asset volatility is fixed to $\sigma^s = 0.2$. The correlation between the asset and interest rate risk is $\rho = 0.05$. Under the Vasicek model, to obtain r, we use the parameters $\kappa = 8\%$, $r_0 = 4\%$, $\theta = 4\%$, and $\lambda = 0$. Based on these assumptions and using equation (2), the death benefit is 14,089 currency units. Table 2 summarizes the initial parameters.

B	constant premium	1,200
γ	death benefit	14,089
x	initial age	30
T	time to maturity	10
κ	interest reversion speed	8%
σ^s	asset volatility	20%

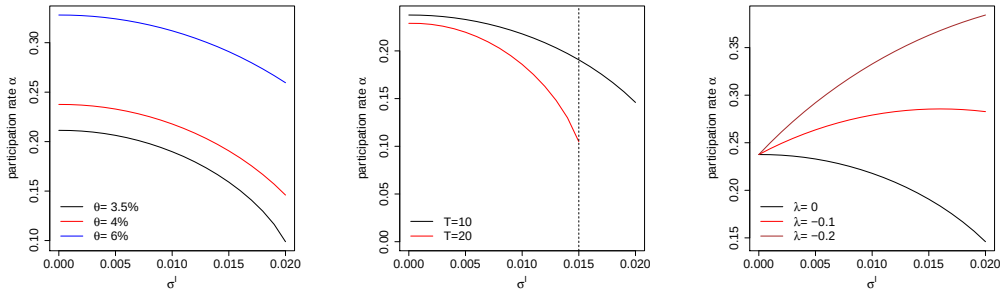
ρ	correlation	5%
g	guarantee rate	3%
r_0	initial interest rate	4%
θ	long term interest mean	4%
λ	market price of risk	0%

Table 2: Parameter table for the base case

Figure 1 shows the relationship between the participation rate, α, and interest rate volatility, σ^I, under different conditions. The participation rate, α, is derived such that the contract is fair at $t = 0$ (cf. $\Pi^{\emptyset} = 0$ in quation (9)). Figure 1(a) shows a clear trend: the higher σ^I is, the lower α gets. When σ^I increases – for the same contract with the same interest rate guarantee – insurers face higher risk and thus must lower the participation rate to ensure a risk-adequate return for shareholders. In addition, the curves with different values of θ move in parallel. For all σ^I from 0 to 2%, the model supports the same conclusion as drawn in Schmeiser and Wagner (2011): higher interest rates lead to a higher policyholder participation rate, α, under fair pricing condition.

In Figure 1(b), α decreases when the time to maturity of the contract, T, is set to 20 years. Insurers face higher risk as T increases. Thus, in order to achieve risk-adequate returns for shareholders, the participation rate, α, must be reduced. When σ^I increases to more than 1.5% for $T = 20$, there is no α with $0 \leq \alpha \leq 1$ that satisfies the fairness condition.

When $\lambda = 0$, we assume a risk-neutral market and hence no risk shift occurs when moving from the empirical to the risk-neutral measure. Whenever $\lambda < 0$, market participants are assumed to be risk averse. In this case, policyholders demand a higher participation rate, α, and/or a higher investment guarantee, g, whenever σ^I increases. Figure 1(c) shows that when $\lambda < 0$, a larger α is required compared to the case $\lambda = 0$.



(a) α and σ^I for different θ (b) α and σ^I for different T (c) α and σ^I for different λ

Figure 1: Relationship between participation rate (α) and interest rate volatility (σ^I) for different values of long-term interest mean (θ), time to maturity (T), and market price of risk (λ)

4.2 Premium Payment Option Value

Following we discuss the results of the different premium payment options. Option's value, ϑ, is defined as the NPVs of the premium payment option. The value/premium ratio (V/P) compares the option value and PV of the premium payments, if the premium payment options are exercised. In formal terms, we have:

$$V/P = \frac{\sum_{n=1}^{N} {}^{n}\vartheta}{\sum_{n=1}^{N} {}^{n}\text{PV Premium}} \tag{18}$$

with ${}^{n}\text{PV Premium} = \sum_{t=0}^{\min(T,\theta)} B'_{tt} p_x (1 + {}^{n}r_t)^{-t}$, where $B'_t = B$ or $B'_t = 0$ depending on the option exercise points.

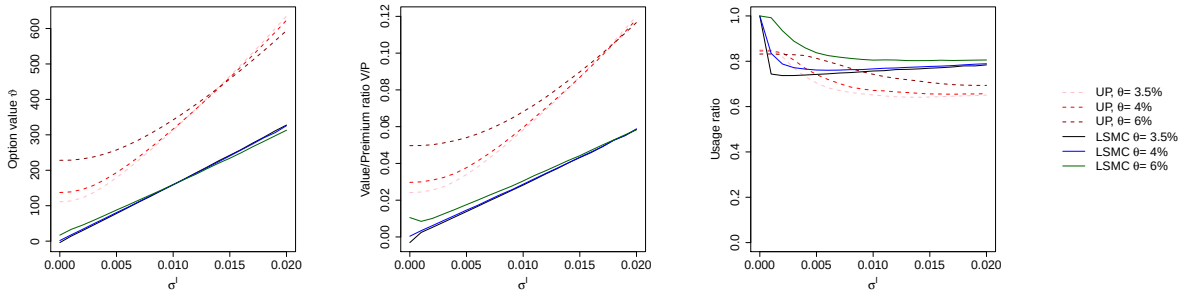
For certain paths n in a Monte Carlo simulation, the option value may be negative for the whole contract period. When calculating the upper limit or PV given perfect information, as this option is never ITM, the option for these paths will never be exercised. Therefore, the usage ratio with the upper limit perspective shows, for the entire simulation, how many paths have a positive option value during the contract period. In the case of LSMC, the usage ratio show how many paths in which it is optimal to exercise the options before the contract matures.

4.2.1 Result for Single Premium Payment Option

Paid-up Option ϑ^P

Figure 2 demonstrates the results for the single paid-up option. In Figure 2(a) with $\sigma^I = 0$ (deterministic term structure), the upper limit exhibits a similar structure to that presented in Schmeiser and Wagner (2011): A higher spot rate, θ, leads to a higher upper limit for the paid-up option. However, this interest-rate effect decreases as σ^I increases.

If $\sigma^I = 0$, its option values are close to zero for all three different θ. As σ^I increases, the option value of the paid-up option also increases. The difference in value among various θ is small. From Figure 2(b), as σ^I increases, V/P increases from 0% to 5.8% when using LSMC approach while the upper limit grows reaching 11.8%. From Figure 2(c), almost 80% of paths have positive paid-up options value during the contract period when $\sigma^I = 0\%$ and more than 60% with large σ^I. When following LSMC, the usage ratio is almost 100% when $\sigma^I = 0\%$ (while the option value is close to zero) and almost 80% with large σ^I.

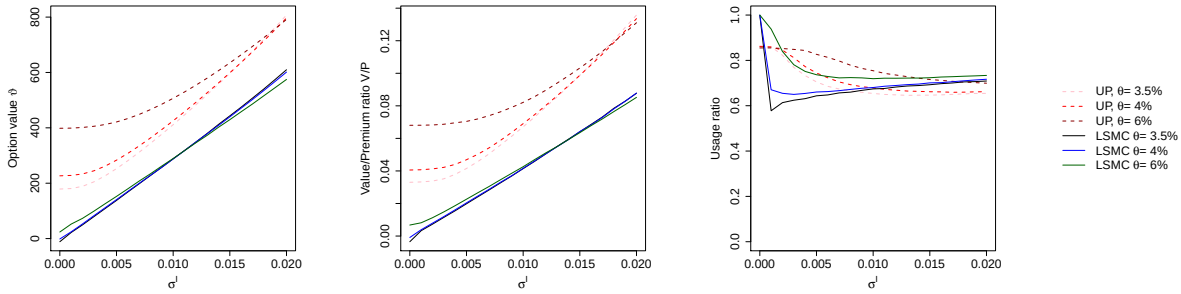


(a) Option value ϑ^P (b) V/P ratio (c) Usage ratio

Figure 2: Paid-up option results for different σ^I

Surrender Option ϑ^S

Like the paid-up option, ϑ^S increases with increasing σ^I. The usage ratio is almost 100% when $\sigma^I = 0$ and around 70% with larger σ^I when following LSMC. Additionally, surrender options are generally more valuable than paid-up options. If, e.g., $\sigma^I = 2\%$, the V/P ratio reaches 8.78% following the LSMC strategy with the upper limit nearly 13.35% (for $\theta = 4\%$).

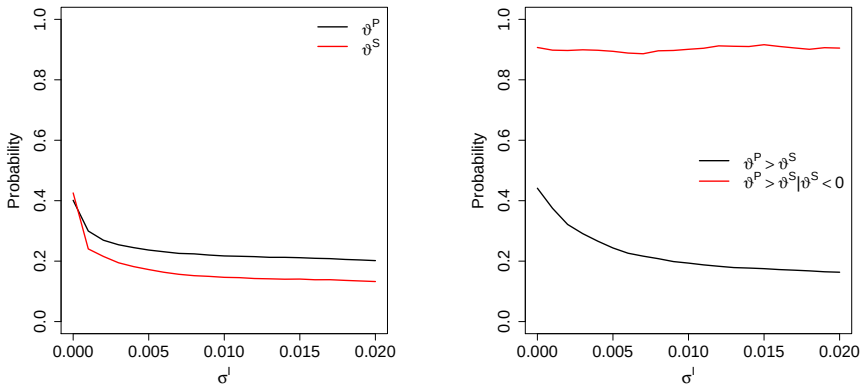


(a) Option value ϑ^S (b) V/P ratio (c) Usage ratio

Figure 3: Surrender option results for different σ^I

As the option value is not determined until the contract matures, it is not possible to perfectly predict if the option is OTM or ITM when it is exercised. Figure 4(a) shows for the two single options (paid-up and surrender), the probability that, when following LSMC, these options are exercised at OTM. This portion is much higher when $\sigma^I = 0\%$. As shown in Figure 2(c) and 3(c), LSMC tends to overuse these single options when $\sigma^I = 0$, resulting in the little value compared to their upper limit.

The surrender option triggers higher cash flow variance. Thus $\vartheta^S > \vartheta^P$ when this put option is ITM. Otherwise, it is more likely that $\vartheta^P > \vartheta^S$ when a put option is OTM. Figure 4(b) shows the portion when $\vartheta^P > \vartheta^S$. For the paths that $\vartheta^S < 0$, ϑ^P is generally higher than ϑ^S (even though the ϑ^P is also likely to be negative in such paths.)



(a) Probability that an option exercised at OTM.

(b) Probability that $\vartheta^P > \vartheta^S$

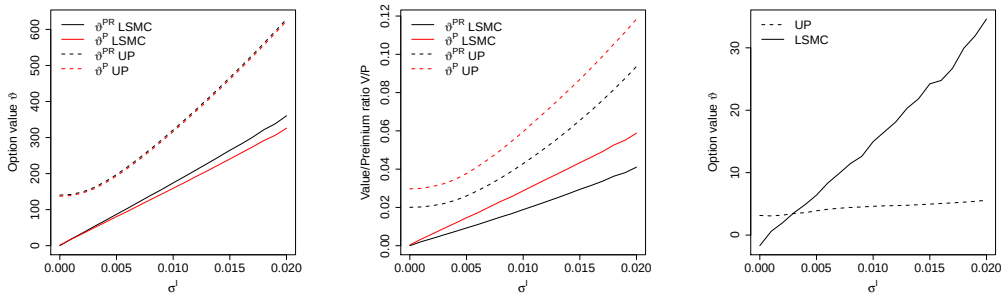
Figure 4: Comparison between single paid-up ϑ^P and single surrender option ϑ^S

4.2.2 Combined Premium Payment Option Value- Two Option Case

Combined Paid-up and Resumption Option ϑ^{PR}

Figure 5(a) and 5(b) compare the combined option ϑ^{PR} with single paid-up option ϑ^P for $\theta = 4\%$. The option value difference is very small especially for small σ^I. Thus, the remaining resumption value, ϑ'^R, as the difference between these two options shown in Figure 5(c) has negligible value or even negative especially when $\sigma^I = 0$. The upper limits for ϑ^P and ϑ^{PR} are almost the same. If it is possible to see the future, the chance is rare that both put and call options have positive value ever in ten years (the contract period).

Moreover, premium payments resume with the exercise of resumption option. PV of these resumed payments is higher than the resumption option value. Thus, the V/P ratios for both upper limit and LSMC decreases when adding additional resumption option. With LSMC strategy, this V/P ratios difference is small for $\sigma^I = 0$ but enlarges when σ^I increases.

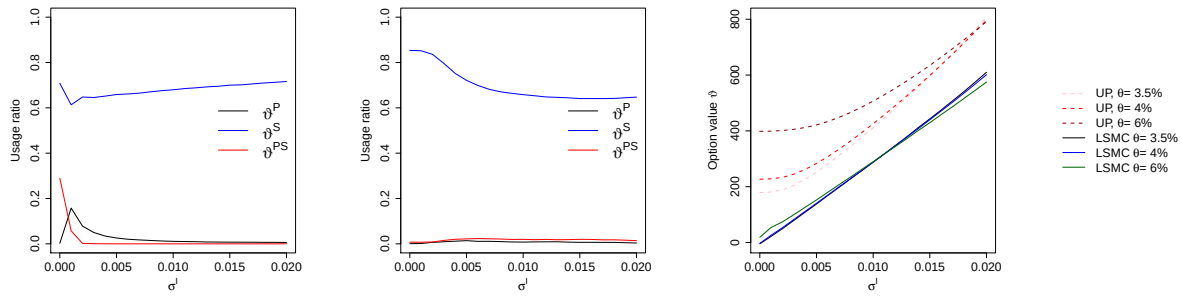


(a) Combined option value ϑ^{PR} (b) V/P ratio (c) Resumption option value ϑ^{IR}

Figure 5: Combined paid-up and resumption option ϑ^{PR} results for different σ^I

Combined Paid-up and Surrender Option ϑ^{PS}

ϑ^{PS} in Figure 6(c) shows almost the same structure as ϑ^{S} in Figure 3(a). Figure 6(a) displays the option usage with LSMC strategy. Paid-up and surrender options are both put options and the surrender option usually values more than the paid-up option especially when σ^I is large. Hence, the optimal strategy is to exercise the surrender option only with large σ^I. It is better to exercise some paid-up option when σ^I is small as the chance that a put option is exercised at OTM is higher. For the upper limit perspective in Figure 6(b), this combined option in more than 60% of paths is positive. While in these paths, the single surrender option values the most.



(a) Usage ratios for LSMC method ($\theta = 4\%$) (b) Usage ratio for upper limit ($\theta = 4\%$) (c) Combined option value

Figure 6: Combined option comparison for different σ^I

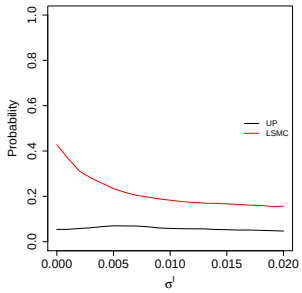
4.2.3 Combined Premium Payment Option Value- Three Option Case

Combined Paid-up, Resumption and Surrender Option ϑ^{PRS}

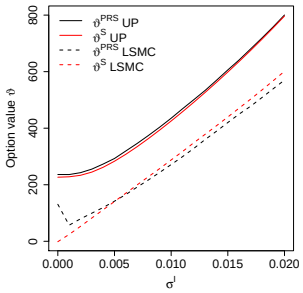
Previous combined options show that the extra option adds little value to the existing single option. More specifically, the remaining resumption option does not improve the value to the paid-up option significantly. With the combined paid-up and surrender option, it is optimal to only exercise the surrender option in most of cases. In addition, the surrender option is the most valuable option among all these premium payment options. When combing three premium payment options, the following shows if the single surrender option still dominates other options.

Figure 7(a) compares the combined option ϑ^{PR} and the single surrender option ϑ^{S}. As ϑ^{PR} generates little extra value compared with ϑ^{P}, while ϑ^{S} is much more valuable than ϑ^{P}, the probability is close to zero that $\vartheta^{PR} > \vartheta^{S}$ regardless σ^{I} under the upper limit viewpoint. Following the LSMC method, $\vartheta^{PR} > \vartheta^{S}$ for at least 20% of the simulation paths. When σ^{I} is small, this portion reaches even higher to more than 40%. This high proportion for small σ^{I} can be considered as the higher time value for the remaining resumption option. More specifically, for small σ^{I}, it is likely to exercise either the paid-up or the surrender option at OTM (cf. Figure ,4(a)). With the resumption option, it is possible to adjust this mistake in the coming years before the contract matures. Figure 7(d) demonstrates that the optimal strategy is to use at least two of the combined option when σ^{I} is small. From Figure 7(b), though the upper limits for the combined option and the single surrender option are almost the same, with small σ^{I}, ϑ^{PRS} is much larger than ϑ^{S} following LSMC. Contrary to the conclusion in Schmeiser and Wagner (2011) that the combined option values little, combing three options, ϑ^{PRS} value enlarges significantly when $\sigma^{I} = 0$.

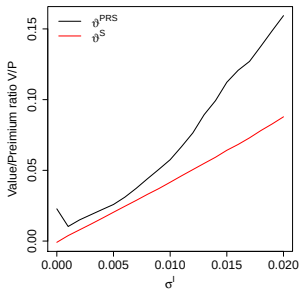
However, when σ^{I} is large, LSMC overestimates the resumption option. Additionally, the time value for the resumption option increases with an early exercise point for the paid-up option. Hence, the paid-up option will be exercised more often and earlier than its optimal exercise point if this option is treated as a single option. Nevertheless, this overestimate and overuse of ϑ^{PR} has relative little impact on the combined option value. When σ^{I} is high, it is optimal to use the single surrender option for the 60% paths (cf. Figure 7(d)). On the other hand, the early usage of the paid up option decreases the premium payment. Therefore, V/P ratio is much higher for the combined option especially when σ^{I} is large (cf. Figure 7(c)).



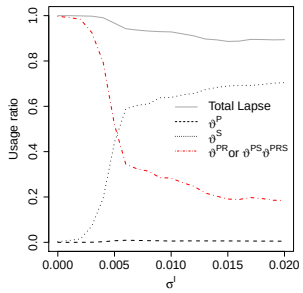
(a) Probability that $\vartheta^{PR} > \vartheta^{S}$



(b) Comparison between ϑ^{PRS} and ϑ^{S}



(c) V/P ratio for ϑ^{PRS} and ϑ^{S}



(d) Usage for ϑ^{PRS}

Figure 7: Combined option comparison for different σ^I ($\theta = 4\%$)

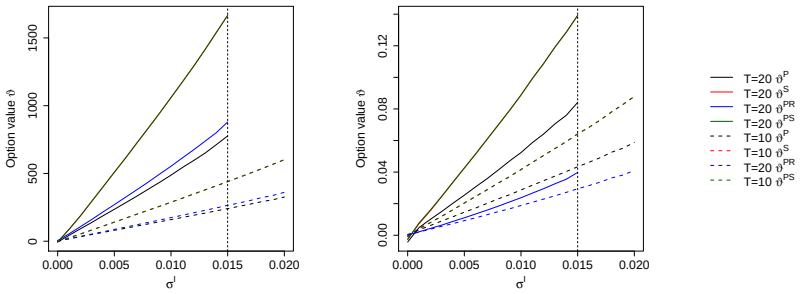
4.3 Sensitivity of Premium Payment Option

This section illustrates the influence of the impact of contract duration, T. Moreover, for the current low-interest rate environment, we also run the model with different parameters controlling interest rates. The results show that as long as there is α such that $\Pi^{\emptyset} = 0$, i.e. it is possible to offer a fair contract, the relation between σ^I and the option value, and the relation among different options' values remain unchanged.

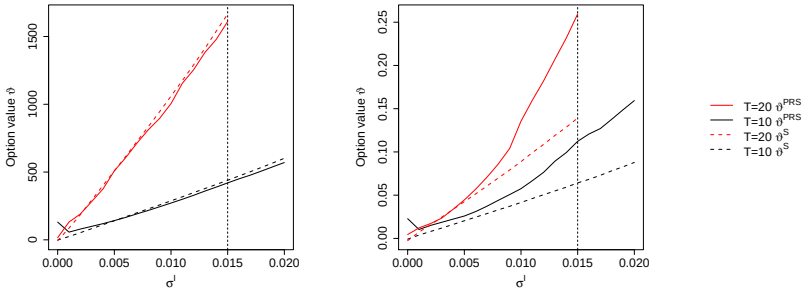
4.3.1 Influence of Contract Duration (T)

Figure 8 shows the option value and V/P ratio of all option combination discussed for $T = 10$ and $T = 20$. For $T = 20$, no data is available for $\sigma^I \geq 1.5\%$ as there exists no α with $0 \leq \alpha \leq 1$ that satisfies the fair contract condition (cf. Figure 1(b)). The option value and V/P ratios increase dramatically when expanding the contract duration to $T = 20$. Figure 8(a) and 8(b) shows no results of

the surrender option as these results overlap with the results of the combined option of paid-up and surrender. From Figure 8(d), as concluded in the case of $T = 10$, while ϑ^S and ϑ^{PRS} have similar value, ϑ^{PRS} has much higher V/P ratio compared that of ϑ^S



(a) Option value for ϑ^P, ϑ^S, ϑ^{PR}, ϑ^{PS} (b) V/P ratio for ϑ^P, ϑ^S, ϑ^{PR}, ϑ^{PS}



(c) Option value for ϑ^S and ϑ^{PRS} (d) V/P ratio for ϑ^S and ϑ^{PRS}

Figure 8: Option value and V/P ratio comparison between T=10 and T=20 for $\theta = 4\%$

4.3.2 Low-Interest Rate case

Adjusting to the current low-interest rate environment, we reset our variables with guarantee ratio, $g = 0\%$, initial interest rate and long term interest rate $r_0 = \theta = 0.5\%$ as described in Table 3a.

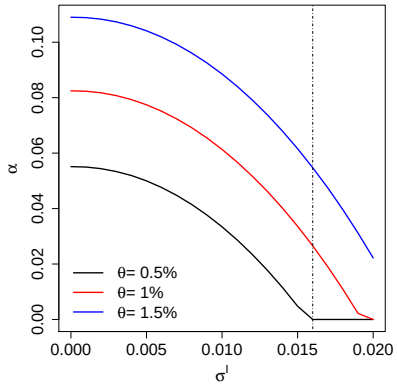
Table 3b shows the participation ratio, α with interest rate $\theta = 1.5\%$, 1% and 0.5%. For $\theta = 0.5\%$ and $\sigma^I \geq 1.6\%$, there exists no α (with $0 \leq \alpha \leq 1$) for the fair contract condition.

Figure 9 presents the option value and V/P ratio for the all option combination. The combined option of all three options has highest V/P ratio, reaching almost 12%. However, the surrender

option (and the combined option of surrender and paid-up as the extra paid-up option has little value) has the highest option value.

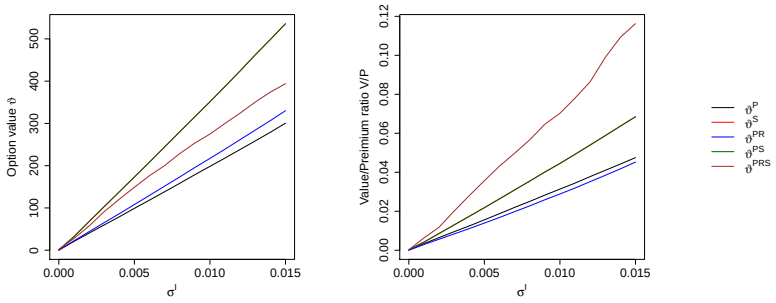
B	constant premium	1200
γ	death benefit	11944.96
g	guarantee rate	0%
r_0	initial interest rate	0.5%
θ	long term interest mean	0.5%/1%/1.5%

a Parameter table for the base case



b α and σ^I for different θ

Table 3: Parameter table and the participation ratio α



(a) Option value for ϑ^P, ϑ^S, ϑ^{PR}, ϑ^{PS} with $\theta = 0.5\%$

(b) V/P ratio for ϑ^P, ϑ^S, ϑ^{PR}, ϑ^{PS} with $\theta = 0.5\%$

Figure 9: Option value and V/P ratio comparison under the low interest rate environment

5 Economic Interpretation and Outlook

The numerical results show that, when stochastic interest rates are taken into account, the fair values of premium payment options can be substantial. When interest rate is deterministic, additional resumption option also increases the combined option value of paid-up and surrender. Furthermore, insurance companies face - in addition to pure random risk - extensive model and parameter risk

in respect to the investigated options. For example, Swishchuk (2004) suggests the risk be even underestimated as volatility is assumed constant. Considering these factors, insurers may need to charge higher premiums than those proposed by the fair pricing concept shown in this paper.

Practitioners may argue that policyholders typically do not exercise premium payment options in a rational way (i.e., in the sense laid down in Chapter 3) and thus lower option prices based on observed exercise behavior could be sufficient. However, in such a case, insurance companies face some additional risk - policyholders could be advised about optimal exercise procedure and hence change their future behavior.

In most cases, insurance companies are not free to choose whether to offer premium payment options or not. For instance, a life insurance contract must have a surrender option by law in all insurance markets we are aware of. Under the assumptions taken in this paper, insurers must charge - in addition to the savings premium and premium for the term life part - substantial premiums for payment options to finance adequate risk management measures. According to our numerical results, with the existence of surrender option, an additional paid-up option can be basically offered for free. However, the combined option of paid-up, resumption, and surrender may be more expensive than the single surrender option. This additional charge reduces the attractiveness and hence the demand for life insurance contracts, given a competitive market with alternative products in the field of old-age provision, such as fixed-income mutual funds in Koijen et al. (2009). In addition, some policyholders may be convinced that they will never use any of the premium payment options, resulting in no willingness to pay for these contract features.

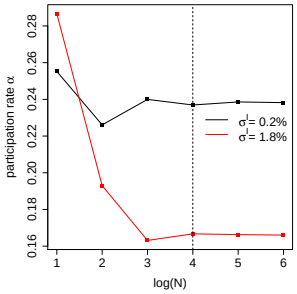
In practice, insurance companies charge policyholders a fee whenever an option is exercised to discourage policyholders from exercising these options. The advantage here is that only those policyholders who exercise a premium payment option would need to pay. However, regulatory bodies are currently attempting to set minimum levels for surrender values in Europe to thwart such an approach. It could negatively influence the financial stability of life insurance companies if a large proportion of policyholders surrender their contracts at the same time (insurance run scenario). Moreover, though the premiums could be charged lower, with a 5% penalty, the surrender option value is still almost 6.9% of the PV of premium when policyholders following LSMC (see Figure A3(a) in Appendices). Insurers may also introduce a lockup period, during which policyholders are not allowed to exercise the premium options. With initial three year as lockup period, surrender value decreases but is still about 7.7% of the premium present value (see Figure A3(b) in Appendices).

Another way to tackle this problem from the insurer's point of view is not to base adjustment of benefits once an option is exercised on an ex ante fixed actuarial framework. For example, for the surrender option, the surrender amount becomes the market values under any condition. The insurer would face no risk from premium payment options and need not charge any additional premium (because the option can never have a positive value). On the other hand, the insurer would then be unable to promise policyholders a fixed payback at certain points in time whenever a premium payment option is exercised.

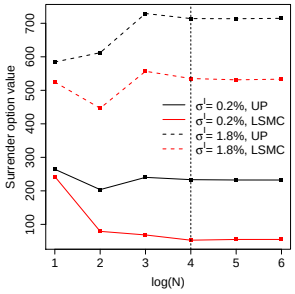
Appendices

A Monte Carlo Convergence

Figure A1 shows the speed of the convergence rate. We ran a Monte Carlo simulation for different N (from 10^1 to 10^6) with $\theta = 4\%$, $\sigma^I = 0.2\%$, and $\sigma^I = 1.8\%$. Both option values and the participation rate, α, stabilize when the simulation number of paths N reaches 10^4.



(a) Participation rate α with different N



(b) Surrender value ϑ^S with different N

N	$\sigma^I = 0.2\%$	$\sigma^I = 1.8\%$
10	0.25523	0.28663
100	0.22609	0.19299
1000	0.23998	0.16312
10000	0.23693	0.16673
100000	0.23859	0.16631
1000000	0.23822	0.16604

(c) Participation rate α with different values of σ^I and N

Figure A1: Convergence speed of participation rate, α and ϑ^S as simulation number of paths N increases

B Least-squares Monte Carlo Method (LSMC)

Figure A2 compares the LSMC strategy discussed in this paper and named "adjusted LSMC" to the Longstaff and Schwartz (2001) method. For our numerical example, we demonstrate that the option value determined with the adjusted LSMC is slightly higher.

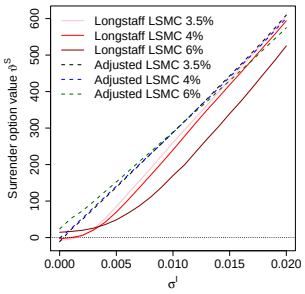


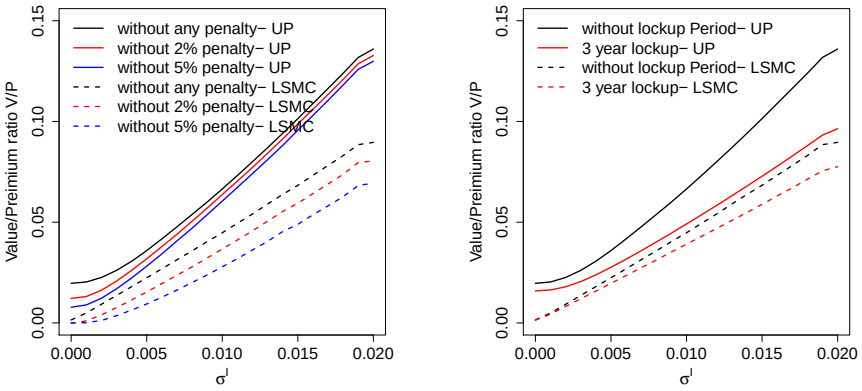
Figure A2: Comparison of surrender option value using different LSMC methods

To check the stability of LSMC approximation, we ran the first simulation and derived $\hat{\alpha}$ and $\hat{\alpha}'$ from equation (14) and (16). We then generated new simulation paths and determined their optimal exercise points using the derived $\hat{\alpha}$ and $\hat{\alpha}'$. The original result (the first simulation) and the second result (the new simulation) as out of sample (OoS) are compared in Table 4. We found no substantial differences, especially as σ^I increases.

		Paid-up Option			Surrender Option		
	σ^I	First	OoS	Diff	First	OoS	Diff
1	0.00%	1.65	0.74	75.50%	-2.18	-1.43	-41.33%
2	0.10%	17.73	17.12	3.51%	25.36	24.35	4.06%
3	0.20%	33.22	32.72	1.52%	54.28	52.80	2.76%
4	0.30%	49.22	47.83	2.86%	83.59	82.39	1.45%
5	0.40%	64.68	62.97	2.68%	112.60	111.44	1.04%
6	0.50%	80.60	78.56	2.57%	141.58	140.67	0.64%
7	0.60%	95.67	93.15	2.68%	170.79	170.20	0.35%
8	0.70%	111.83	109.29	2.30%	199.59	199.55	0.02%
9	0.80%	127.78	125.39	1.89%	229.09	228.70	0.17%
10	0.90%	143.37	141.18	1.54%	258.79	257.52	0.49%
11	1.00%	159.20	156.66	1.61%	287.76	287.70	0.02%
12	1.10%	175.07	172.22	1.64%	318.23	317.43	0.25%
13	1.20%	191.53	187.79	1.97%	348.17	348.55	0.11%
14	1.30%	207.31	204.13	1.54%	378.78	378.43	0.09%
15	1.40%	223.97	220.84	1.41%	409.23	408.73	0.12%
16	1.50%	240.46	236.73	1.57%	439.66	439.65	0.00%
17	1.60%	257.27	254.08	1.25%	471.43	471.09	0.07%
18	1.70%	273.95	270.29	1.34%	503.56	502.38	0.23%
19	1.80%	291.64	288.15	1.21%	535.32	534.78	0.10%
20	1.90%	306.16	305.61	0.18%	566.82	567.74	0.16%
21	2.00%	325.83	322.42	1.05%	600.45	601.34	0.15%

Table 4: Out of sample check for paid-up option and surrender option values

C Surrender option with Fee Structure and Lockup Period



(a) Result for different fee structures (b) Result for different lockup periods

Figure A3: V/P for surrender option with fee structure or lockup period with $\theta = 1\%$

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