SECOND-ORDER UNCERTAINTY AND NAIVE DIVERSIFICATION

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Abstract. This paper introduces a new model for decision making under ambiguity called second-order uncertainty. The decision maker considers all probability distributions associated with the given options, and chooses the allocation that minimizes the variability of outcome across all probabilities. Optimal allocation under second-order uncertainty is the equal weighted allocation, also called naive diversification.

Keywords: second-order uncertainty, ambiguity, naive diversification, principle of indifference
JEL Classification: D01, D80, D81

1. Introduction

The standard paradigm of choice requires individuals to state their preferences between alternatives characterized by well-specified probabilities and outcomes. However, decision making problems when one has no real notion of probabilities and outcomes are more common and arguably more realistic. Although there are several frameworks addressing choice under complete ignorance (Arrow and Hurwicz 1972, Cohen and Jaffray 1980, Bossert, Pattanik, and Xu 2000), they are complex for the most naive decision maker. In the context of portfolio construction, behavioral research suggests that the task may be too complex for decision makers to perform (Barberis and Thaler 2003). Consequently, financial decision makers adapt various types of simplified rules of thumb in practice. A prominent example is *naive diversification*, which stipulates that the agent allocates equal decision weights to given choice alternatives. This diversification paradigm is closely related to the *principle of indifference*, which states that in the absence of any relevant evidence, agents distribute their "degrees of belief" equally among the possible outcomes.

This paper provides a theoretical framework for decision making under complete uncertainty that is consistent with naive diversification, underpinned by a new theoretical concept and focused on decision making in the context of portfolio allocation. The proposed model is, on one hand, independent of beliefs and of tastes for risk, and on the other hand quantifies uncertainty without

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Date: March 21, 2022.

ranking or evaluation of possible outcomes, using the entire range of possible probability distributions. The crux of the model is the notion of *second-order uncertainty*, whose underlying idea is that a naive decision maker who is averse to complete uncertainty seeks to be as close as possible to certainty by minimizing the variability of the expected outcome, stemming from the fact that probabilities are unknown. This second-order uncertainty is in contrast to stochastic variability, which arises as a consequence of known probability assignments. For each possible decision, there is a number of outcomes depending on the probability assignment. The naive decision maker who is second-order uncertainty averse summarizes these evaluations of all possible probability distributions by choosing the allocation that leads to the smallest second-order uncertainty.

The model of decision making under second-order uncertainty is then shown to result in *naive diversification*, which implies a preference of equality over inequality in the choice weights. Although widely applied in practice, it violates standard economic theory, which postulates that an investor should maximize portfolio return while minimizing portfolio risk (Markowitz 1952). Experimental work in the decades after the emergence of Markowitz's theory has shown that economic agents in reality systematically violate the traditional diversification assumption when choosing among risky gambles. For example, the experimental evidence of Benartzi and Thaler (2001) suggests that some people spread their contributions evenly across the investment options irrespective of the particular mix of options. Similarly, Huberman and Jiang (2006), Baltussen and Post (2011), and Fernandes (2013) find strong evidence for the naive diversification heuristic. It is documented that even Harry Markowitz used the simple $1/n$ heuristic when he made his own retirement investments. He justifies his choice on psychological grounds: "My intention was to minimize my future regret. So I split my contributions fifty-fifty between bonds and equities" (Gigerenzer 2010). Naive diversification is thus largely considered a simple and practical heuristic without the economic foundation guaranteeing its optimality. This paper thereby provides an intuitive decision-theoretic setting under which naive diversification optimizes the utility of the second-order uncertainty averse decision maker.

2. Theoretical background

2.1. **The choice setup.** Let Ω be the set of all possible states of the world. I shall assume that Ω itself is known, but that the probabilities of the various states occurring are completely unknown. The decision maker is assumed to form an allocation from a finite subset of the countable set $\mathcal{X}$ of random outcomes $X_1, \ldots, X_n$. For $i = 1, \ldots, n$, each $X_i : \Omega \to \mathbb{R}$ is a real-valued random variable on the sample space Ω. The expected payoff for a given random variable X is denoted by $\mathbb{E}_{\mathbb{P}}[X]$,

for some unknown probability distribution $\mathbb{P}$. The decision maker is assumed to be able to form compound choices $\alpha X_1 + (1-\alpha)X_2$ for $X_1, X_2 \in \mathcal{X}$ and $\alpha \in [0,1]$, whose random outcomes are defined by $\alpha\, X_1(\omega) + (1-\alpha)\, X_2(\omega)$ for $\mathbb{P}$-almost all $\omega \in \Omega$.

Denote by I a subset of the unit interval $[0,1]$ necessarily including the limits 0, 1, and $1/n$, and by $\mathcal{D}$ the set of all vectors $\Delta = (\delta_1, \ldots, \delta_n)$ satisfying $\delta_i \in I$ and $\sum_i \delta_i = 1$, for $i = 1, \ldots, n$. Then $\Delta = (\delta_1, \ldots, \delta_n) \in \mathcal{D}$ represents the allocation that a decision maker assigns to alternatives $X_1, \ldots, X_n$. For $\Delta = (\delta_1, \ldots, \delta_n) \in \mathcal{D}$, a decision maker devoting a fraction δ_i to the alternative X_i would obtain the random outcome represented by the random variable $X_\Delta = \sum_{i=1}^n \delta_i X_i$. The decision problem in this paper is that of choosing an allocation vector in $\mathcal{D}$. Note that both the possibilities of allocation vectors as well as the possibility of forming compound choices in the form of convex combinations is meant to emulate the portfolio allocation setup.

A weak preference relation on $\mathcal{D}$ is a binary relation $\succsim$ satisfying:

AXIOM 1 (Completeness): For all $\Delta, \Delta' \in \mathcal{D}$, $\Delta \succsim \Delta' \vee \Delta' \succsim \Delta$.

AXIOM 2 (Transitivity): For all $\Delta, \Delta', \Delta^* \in \mathcal{D}$, $\Delta \succsim \Delta' \wedge \Delta' \succsim \Delta^* \Rightarrow \Delta \succsim \Delta^*$.

Every weak preference relation $\succsim$ on $\mathcal{D}$ induces an indifference relation $\sim$ on $\mathcal{D}$ defined by $\Delta \sim \Delta' \Leftrightarrow (\Delta \succsim \Delta') \wedge (\Delta' \succsim \Delta)$. The corresponding strict preference relation $\succ$ on $\mathcal{D}$ is defined by $\Delta \succ \Delta' \Leftrightarrow \Delta \succsim \Delta' \wedge \neg(\Delta \sim \Delta')$. A numerical or utility representation of the preference relation $\succsim$ is a real-valued function $u\colon \mathcal{D} \to \mathbb{R}$ for which $\Delta \succsim \Delta'$ if and only if $u(\Delta) \geq u(\Delta')$.

2.2. **Majorization as a measure of variability.** The theory of majorization from linear algebra is adopted to measure equality and variability throughout this paper. Majorization, which was formally introduced by Hardy, Littlewood, and Pólya (1934), captures the idea that the components of a weight vector $\boldsymbol{\alpha} \in \mathbb{R}^n$ are less spread out or more nearly equal than the components of a vector $\boldsymbol{\beta} \in \mathbb{R}^n$. For any $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, let $\alpha_{(1)} \geq \cdots \geq \alpha_{(n)}$ denote the components of $\boldsymbol{\alpha}$ in *decreasing* order, and let $\boldsymbol{\alpha}_\downarrow = (\alpha_{(1)}, \ldots, \alpha_{(n)})$ denote the decreasing rearrangement of $\boldsymbol{\alpha}$. Attention is restricted to non-negative weights.

Definition 2.1 (Majorization)**.** For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$, $\boldsymbol{\beta}$ is said to *(weakly) majorize* $\boldsymbol{\alpha}$ (or, equivalently, $\boldsymbol{\alpha}$ is *majorized by* $\boldsymbol{\beta}$), denoted by $\boldsymbol{\beta} \geq_m \boldsymbol{\alpha}$, if

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i$$

and for all $k = 1, \ldots, n-1$,

$$\sum_{i=1}^{k} \alpha_{(i)} \leq \sum_{i=1}^{k} \beta_{(i)} \ .$$

Majorization is a preorder on the vectors in $\mathbb{R}^n$ and a partial order on $\mathbb{R}^n_{\downarrow}$, which contains all vectors whose components are ordered in decreasing order. It is trivial but important to note that all vectors majorize the uniform vector $\boldsymbol{u}_n = (\frac{1}{n}, \ldots, \frac{1}{n})$, since the uniform vector is the vector with minimal differences between its components.

It is well known that so-called *Schur-convex* and *Schur-concave* functions preserve the vector ordering induced by majorization (Schur 1923, Ostrowski 1952). A natural interpretation of majorization is that if a vector $\boldsymbol{\alpha}$ majorizes another $\boldsymbol{\beta}$ then $\boldsymbol{\alpha}$ is less spread out than $\boldsymbol{\beta}$. So it is natural to ask if statistical measures of variability are Schur-convex. The variance and standard deviation are Schur-convex functions, while the median absolute deviation is not. A typical example of a Schur-convex function in economics is the Gini-coefficient, which measures the inequality among values of a frequency distribution, for example levels of income or wealth.

3. A model of second-order uncertainty

Suppose that the joint probability distribution of the random variables X_i (for $i = 1, \ldots, n$) is unknown to the decision maker. Denote by $\mathcal{P}$ a given finite set of probability measures $\mathbb{P} : \mathcal{F} \to [0, 1]$ defined on top of the sample space Ω, where $\mathcal{F} \subseteq 2^{\Omega}$ is the σ-algebra of events. Towards the end of introducing the paper's notion of naive regret under complete uncertainty, the following two key assumptions will be made. Both assumptions emulate the diversification setting and thus reflect the decision of portfolio allocation to a selection of assets.

First, the decision to be made is solely that of choosing an allocation $\Delta \in \mathcal{D}$ to a given finite collection of random alternatives $X_1, \ldots, X_n \in \mathcal{X}$. Faced with a finite number of assets, the economic agent decides on the weights allocated to each asset. If the joint probability distribution associated with the random variables $X_1, \ldots, X_n$ was known, the choice problem reduces to one under risk, and utility could, for instance, be given by pure expectation or expectation discounted by known stochastic variation. In the absence of this knowledge, however, an objective quantification should account for all possible probability measures.

The second key assumption is that the decision maker is concerned with relative rather than absolute payoff. Formally, the *relative payoff* of the random variable X_Δ can be either represented

as the ratio random variable defined by

$$R_\Delta = \frac{X_\Delta}{\sum_{\Delta' \in \mathcal{D}} X_{\Delta'}},$$

or as the difference

$$R^d_\Delta = X_\Delta - \sum_{\Delta' \in \mathcal{D}} X_{\Delta'},$$

where $X_\Delta = \sum_{i=1}^{n} \delta_i X_i$ is the compound random variable based on allocation $\Delta = (\delta_1, \ldots, \delta_n) \in \mathcal{D}$. For a given allocation Δ, R_Δ represents the ratio of payoff X_Δ with respect to the sum of all other payoffs $X_{\Delta'}$ under all other possible allocations $\Delta' \in \mathcal{D}$, and R^d_Δ the difference between payoff X_Δ and the the sum of all other payoffs $X_{\Delta'}$. Relative payoffs thus compare the payoff of an allocation Δ to the payoff of all other possible allocations $\Delta' \in \mathcal{D}$. Note that the expected value of the quotient R_Δ is not identical to the quotient of the corresponding expected values, and there is no general statistical formula for evaluating the ratio of two random variables.

The intuition underlying the focus of the decision maker on relative rather than absolute return rests on the idea that decision makers in financial investing settings measure the return of an investment portfolio relative to a theoretical passive reference portfolio or benchmark. The simplest such benchmark would be one representing the market overall. When the relative payoff is positive, the portfolio is said to outperform the benchmark. Conversely, when the relative payoff is negative, the portfolio is said to underperform the benchmark. In the context of the allocation of financial assets, decision makers are largely concerned with relative rather than absolute return, since one may always passively invest in the market benchmark.

Definition 3.1 (Second-order uncertainty)**.** Given an allocation $\Delta \in \mathcal{D}$ with corresponding relative random payoff R_Δ and $\mathcal{P}$ the set of all possible probability measures, the *second-order uncertainty* $\Phi(R_\Delta)$ associated with R_Δ is defined by

$$\Phi(R_\Delta) = d\left(\langle \mathbb{E}_\mathbb{P}[R_\Delta] \rangle_{\mathbb{P} \in \mathcal{P}}\right),$$

where d is a Schur-convex function.

Here, for a given allocation Δ, $\langle \mathbb{E}_\mathbb{P}[R_\Delta] \rangle_{\mathbb{P} \in \mathcal{P}}$ is the vector of expected relative payoffs under each possible probability assignment $\mathbb{P} \in \mathcal{P}$. The inequality or dispersion d in these outcomes stemming from the fact that probabilities are unknown is essentially second-order uncertainty. This is in contrast to (first-order) stochastic variability, which arises as a consequence of known probabilistic

assignments. Note that second-order uncertainty can also be defined in terms of the relative payoff R^d_Δ. For simplicity, the remainder of this paper focuses on uncertainty over the payoff R_Δ.

Definition 3.2 (Aversion to second-order uncertainty)**.** Preferences $\succeq$ on $\mathcal{D}$ are said to exhibit *aversion to second-order uncertainty* if for all $\mathbb{P} \in \mathcal{P}$ and $\Delta, \Delta' \in \mathcal{D}$,

$$\Delta \succeq \Delta' \iff \langle \mathbb{E}_\mathbb{P}[R_{\Delta'}] \rangle_{P \in \mathbb{P}} \geq_m \langle \mathbb{E}_\mathbb{P}[R_\Delta] \rangle_{P \in \mathbb{P}} \ .$$

Similarly, preferences $\succeq$ on $\mathcal{D}$ are *second-order uncertainty seeking* if for all $\mathbb{P} \in \mathcal{P}$ and $\Delta, \Delta' \in \mathcal{D}$,

$$\Delta \succeq \Delta' \iff \langle \mathbb{E}_\mathbb{P}[R_\Delta] \rangle_{P \in \mathbb{P}} \geq_m \langle \mathbb{E}_\mathbb{P}[R_{\Delta'}] \rangle_{P \in \mathbb{P}} \ .$$

For each allocation alternative Δ, the decision maker thus considers all possible probabilities and relative expectations. (S)he then summarizes all these evaluations by choosing the allocation that leads to the smallest second-order uncertainty. Formally, second-order uncertainty is exactly captured via majorization of the vectors of expected relative payoffs, and thus by the dispersion or inequality of expected relative payoffs. Intuitively, second-order uncertainty aversion leads to certainty, since the (first-order) uncertainty from not knowing probabilities is reduced by knowing that the expected outcomes are all close to one another.

Note that this implies that preferences exhibit second-order uncertainty aversion if and only if they are represented by the following utility function:

$$U(\Delta) = d\left(\langle \mathbb{E}_\mathbb{P}[R_\Delta] \rangle_{\mathbb{P} \in \mathcal{P}}\right),$$

where d is a Schur-concave function. The proof follows immediately from the definition and functional representation of majorization.

4. Naive diversification

It is next shown that the equal weighted allocation arises as the utility maximizing alternative for the second-order uncertainty averse decision maker. This "naive" choice has traditionally been viewed as an anomaly or a rule of thumb. Viewed through the lens of the model of second-order uncertainty, the following Theorem can be interpreted as a formal rationalization of applying the naive diversification heuristic when probabilities are completely unknown.

Theorem 4.1. *The maximum of the utility function*

$$U(\Delta) = d\left(\langle \mathbb{E}_\mathbb{P}[R_\Delta] \rangle_{\mathbb{P} \in \mathcal{P}}\right),$$

which represents aversion to second-order uncertainty, is attained at the equal allocation $\Delta_n = (\frac{1}{n}, \ldots, \frac{1}{n})$*, and the optimal value of the utility function is* $U(\Delta_n) = 0$*.*

Proof. I will show that the expected relative payoffs R_{Δ_n} of the equal allocation Δ_n coincide for any two probability measures $\mathbb{P}, \mathbb{Q} \in \mathcal{P}$, that is $\mathbb{E}_{\mathbb{P}}[R_{\Delta_n}] = \mathbb{E}_{\mathbb{Q}}[R_{\Delta_n}]$ and are in fact deterministic, which makes any measure of dispersion and thereby the value of the objective function equal to zero. The random variable representing the total payoff across all allocations is

$$Z := \sum_{\Delta \in \mathcal{D}} X_\Delta = \sum_{\Delta \in \mathcal{D}} \sum_{i=1}^{n} \delta_i X_i,$$

where $\Delta = (\delta_1, \ldots, \delta_n)$, and the random variable representing relative payoff of the equal allocation Δ_n is

$$R_{\Delta_n} = \frac{X_{\Delta_n}}{Z} = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i}{\sum_{\Delta \in \mathcal{D}} \sum_{i=1}^{n} \delta_i X_i}.$$

The denominator above can be rewritten as

$$\sum_{\Delta \in \mathcal{D}} \sum_{i=1}^{n} \delta_i X_i = C \sum_{i=1}^{n} X_i$$

for a fixed real constant $C \in \mathbb{R}$. To see this, note that factoring out the X_i term for each i in the sum, one ends up with a long sum of repetitive combinations of all possible δ_j. Denote each of these sums by C_i, that is

$$\sum_{\Delta \in \mathcal{D}} \sum_{i=1}^{n} \delta_i X_i = \sum_{i=1}^{n} C_i X_i.$$

The values C_i are sums of permutations of the same terms, making them all equal. This proves that

$$R_{\Delta_n} = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i}{C \sum_{i=1}^{n} X_i},$$

implying that for any given probability distribution $\mathbb{P} \in \mathcal{P}$, $R_{\Delta_n} = 1/(nC)$, where $C \in \mathbb{R}$. This is a determinisitc value, hence resulting in zero variability in expectation under all probability distributions. □

Theorem 4.1 states that under the assumption of aversion to second-order uncertainty, an equal weighted allocation to all available choice alternatives will yield zero variability in expected frequency in the output. This result also follows intuitively from the model's restrictions: given the restriction of weighting schemes to the unit simplex and the lack of restrictions on the probability measures, the loss minimization imposes symmetry with respect to the different distributions.

With convexity of the objective and symmetry, equal weights arise as the optimal solution. In fact, as the next corollary states, no other allocation yields a deterministic payoff.

Corollary 4.2. *The equal-weighted allocation Δ_n is the unique allocation such that the relative payoff R_{Δ_n} is deterministic.*

Proof. To prove the uniqueness of Δ_n, suppose that there is some other allocation $\Delta^* = (\delta_1^*, \ldots, \delta_n^*) \in \mathcal{D}$ satisfying $R_{\Delta^*} = K$ for some $K \in \mathbb{R}$. If $R_{\Delta^*} = K$, then $\sum_{i=1}^n \delta_i^* X_i = KC \sum_{i=1}^n X_i$, which implies $KC \sum_{i=1}^n X_i - \sum_{i=1}^n \delta_i^* X_i = \sum_{i=1}^n (KC - \delta_i^*) X_i = 0$. Therefore, one must have $KC - \delta_i^* = 0$ for all $i = 1, \ldots, n$, or equivalently $\delta_i^* = KC$. This means that all components δ_i^* of the allocation Δ^* are equal. The only element of the set of all allocations $\mathcal{D}$ satisfying this property is Δ_n. □

Note that, intuitively, this simple Corollary supports the notion that aversion to second-order uncertainty leads to attaining a degree of certainty.

More strongly even, an allocation that is in some sense "closer" to the equal weighted allocation has higher utility. Distance between two allocation vectors in $\mathcal{D}$ is again captured via functions representing the majorization preorder on $\mathcal{D}$.

Proposition 4.3. *Let $f : \mathcal{D} \to \mathbb{R}$ be a real-valued distance function on the set of allocation vectors in $\mathcal{D}$ representing the majorization preorder. For $\Delta, \Delta' \in \mathcal{D}$ with $f(\Delta) < f(\Delta')$, then $U(\Delta) > U(\Delta')$. In words: an allocation closer to Δ_n has lower second-order uncertainty and therefore higher utility.*

Proof. Let $\Delta, \Delta' \in \mathcal{D}$ with $f(\Delta) < f(\Delta')$. For any probability measure $\mathbb{P} \in \mathcal{P}$, $\mathbb{E}_\mathbb{P}[R_\Delta]$ is "closer" to $\mathbb{E}_\mathbb{P}[R_{\Delta_n}]$ than is $\mathbb{E}_\mathbb{P}[R_{\Delta'}]$, in the sense that

$$\left|E_\mathbb{P}[R_\Delta] - \mathbb{E}_\mathbb{P}[R_{\Delta_n}]\right| < \left|E_\mathbb{P}[R_{\Delta'}] - \mathbb{E}_\mathbb{P}[R_{\Delta_n}]\right|.$$

To see this, note that the numerator of $|R_\Delta - R_{\Delta_n}|$ is smaller than that of $|R_{\Delta'} - R_{\Delta_n}|$:

$$\left|R_\Delta - R_{\Delta_n}\right| = \frac{\sum_{i=1}^n |\delta_i - \frac{1}{n}|}{C \sum_{i=1}^n X_i} < \frac{\sum_{i=1}^n |\delta_i' - \frac{1}{n}|}{C \sum_{i=1}^n X_i} = \left|R_{\Delta'} - R_{\Delta_n}\right|.$$

This is because for all $i = 1, \ldots, n$, $f(\Delta) < f(\Delta')$ implies $|\delta_i - \frac{1}{n}| < |\delta_i' - \frac{1}{n}|$. Therefore, since Δ_n is optimal and thus $d\left(\langle \mathbb{E}_\mathbb{P}[R_{\Delta_n}] \rangle_{\mathbb{P} \in \mathcal{P}}\right)$ is minimal for any Schur-concave function d,

$$d\left(\langle \mathbb{E}_\mathbb{P}[R_\Delta] \rangle_{\mathbb{P} \in \mathcal{P}}\right) < d\left(\langle \mathbb{E}_\mathbb{P}[R_{\Delta'}] \rangle_{\mathbb{P} \in \mathcal{P}}\right).$$

□

5. Concluding remarks

This paper introduces a new model for decision making under ambiguity called second-order uncertainty. Faced with a collection of choice alternatives, the decision maker with second-order uncertainty aversion considers all probability distributions associated with the random payoffs, and chooses the allocation that minimizes the variability of outcome across all probabilities. Optimal allocation under second-order uncertainty is the equal weighted allocation. The model of second-order uncertainty thus provides a justification for naive diversification under complete uncertainty, a pradigm widely viewed as an anomaly linked to irrational behavior.

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