

Probabilistic Aging and Overlapping Generations*

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Abstract

The paper develops an overlapping generations model with probabilistic aging of households. We define age as a set of personal attributes such as earnings potential, mortality and other attributes that are characteristic of a person's position in the life-cycle. In assuming a limited number of different states of age, we separate the concepts of age and time since birth. Agents may retain their age characteristics for several periods before they move with a given probability to another state of age. Different generations that share the same age characteristics are aggregated analytically to a low number of age groups. The probabilistic aging model thus allows for a very parsimonious yet rather close approximation of demographic structure and life-cycle differences in earnings, wealth and consumption. Existing classes of overlapping generations models follow as special cases.

Keywords: Overlapping generations, aging, demographics, life-cycle earnings.

JEL-Classification: D58, D91, H55, J21.

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1 Introduction

Many issues of fiscal policy and intertemporal macroeconomics are concerned with the behavior of overlapping generations (OLG) of households. Depending on the specific issues to be analysed, economists have a range of alternative models at hand. At one extreme end, one finds the representative agent model of infinitely lived consumers due to Ramsey (1928), Cass (1965) and Koopmans (1965). Weil (1987) has shown that this model follows from an OLG model if current and future generations are perfectly linked by an operative, altruistic bequest motive. At the other extreme is the two period OLG model without bequests, as pioneered by Samuelson (1958) and Diamond (1965). The model is analytically very tractable but cuts down life-cycle detail to the bare bones. Since the length of a period featuring constant interest rates and constant rates of consumption and investment etc. covers about thirty years in real time, the model is less useful in empirical applications especially if short-run and transitional problems are to be analyzed.

At the expense of analytical tractability, empirical applications thus rely on numerically solved models with a large number of generations and detailed life-cycle patterns of earnings, consumption and savings. These models were pioneered by Auerbach and Kotlikoff (1987) and Hubbard and Judd (1987) for the analysis of tax and fiscal policy including demographic change and social security reform. More recent and refined applications include Altig and Carlstrom (1999), Altig, Auerbach, Kotlikoff, Smetters and Walliser (2001), Imrohoroglu, Imrohoroglu and Joines (1999), Heckman, Lochner and Taber (1998) and Blundell, Costa Dias and Meghir (2003) for tax and social security reform as well as education and labor market policies. Macroeconomists have recently developed stochastic general equilibrium life-cycle models to study volatility and fluctuations at a more disaggregate level (e.g. Gomme et al., 2005, and Rios-Rull, 1996),¹ and to investigate issues of intergenerational risk sharing (Krueger and Kubler, 2006).

Life-cycle models with many generations can take account of detailed differences in

¹Part of the literature has introduced uninsurable labor income risk to study issues of wealth and income heterogeneity in a representative agent framework, see Krusell and Smith (1998).

wealth, marginal propensity to consume and in labor supply and earnings of different agents. But precisely for this reason, they tend to be analytically intractable and must be solved numerically. Models of the Auerbach-Kotlikoff type operate in a state space of as many as 108 dimensions as Laitner (1990) has shown.² Therefore, they are very expensive to implement and the results are difficult to replicate. By way of contrast, the perpetual youth model due to Yaari (1965), Blanchard (1985) and Buiter (1988) allows analytical aggregation. The aggregate version is very tractable and supports a realistic period length. Because of its simple empirical application, it has become a widely used tool of quantitative analysis as well. The main drawback, however, is the rigid demographic assumption of an age independent mortality rate and the absence of life-cycle detail of earnings, consumption and savings.³ For this reason, the model is not well suited for the analysis of aging or old age insurance. The recent extension by Gertler (1999) reconciles the perpetual youth model with an important aspect of life-cycle behavior by allowing for a stochastic transition from work into retirement and from retirement into death.⁴ After aggregation, it includes only two stock variables to represent the household sector. Although somewhat more complex, it is analytically tractable and easily implemented empirically. According to Cooley (1999), however, it is not really an improvement in quantitative terms over existing life-cycle models with many generations.

This paper develops an alternative approach that retains the simplicity of analytically aggregated OLG models and yet succeeds to replicate the rich life-cycle details of the high dimensional finite horizon models. The main challenge is the analytical aggregation of heterogeneous agents with different assets into a low number of age groups. As a sep-

²The state space would double if one considered two skill groups. The high dimensionality is a difficult constraint in many applications, and in real business cycle analysis in particular.

³Heijdra and Romp (2005), however, have included life-cycle earnings and mortality in a continuous time version of the model that still allows for analytical solutions.

⁴Hock and Weil (2006), extending Weil (1999), go one step further by considering three age groups with stochastic transitions from young to working age, from work to retirement, and from retirement to death. They are able to analytically solve the model. The absence of intertemporal savings and investment prevents further heterogeneity and avoids aggregation problems which are solved in this paper.

arate novel insight, our generalized model offers a synthesis of a wide range of existing OLG models. It replicates them as special cases by appropriately choosing two parameters governing the aging and mortality process. After aggregation, the model is easily empirically implemented with a few state variables only and thereby makes it easier to replicate empirical studies on life-cycle effects of public policy and other shocks. The key idea is to separate the concept of “age” from “time since birth”. We define *age* as a set of physical or mental attributes such as earnings potential, mortality, tastes, and other characteristics that may not change every period. Aging thus refers to the change in life-cycle characteristics rather than the passing of calendar time. This is the key difference to existing models, where aging and the passing of time are perfectly synchronized and age is understood as time since birth. In these models, there are as many states of age as there are life-cycle periods, and an age period is identical to a time period. For example, an OLG model with 55 life-cycle periods distinguishes between 55 states of age to replicate as close as possible real world life-cycles.

Our alternative approach builds on the insight that, for empirical purposes, it is enough to distinguish only a few different states of age in order to capture empirically realistic life-cycle differences. Consequently, age and time since birth become very different. Aging occurs much less frequently than the passing of calendar time. Since aging is assumed to occur stochastically, we label the proposed framework the “probabilistic aging” (PA) model. It will usually be sufficient to distinguish only a few different states of age. This gives rise to the concept of an age group which consists of the collection of generations or cohorts that find themselves in the same state of age. The state-space of the aggregate model is thereby drastically reduced, compared to standard life-cycle models with many generations. Despite of distinguishing only a few age states, the PA model nevertheless supports a period length of one year, or a quarter if desired, for realistic short-run and transitional dynamics. Choosing the number of age states and, correspondingly, of age groups is a matter of how closely the empirical life-cycle features should be replicated.

In the PA model, people move stochastically from one age state to another with an

age group specific transition rate. It is a natural generalization of the model by Gertler (1999). He distinguishes only two age groups, workers and retirees. He also assumes, somewhat restrictively, that mortality risk sets in only after retirement while workers face no such risk. We allow for more age groups and find that about eight age states already yield a close approximation of empirical life-cycle properties. Another generalization and extension is that the PA model allows for mortality already in younger age groups and thereby yields a closer approximation of demographic and life-cycle characteristics.

Section 2 explains how an agent's life-cycle is modeled by defining a small number of age states with stochastic transitions, and derives the demographic equilibrium. Section 3 solves the intertemporal optimization problem of households in the presence of mortality and aging risks. It is then shown how the population is analytically aggregated into age groups which collect all agents with diverse biographies leading to the same age state. We also delineate existing classes of intertemporal models as special cases of the PA model. Section 4 presents an illustrative empirical application and section 5 concludes.

2 The Probabilistic Aging Model

We view aging as discrete events that occur stochastically. Some people remain young for many periods, with their labor productivity and other life-cycle attributes untarnished until an aging event moves them to the next state. The less fortunate ones grow old much more rapidly, maybe due to sickness, accidents and other key events. One thus needs two different clocks to measure the passing of real time and the speed of the aging process. We measure real time in regular annual periods, with interest, prices and quantities defined per year. The aging clock runs slower and stochastically.

2.1 Life-Cycle Histories

We define a discrete number of A states of increasing age, and accordingly collect all agents with identical characteristics in the same age group $a \in \{1, \dots, A\}$. People start

life in state $a = 1$ with a given set of attributes. The life-cycle characteristics could include a person's earnings potential or her mortality risk but possibly other attributes as well. Aging means that an individual's life-cycle characteristics change when she grows older by switching to state $a + 1$. If a person does not age from one period to the next, she remains in the same group and keeps her characteristics unchanged. In consequence, households differ not only by their date of birth, but also by their diverse life-cycle histories. An agent's life-cycle history is her biography of aging events. It is represented by a vector α that records the past dates of aging events.⁵ At date t , the set of possible histories of a household that belongs to age group a is

$$\mathcal{N}_t^a \equiv \{(\alpha_1, \dots, \alpha_a) : \alpha_1 < \dots < \alpha_a \leq t\}. \quad (1)$$

A life-cycle history is represented by a vector $\alpha \in \mathcal{N}_t^a$. The element α_i refers to the date when the household who was formerly in state $i - 1$, moved into state i . Denote the unborn state by a virtual state zero. The element α_1 then lists the date of birth. An age group is defined by the mass of agents which experienced the same number of aging events. A member of group one aged only once with no further aging since birth. Nevertheless, different persons of the first age group are heterogeneous since they were born at different moments in the past. The set of possible biographies is $\mathcal{N}_t^1 = \{(\alpha_1) : \alpha_1 \leq t\}$. By the same logic, α_2 is the date when an agent moved from group 1 into 2. People in age group 2 have aged twice. The set of life-cycle histories is $\mathcal{N}_t^2 = \{(\alpha_1, \alpha_2) : \alpha_1 < \alpha_2 \leq t\}$. The vectors α , describing the biography of people in group a , have a elements since such persons have aged a times.

Individual biographies are updated when a person experiences an aging event. This updating is essential in the aggregation procedure below. Consider a person in age group $a - 1$ with a specific biography $\alpha = (\alpha_1, \dots, \alpha_{a-1})$. When the next aging shock occurs at the end of period t , she arrives in state a next period. Her biography is appended by the

⁵Rios-Rull (1996, p.467) introduced a set of histories referring to macroeconomic shocks. He solved for the full life-cycle economy. Here, a history refers to the set of possible aging events. It is needed to aggregate households into a low number of life-cycle groups. There is no uncertainty in the aggregate.

entry $t + 1$ and reads $(\alpha_1, \dots, \alpha_{a-1}, t + 1)$. Accordingly, the set $\mathcal{N}_t^a$ of biographies of age group a will be augmented next period by all the biographies $\alpha' \in \mathcal{N}_t^{a-1} \times (t + 1)$ that have $t + 1$ as their last entry and refer to people who currently switch from group $a - 1$ to a . The set of biographies $\mathcal{N}_{t+1}^a$ is thus divided into two disjoint sets where the first refers to all those who were already in group a in period t while the second refers to the newly aged people who were in group $a - 1$,

$$\mathcal{N}_{t+1}^a = \mathcal{N}_t^a \cup \mathcal{N}_t^{a-1} \times (t + 1), \quad a \in \{1, \dots, A\}. \quad (2)$$

To model demographics, we allow for mortality among younger age groups. At any moment in time, an individual with an arbitrarily given life-cycle history faces the risk of aging and dying. She must reckon with three possible events: (i) she dies with probability $1 - \gamma^a$; (ii) she survives and remains in the same age group with probability $\gamma^a \omega^a$, and (iii) she survives and ages and belongs to age group $a + 1$ next period with probability $\gamma^a (1 - \omega^a)$. It is a central assumption that these transition probabilities are independent of past aging history α . Individuals in the last age group have exhausted the aging process. They may either survive with probability γ^A within group A or die with probability $1 - \gamma^A$. Observe that only the last age group behaves according to the mortality and demographic assumptions of Blanchard's (1985) perpetual youth model.

Since the characteristics of people such as their earnings potential differ across age groups, an agent's consumption, assets and other economic variables will generally depend on her particular life-cycle history. For example, assets depend on the agent's past earnings history which, in turn, is linked to her aging trajectory. To keep track of the population's heterogeneity, one must thus very carefully identify each agent by her age group as well as her aging biography α . The number of agents at date t , in state of life a and with aging history α is given by $N_{\alpha,t}^a$. This is the smallest unit of population with all members being identical. They thus face the same probability of moving to one of the following possible states: (i) death; (ii) no aging; and (iii) aging. The biography α remains unchanged in case (ii) while switching to the next group $a + 1$ in case (iii) adds another event in a

person's life-cycle history α and thereby results in a new biography α' :

$$\begin{aligned}
(i) \quad N_{\alpha^\dagger, t+1}^\dagger &= N_{\alpha, t}^a \cdot (1 - \gamma^a), & \text{death,} \\
(ii) \quad N_{\alpha, t+1}^a &= N_{\alpha, t}^a \cdot \gamma^a \omega^a, & \text{no aging,} \\
(iii) \quad N_{\alpha', t+1}^{a+1} &= N_{\alpha, t}^a \cdot \gamma^a (1 - \omega^a), & \text{aging.}
\end{aligned} \tag{3}$$

2.2 Demographic Structure

People with the same biography are identical. At date t , age group a includes a number $N_{\alpha, t}^a$ of identical agents with the same biography α . The total number of people in age group a is obtained by adding up over all possible histories α ending up in this group a ,

$$N_t^a \equiv \sum_{\alpha \in \mathcal{N}_t^a} N_{\alpha, t}^a. \tag{4}$$

The aggregation formula (4) takes the sum over all possible biographies with varying dates of birth that could conceivably lead to the same age group a in period t . Given identical death and aging probabilities within a given group, one can now use the law of large numbers for analytical aggregation. The aggregate population groups evolve deterministically over time. $N_{(t+1), t+1}^1$ refers to the mass of newborns who arrive at the beginning of period $t+1$ in the first age group. Proofs are found in Appendix A.

Proposition 1 (*Demographic Structure*) *The laws of motion for age groups are*

$$\begin{aligned}
(a) \quad & N_{t+1}^a = \gamma^a \omega^a \cdot N_t^a + \gamma^{a-1} (1 - \omega^{a-1}) \cdot N_t^{a-1}, \quad \omega^A = 1, \\
(b) \quad & N_{t+1}^1 = \gamma^1 \omega^1 \cdot N_t^1 + N_{(t+1), t+1}^1, \\
(c) \quad & N_{t+1} = \sum_{a=1}^A \gamma^a N_t^a + N_{(t+1), t+1}^1, \quad N_t \equiv \sum_{a=1}^A N_t^a.
\end{aligned} \tag{5}$$

The key demographic parameters are the birth rate, the transition rates to successive age groups, and the mortality rates. Since all are exogenous, the demographic subsystem is independent of economic influences and evolves autonomously according to (5.a-c). Hock and Weil (2006) have been able to study aging patterns analytically for the case

of three age groups.⁶ The demographic steady state results from the requirement that inflows and outflows of any age group must balance to yield constant group size.

3 Life-Cycle Economies

3.1 Consumption and Savings

Probabilistic aging is now combined with an intertemporal model of life-cycle decisions and perfect foresight. A key ingredient of applied OLG models is the life-cycle pattern of labor income y_t^a , consisting of earnings during active work and pension income in retirement. As a result of intertemporal consumption smoothing, the pattern of labor income induces life-cycle profiles of assets. The PA model rests on the key assumption that per capita labor income y_t^a differs across age groups but is assumed identical within the same group. It does not depend on history α which is generally different for people in the same age group. In contrast, all variables such as financial assets A or consumption C that represent accumulating stocks or depend on them, are specific to individual aging histories. Given current financial wealth and labor income, and an annual interest factor $R = 1 + r$, agents accumulate assets according to

$$\gamma^a A_{\alpha,t+1}^a = R_{t+1} [A_{\alpha,t}^a + y_t^a - C_{\alpha,t}^a], \quad A_{\alpha,t+1}^a = A_{\alpha',t+1}^{a+1}, \quad (6)$$

The value of assets is measured at the end of the period and earns, together with new savings $y - C$, an annual interest r_{t+1} until the end of next period. The second equation says that a person has the same asset wealth in period $t + 1$ from savings in period t

⁶The discrete system in (5) corresponds to the continuous time demographic model of Hock and Weil (2006), equations (1-3) and footnote 4, for the case of three age groups. The system could have been postulated directly without explicit aggregation, as in Hock and Weil, since transition rates are assumed independent of past history. However, assets and other variables depend on past earnings which, in turn, reflects the aging history. We must thus develop an explicit aggregation procedure which is circumvented by these authors since they postulate a largely linear economic structure.

irrespective of whether he ages or not. The term γ^a on the left-hand side arises due to the assumption of reverse life-insurance. In the absence of a bequest motive, the agent wants to get compensated during her life-time for any accidental bequests that she leaves upon death. Such compensation is assumed to be available from a competitive insurance sector. It is assumed that an actuarially fair, group specific insurance scheme is available to all age groups. Suppose the agent has assets of $S_{\alpha,t}^a$ at the end of period t , equal to the square bracket in (6). Aggregate end of period assets of this group are S^a . Since a fraction $1 - \gamma^a$ dies, the insurance sector collects assets with a total value of $(1 - \gamma^a) S^a$. On the other hand, premiums must be paid to those who survive, adding up to $\pi^a \gamma^a S^a$. The insurance sector breaks even with a premium of $\pi^a = (1 - \gamma^a) / \gamma^a$ or $1 + \pi^a = 1 / \gamma^a$. If such insurance is available, the agent's assets next period are $A_{\alpha,t+1}^a = R_{t+1} (1 + \pi_t^a) S_{\alpha,t}^a$ if she survives, or $\gamma^a A_{\alpha,t+1}^a = R_{t+1} S_{\alpha,t}^a$ as is stated in (6).

Preferences are parameterized according to CES non-expected utility theory as proposed by Farmer (1990) and Weil (1990) and recently reviewed by Backus, Routledge and Zin (2005). These preferences restrict individuals to be risk neutral with respect to variations in income but allow for an arbitrary intertemporal elasticity of substitution. Let β be a subjective discount factor reflecting the pure rate of time preference. Agents maximize expected welfare over the remaining life-time. Except for the last age group, all agents are potentially subject to an aging shock. Hence, a person's expected utility next period, conditional on surviving, is

$$\bar{V}_{\alpha,t+1}^a \equiv \omega^a V_{\alpha,t+1}^a + (1 - \omega^a) V_{\alpha',t+1}^{a+1}. \quad (7)$$

With probability ω^a , the agent is not aging and expects welfare $V_{\alpha,t+1}^a$. With probability $1 - \omega^a$, she ages, switches to the next age state and expects welfare $V_{\alpha',t+1}^{a+1}$. The recursive form of the value function yields the Bellmann equation

$$V(A_{\alpha,t}^a) = \max_{C_{\alpha,t}^a} [(C_{\alpha,t}^a)^\rho + \gamma^a \beta (\bar{V}_{\alpha,t+1}^a)^\rho]^{1/\rho}, \quad (8)$$

where $\rho = 1 - 1/\sigma$ reflects the constant elasticity of intertemporal substitution σ .

The solution yields a modified Euler equation which states that desired consumption growth is a function of the expected interest rate relative to the subjective discount rate which is implicit in the factor β . A higher interest tilts the consumption profile towards the future, relative to a given life-cycle income profile, and thus implies higher savings. The sensitivity with respect to interest depends on the elasticity of intertemporal substitution. One obtains a closed form solution for consumption and indirect utility in terms of the agent's financial assets and the present value of her expected future labor income, see Appendix A for a proof.

Proposition 2 (*Consumption*) *Optimal consumption $C_{\alpha,t}^a$ and indirect utility $V_{\alpha,t}^a$ are*

$$\begin{aligned}
(i) \quad C_{\alpha,t}^a &= (1/\Delta_t^a) (A_{\alpha,t}^a + H_{\alpha,t}^a), \quad \sigma = 1/(1-\rho), \\
(ii) \quad V_{\alpha,t}^a &= (\Delta_t^a)^{1/\rho} C_{\alpha,t}^a, \\
(iii) \quad \Delta_t^a &= 1 + \gamma^a \beta^\sigma (\Omega_{t+1}^a R_{t+1})^{\sigma-1} \Delta_{t+1}^a, \\
(iv) \quad \Omega_{t+1}^a &= \omega^a + (1-\omega^a) (\Lambda_{t+1}^a)^{1-\rho}, \quad \Lambda_{t+1}^a = (\Delta_{t+1}^{a+1}/\Delta_{t+1}^a)^{1/\rho}, \\
(v) \quad H_{\alpha,t}^a &= y_t^a + \gamma^a \bar{H}_{\alpha,t+1}^a / (\Omega_{t+1}^a R_{t+1}), \\
(vi) \quad \bar{H}_{\alpha,t+1}^a &= \omega^a H_{\alpha,t+1}^a + (1-\omega^a) (\Lambda_{t+1}^a)^{1-\rho} H_{\alpha',t+1}^{a+1}.
\end{aligned} \tag{9}$$

The inverse of Δ_t^a is the marginal propensity to consume and H denotes human capital equal to the present value of future labor income.

The last group is conceptually different from earlier ones because no further aging is possible ($\omega^A = 1$). A person may either survive to the next period within the same age group, or die, hence $\Omega^A = \Lambda^A = 1$. Imposing these restrictions yields the optimal consumption policy of the last age group from (9). In particular, human wealth and the inverse of the marginal propensity to consume follow $H_{\alpha,t}^A = y_t^A + \gamma^A H_{\alpha,t+1}^A / R_{t+1}$ and $\Delta_t^A = 1 + \gamma^A \beta^\sigma (R_{t+1})^{\sigma-1} \Delta_{t+1}^A$.⁷ The behavior of the last group is fully described by the perpetual youth model of Blanchard (1985) albeit with one important difference. Since the last is only one of several age groups, average time spent in this group is rather

⁷The reader may verify the solution in (9) for the oldest life-cycle group by solving the problem $V(A_{\alpha,t}^A) = \max_{C_{\alpha,t}^A} [(C_{\alpha,t}^A)^\rho + \gamma^A \beta (V_{\alpha,t+1}^A)^\rho]^{1/\rho}$ subject to $\gamma^A A_{\alpha,t+1}^A = R_{t+1} [A_{\alpha,t}^A + y_t^A - C_{\alpha,t}^A]$.

limited. The mortality rate must thus be chosen much larger which implies a much higher marginal propensity to consume, compared to younger age groups.

If all age classes face the same survival probability $\gamma^a = \gamma$ and thereby differ only in their earnings, then the marginal propensity to consume is invariant across age classes. This is most easily seen in case of a unit elasticity $\sigma = 1$ which yields $1/\Delta = 1 - \beta\gamma$ by (9.iii) but is also true more generally. Suppose $\Delta^{a+1} = \Delta^a$, then Λ^a as well as Ω^a are both equal to one, implying $\Delta_t^a = 1 + \gamma\beta^\sigma (R_{t+1})^{\sigma-1} \Delta_{t+1}^a$. Hence, Δ^a must indeed be the same for all groups which, by (9.iii), is possible only with a uniform survival rate γ . Age classes then differ only in terms of human wealth and consumption levels but all choose the same intertemporal consumption structure. The arguments show that a change in the marginal propensity to consume from one age class to the next exclusively reflects a change in the mortality rate.

It is a hard fact that the mortality rate increases between any two consecutive groups. Again, the Cobb Douglas case shows most transparently that the marginal propensity to consume $1/\Delta^a = 1 - \gamma^a\beta$ rises with a lower survival rate γ^a . It would become equal to one if there were a truly last group with $\gamma^A = 0$, as in OLG models with a fixed finite life. Given that the ratio Δ^{a+1}/Δ^a falls below unity, and that $\rho < 0$ for an intertemporal substitution elasticity $\sigma < 1$, the factor Λ^a exceeds one and thereby raises Ω^a above one as well. Agents start to discount the future more heavily, at an effective rate $\Omega^a R$, as the end of life becomes a more probable event on account of an increased mortality rate. Consequently, the marginal rate of substitution across age states, as defined prior to (A.4), becomes larger when mortality is more common. With a higher probability of extinction, agents are less inclined to postpone consumption and thus consume a larger fraction of their resources immediately. Quite apparently, saving for the future makes less sense. Agents value consumption in state $a + 1$ relatively more than in state a which explains the increase in the marginal rate of substitution across age classes.

3.2 Aggregation

The simplicity and tractability of the perpetual youth model with a constant mortality rate rests on the fact that it allows for analytical aggregation. The same applies to the PA model. The advantage of approximating actual demographic and life-cycle properties with a low dimensional system is possible only if age groups can be aggregated analytically. In the PA model, agents are uniquely identified by their biography α . In age group a , one records at date t a mass of agents $N_{\alpha,t}^a$ with identical life-cycle history. These agents are economically identical and thus consume the same quantity $C_{\alpha,t}^a$. Section 2 emphasized, however, that there is a large heterogeneity of biographies that can lead to state a in period t . This heterogeneity is eliminated by aggregation. Summing up consumption over all people with biographies α that lead to state a in period t yields aggregate consumption of age group a . Adding over all age groups yields total, economy wide consumption:

$$C_t^a \equiv \sum_{\alpha \in \mathcal{N}_t^a} C_{\alpha,t}^a N_{\alpha,t}^a, \quad C_t \equiv \sum_{a=1}^A C_t^a. \quad (10)$$

The same principle applies to other static variables. When aggregating expressions that multiply with variables identical within an age group, or identical over the entire population, these constant terms drop out of the summation. For example, aggregating labor income in (6) yields $Y_t^a = \sum_{\alpha} y_t^a N_{\alpha,t}^a = y_t^a N_t^a$. Similarly, when aggregating consumption of age group a as stated in (9.i), we observe that Δ_t^a , Λ_t^a and Ω_t^a are identical within each group. They implicitly depend only on the common interest rate and demographic parameters that differ across but are constant within each group. Hence, aggregate consumption is simply $C_t^a = (A_t^a + H_t^a) / \Delta_t^a$.

Consider now the aggregation of dynamic variables such as human capital and assets. Human capital is particularly simple since, by assumption, labor income and, thus, human capital per capita is the same for all persons within each group. This is most evident for human capital of the last group. Solving forward $H_{\alpha,t}^A = y_t^A + \gamma^A H_{\alpha,t+1}^A / R_{t+1}$ yields the same present value of future income, discounted at a common rate, irrespective any past biography. Writing the per capita value as $H_{\alpha,t}^A = h_t^A$, aggregate human capital is

$H_t^A = h_t^A N_t^A$. The same holds for all other groups as is obvious from (9),

$$H_t^a = h_t^a N_t^a, \quad h_t^a = y_t^a + \gamma^a \cdot \frac{\omega^a h_{t+1}^a + (1 - \omega^a) (\Lambda_{t+1}^a)^{1-\rho} h_{t+1}^{a+1}}{\Omega_{t+1}^a R_{t+1}}. \quad (11)$$

The evolution of human capital per capita is repeated from (9.v-vi). The equation for the oldest group follows on account of the restriction $\omega^A = 1$, implying $\Omega^A = 1$.

Proposition 3 (*Asset Accumulation*) *Using the aggregator in (10), $A_t \equiv \sum_a A_t^a$ and $A_t^a \equiv \sum_{\alpha \in \mathcal{N}_t^a} A_{\alpha,t}^a N_{\alpha,t}^a$, aggregate assets evolve as*

$$\begin{aligned} (a) \quad & A_{t+1}^1 = \omega^1 S_t^1, \quad S_t^a \equiv R_{t+1} [A_t^a + Y_t^a - C_t^a], \\ (b) \quad & A_{t+1}^a = \omega^a S_t^a + (1 - \omega^{a-1}) S_t^{a-1}, \\ (c) \quad & A_{t+1} = R_{t+1} [A_t + Y_t - C_t], \end{aligned} \quad (12)$$

where $Y \equiv \sum_a Y^a$ is aggregate labor income.

The proof is in Appendix A. Upon aggregation, the survival factor γ^a cancels out. The insurance mechanism redistributes, without any net loss in the aggregate, the assets unintentionally left behind by the deceased to the surviving fraction in each age group.

3.3 A Synthesis of Models

The PA framework encompasses a wide range of intertemporal models as special cases and thereby shows how they are related by specific assumptions with respect to aging and mortality. The simplest case is the Ramsey model with an infinitely lived representative agent. Weil (1987) showed that it follows from an OLG model of households linked by an operative, altruistic bequest motive. It emerges by excluding both aging and mortality, $\omega^a = \gamma^a = 1$ for all a which implies $\Omega^a = 1$ as well. With a constant population, savings and consumption result from the problem $V(A_t) = \max [C_t^\rho + \beta V(A_{t+1})^\rho]^{1/\rho}$ subject to $A_{t+1} = R_t [A_t + y_t - C_t]$. The standard decision rules are easily recovered from (9) under the given parameter restriction. Indices a and α become irrelevant. The marginal

propensity to consume Δ_t and human wealth obviously remain independent of any age characteristics. Human wealth is the simple present value of future earnings derived from $H_t = y_t + H_{t+1}/R_{t+1}$. The Euler equation (A.5) reduces to $C_{t+1} = (\beta R_{t+1})^\sigma C_t$ and requires the interest rate to satisfy $\beta R = 1$ in a stationary long-run equilibrium. The PA model can also represent Weil's (1989) model of overlapping families of infinitely lived agents by introducing a positive birth rate while setting the mortality rate to zero.

The perpetual youth model of Blanchard (1985) is a model with an infinity of cohorts but only one age group. It follows by setting $\gamma^a = \gamma < 1$ and $\omega^a = 1$. In any period, the event space is reduced to the first two events in (3) where the instantaneous mortality rate is invariant with respect to time and age. In the absence of aging, the Bellmann equation in (8) becomes independent of age characteristics a . An agent's life-cycle history reduces to a single event which is birth at date $\alpha = \alpha_1$. Note that time since birth does not translate into aging which we defined as a change in life-cycle characteristics. Hence, this model is quite appropriately termed a "perpetual youth" model. The constant mortality rate simply increases the effective discount factor to $\gamma\beta$. With $\omega^a = \Omega^a = 1$, the marginal propensity to consume, the inverse of (9.iii), is independent of age and the same for young and old generations since they all expect the same remaining life-time. However, it is higher compared to the Ramsey model and thereby reflects the finiteness of life. For the same reason, future wage income is less valuable since it might not be available in case of premature death, and is thus discounted at a higher effective rate. Human capital follows from $H_t = y_t + \gamma H_{t+1}/R_{t+1}$ and is independent of age, see (9.v-vi), since all generations receive the same wage. Agents differ only in the amount of financial assets that were accumulated since birth, and the level of consumption.

Extensions of the basic model have also made labor income dependent on time since birth to approximate the empirical life-cycle pattern of wage income. While such extensions introduce some heterogeneity with respect to human wealth, they do not improve on the life-cycle pattern of the marginal propensity to consume which remains age independent as long as the survival rate remains constant. The PA model can reproduce this case

by assuming annual aging, $\omega^a = 0$, where aging occurs each single period. An age group becomes identical to a cohort so that the index for the age group measures time since birth. One need not keep track of life-cycle history separately. The Bellmann equation simply becomes $V(A_t^a) = [(C_t^a)^\rho + \gamma\beta (V_{t+1}^{a+1})^\rho]^{1/\rho}$. The decision rules are again recovered from (9). Since the survival probability is identical in all age groups, $\gamma^a = \gamma$, all agents face the same remaining life-horizon, leading to $\Delta_{t+1}^a = \Delta_{t+1}^{a+1}$ and thus $\Lambda = 1$ and $\Omega = 1$. The marginal propensity to consume remains age independent as long as the survival factor is constant, $\Delta_t = 1 + \gamma\beta^\sigma (R_{t+1})^{\sigma-1} \Delta_{t+1}$. Human wealth follows $H_t^a = y_t^a + \gamma H_{t+1}^{a+1}/R_{t+1}$ with y_t^a now depending on time since birth, as indexed by a .

Gertler's (1999) extension of the perpetual youth model is easily reproduced by allowing for only two age groups corresponding to two aging events, birth and retirement.⁸ Agents are born as workers in age group one and then switch with constant probability into retirement. Mortality sets in only after retirement. The demographic assumptions are $\gamma^1 = 1$, $\gamma^2 < 1$, $\omega^1 < 1$ and $\omega^2 = 1$, leading to two age groups and an infinity of cohorts. This model not only introduces heterogeneity in human wealth. Retirees have a higher marginal propensity to consume than workers since they face a higher mortality rate. The PA model is a natural extension and generalization of this basic approach.

The PA model also encompasses the original Auerbach Kotlikoff (1987) model with an annual, although degenerate aging process. Each year since birth is taken as a separate age group a . In that sense, aging occurs every single period. The number of age groups thus coincides with the number of cohorts and life-cycle periods. People are assumed to live with certainty until a fixed finite period and then die with certainty:⁹ $\gamma^a = 1$ for $a < A$ and $\gamma^A = 0$. The aging process is degenerate in the sense that people switch

⁸Hock and Weil (2006) consider three age groups to study demographic issues but do not include an intertemporal household model.

⁹Recent variations of Auerbach Kotlikoff type OLG models also account for demographic change by shrinking the size of cohorts from period to period. The PA model could replicate this with $\gamma^a < 1$ for $a < A$ and $\gamma^A = 0$ if the assumption of an actuarially fair, group specific reverse life-insurance were given up and replaced by another mechanism for intergenerational redistribution of accidental bequests.

with certainty to the next higher age group every period, $\omega^a = 0$ all a , except $\omega^A = 1$. Age is necessarily updated to $a + 1$ in the next period which implies $\Lambda = 1$ and $\Omega = 1$. Time since birth fully describes an agent's biography which can thus be suppressed. Both human wealth and the marginal propensity to consume become age dependent. From (9), we recover $\Delta_t^a = 1 + \beta^\sigma (R_{t+1})^{\sigma-1} \Delta_{t+1}^a$ and $H_t^a = y_t^a + H_{t+1}^{a+1}/R_{t+1}$ (see the proof of Proposition 4 in Appendix A). However, life-time now ends with certainty after A years which introduces the end point restrictions $\Delta_t^A = 1$ and $H_t^A = y_t^A$ on account of $\gamma^A = 0$. The marginal propensity to consume increases up to unity when the last period of life approaches. As Table 1 below demonstrates, the PA model approximates this feature by means of a very low survival rate in the last age groups.

Finally, the PA model collapses to the two period OLG model of Diamond (1965) and Samuelson (1958) when $\gamma^1 = 1$, $\gamma^2 = 0$, $\omega^1 = 0$ and $\omega^2 = 1$ and $y_{t+1}^2 = 0$. There is no life-time uncertainty ($\gamma^1 = 1$). People die with probability one after two periods, $\gamma^2 = 0$. Aging occurs after one period, $\omega^1 = 0$, when people retire with zero labor earnings, $y_{t+1}^2 = 0$. Agents are necessarily in age group 1 in their first period of life and in group 2 in their retirement period. Cohorts and age groups are thus exactly identical. In this simplest case with periodic aging, people are fully identified by their age group which makes the indexing of life-cycle history superfluous. Agents are not endowed with any assets in their first period. This leaves the familiar problem of maximizing $V_t^1 = [(C_t^1)^\rho + \beta (C_{t+1}^2)^\rho]^{1/\rho}$ subject to $C_{t+1}^2 = R_{t+1} [y_t^1 - C_t^1]$. We summarize:

Proposition 4 (*Synthesis of Models*) *The PA model includes existing OLG models as special cases reflecting different assumptions on aging frequency, mortality and period length: (a) Infinitely lived agents (Weil, 1987, 1989): $\omega^a = \gamma^a = 1$; (b) Perpetual youth (Blanchard, 1985): $\gamma^a = \gamma < 1$ and $\omega^a = 1$ without and $\omega^a = 0$ with wage profiles; (c) Perpetual youth with two age groups (Gertler, 1999): $\gamma^1 = 1$ and $\omega^1 < 1$ for workers and $\gamma^2 < 1$ and $\omega^2 = 1$ for retirees; (d) Life-cycle models with many periods (e.g. Auerbach and Kotlikoff, 1987): $\omega^a = 0$ and $\gamma^a = 1$ and $\gamma^A = 0$; (e) Two period OLG model (e.g. Diamond, 1965): $\gamma^1 = 1$, $\omega^1 = 0$ for workers and $\gamma^2 = 0$, $\omega^2 = 1$ for retirees.*

3.4 General Equilibrium

In economic applications, a key distinction between age groups is the definition of wage related income,

$$y_t^a = \begin{cases} (1 - \tau) w_t \theta_t^a & : a \in \{1, \dots, a^R - 1\}, \\ p_t & : a \in \{a^R, \dots, A\}. \end{cases} \quad (13)$$

There are several states of work and retirement life where a^R denotes the retirement date in a person's biography. Wage related income per capita, y_t^a , differs across age groups but is assumed identical for all agents within the same group. Importantly, it does not depend on life-cycle history α which is generally different for people in the same age group. Each worker in group a is thus endowed with θ^a efficiency units of labor which earn a wage of w_t per unit. Consequently, a worker's gross wage per unit of labor is $w_t^a = w_t \theta_t^a$. The government levies a proportional wage tax τ to finance pensions and other public spending. Retirees receive a pension of p_t per capita from pay-as-you-go (PAYG) social security. Denoting labor supply in efficiency units by L^S and the number of retirees by N^R , aggregate labor income amounts to

$$Y_t^a = y_t^a N_t^a, \quad Y_t = (1 - \tau) w_t L_t^S + p_t N_t^R, \quad L_t^S \equiv \sum_{a=1}^{a^R-1} \theta_t^a N_t^a, \quad N_t^R \equiv \sum_{a=a^R}^A N_t^a. \quad (14)$$

Firms combine capital K_t and efficiency units of labor L_t to produce $Q_t = F(K_t, L_t)$. Technology is linear homogeneous and quasiconcave. For realistic dynamics of investment in a small open economy, investment costs J are assumed to increase progressively with the rate of gross investment I , reflecting installation costs of capital. The installation technology is linear homogeneous and satisfies $\phi_I > 0$, $\phi_{II} > 0$, and $\phi_K < 0$. Capital depreciates at rate δ . As a normalization, we may specify $\delta = \phi(\delta, 1)$, implying that $J = I$ in a steady state. Dividends χ_t are

$$\chi_t = Q_t - w_t L_t - J_t, \quad J = \phi(I, K), \quad K_{t+1} = I_t + (1 - \delta) K_t. \quad (15)$$

Defining the end of period, cum dividend value of capital by V^K , optimal investment and employment policies follow from $V^K(K_t) = \max_{I_t, L_t} \chi_t + V^K(K_{t+1})/R_{t+1}$. Dis-

counting with the households' interest factor R_{t+1} assures that the investors' no arbitrage-condition is satisfied, i.e. the return on dividend generating capital must equal the market rate of interest available on alternative assets. Denoting the shadow price of capital by $\lambda_t^K \equiv \partial V_t / \partial K_t$, the optimality and envelope conditions are $\phi_I = \lambda_{t+1}^K / R_{t+1}$, $w_t = F_L$ and $\lambda_t^K = F_K - \phi_K + (1 - \delta) \lambda_{t+1}^K / R_{t+1}$. It is well known since Hayashi (1982) that the marginal and average shadow prices of capital are identical, $V_t^K \equiv \lambda_t^K K_t$. Finally, note that in a steady state, $I = \delta K$. Due to our normalization of adjustment costs ϕ , we have $\phi_I = 1$ and $\phi_K = 0$ in a steady state which yields the familiar condition $F_K = r + \delta$.

The model is closed by imposing the government budget constraint. Wage taxes must finance public consumption G and PAYG pensions where L^S and N^R are given in (14),

$$\tau w_t L_t^S = p_t N_t^R + G_t. \quad (16)$$

Finally, we state the capital and labor market clearing conditions. In equilibrium, demand for efficiency units of labor corresponds to aggregate household sector labor supply, scaled by worker productivity, $L_t = L_t^S$. Further, accumulated household sector financial wealth $A_{t+1} = V_{t+1}^K + D_{t+1}$ absorbs the value of domestically issued equity, V_{t+1} , and foreign bonds D_{t+1} . When assets are perfectly substitutable, they must earn an identical rate of return equal to the market interest r_{t+1} . Given optimal household and firm behavior, and with all budget constraints fulfilled, Walras' Law implies the current account

$$D_{t+1} = R_{t+1} [D_t + Q_t - J_t - G_t - C_t], \quad (17)$$

where the stock of foreign net assets D_t is measured at the end of the period.

4 An Illustrative Application

We highlight the usefulness of the PA model with a simple policy experiment that would be difficult to analyse with a standard perpetual youth model due to its lack of life-cycle structure. We consider the consequences of an increase in pay as you go pensions financed

by a wage tax on the active population which corresponds to life-cycle redistribution from young to old. The next subsection first discusses the empirical implementation of the PA model on real population data.

4.1 Demographic Structure and Wage Profiles

For the sake of concreteness, assume that total life-time consists of $T = 70$ periods as in Table 1 which considers the active population starting at age 20 and living until age 90. The stationary composition of the population results from recursively applying $\tilde{N}^t = \tilde{\gamma}^{t-1} \tilde{N}^{t-1}$ where the survival rates $\tilde{\gamma}^t$ of vintage t is taken from official mortality tables. A tilde refers to annual cohorts. The total population is divided into a cross section of eight groups. The first six are equally spaced and contain 10 cohorts each, the very old are subdivided into two smaller groups with five cohorts each. The second group, for example, contains vintages $N^2 = \sum_{t=30}^{39} \tilde{N}^t$. Line 3 of Table 1 reflects actual, non-stationary population data. Line 4 reports the population shares that reflect a demographic steady state with observed mortality rates.

Table 1: Demographic and Life-Cycle Parameters

1. Age groups	1	2	3	4	5	6	7	8
2. Cohorts	20-29	30-39	40-49	50-59	60-69	70-79	80-84	85-89
3. Data N^a/N	0.168	0.222	0.192	0.168	0.120	0.089	0.025	0.016
4. Model N^a/N	0.179	0.177	0.175	0.168	0.148	0.107	0.031	0.016
5. Labor prod. θ^a	1.000	1.362	1.561	1.582	1.295	0.000	0.000	0.000
6. Prob. $1 - \gamma^a$	0.001	0.001	0.004	0.012	0.028	0.042	0.096	0.200
7. Prob. $1 - \omega^a$	0.099	0.099	0.096	0.089	0.074	0.061	0.115	0.000
8. Propens. $1/\Delta^a$	0.047	0.052	0.059	0.069	0.086	0.110	0.168	0.230

Notes: θ^a life-cycle labor productivity determines wage $w^a = w\theta^a$, $1 - \gamma^a$ probability of dying, $1 - \omega^a$ probability of aging, $1/\Delta^a$ marginal propensity to consume. Data sources: BFS (2004), and own calculations.

Aggregation collects a subgroup of neighboring cohorts into an age group and takes the average wage and other average attributes to define the characteristics of this group.

Taking the first age group as an example, we have three restrictions to be fulfilled: (i) the mass of agents in age group one must be identical with the mass of people in cohorts 20-29; (ii) the average wage of cohorts 20-29 must correspond to the uniform wage in age group one; and (iii) the expected duration that an agent spends in a given age group corresponds to the number of vintages that define this group. Since cohorts 20-29 define the first group, the expected duration in that group is ten years. For the last two age groups in Table 1, the expected duration is assumed 5 years to approximate more closely the increase in mortality rates in the last phase of life.

We now show how these restrictions identify the parameters of the PA model. Setting the date of birth at $\alpha_1 = 20$, the aggregation key chosen in Table 1 is represented by a particular biography $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_A) = (20, 30, 40, \dots, 80, 85)$. Having chosen the aggregation key, one may aggregate T cohorts into A age groups by

$$N^a = \sum_{t=\alpha_a}^{\alpha_{a+1}-1} \tilde{N}^t, \quad w^a N^a = \sum_{t=\alpha_a}^{\alpha_{a+1}-1} \tilde{w}^t \tilde{N}^t. \quad (18)$$

The life-cycle profiles of annual wages $\tilde{w}^t$ are readily available from cohort data. We require that the aggregate wage bill of a given age group is the same as the accumulated wage bill from all cohorts that define this group. This restriction determines the group specific wages w^a as an average of cohort specific wages $\tilde{w}^t$, taking the shares $\tilde{N}^t/N^a$ of the cohorts in that age group as weights.

Figure 1 illustrates how the PA model approximates the empirical wage profile. The solid, ragged line shows actual wage data for each cohort in Switzerland 2000. The solid step function with 10 year intervals represents life-cycle wages $w^a = w\theta^a$ and induces the labor productivity profile θ^a , given in line 5 of Table 1, upon specifying a wage w per efficiency unit of labor. Note that the wage of age group 5 is an average of the relatively high wage prior to retirement and the zero wage after retirement when people are in their late sixties. The dashed line with 20 year intervals represents only 4 age groups with the first three comprising 20 cohorts and the last one 10 cohorts. Obviously, the approximation is rather crude for periods early and late in the life-cycle while it makes not much difference for agents in their mid-life. Finally, the dashed line with 5 year spacing

comes considerably closer to the true wage profile by taking averages over fewer cohorts. The gains in empirical approximation are highest for the youngest cohorts facing a steep wage profile, and for the cohorts close to retirement.

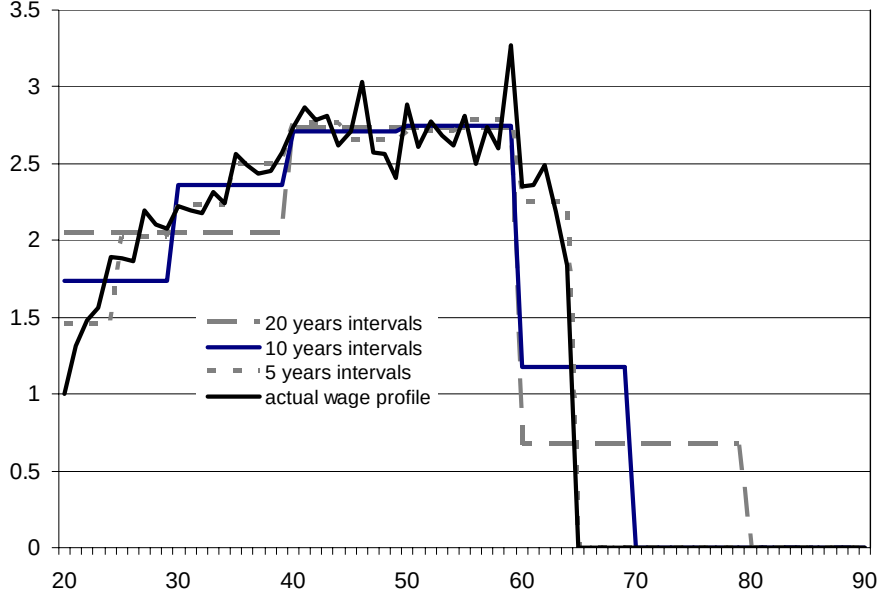


Fig. 1: Life-Cycle Wages. Data source: BFS (2004)

Finally, the demographic parameters ω^a and γ^a must be set to replicate the population shares listed in line 4 of Table 1. They must satisfy two restrictions, the expected duration in group a and the size of group a . Linking the aggregation of cohorts to a life-cycle history as in (18) implies that an agent expects to remain $\alpha_{a+1} - \alpha_a$ periods in age state a . For example, the average time spent in the first age group would be 10 years while expected duration in the last group is five years. Given the instantaneous probability $\omega^a \gamma^a$ of staying as in (3.ii), the expected duration in group a is

$$\alpha_{a+1} - \alpha_a = 1 / (1 - \omega^a \gamma^a). \quad (19)$$

One can now recover the demographic parameters of the PA model from our knowledge of aggregated population data N^a and duration in group a . In a demographic steady state, each age group must fulfill the restriction (5.a) which is

$$(1 - \gamma^a \omega^a) \cdot N^a = (\gamma^{a-1} - \gamma^{a-1} \omega^{a-1}) \cdot N^{a-1}. \quad (20)$$

At this stage, one knows N^a from aggregated population data and $\gamma^a \omega^a$ from the age group duration as implied by the chosen aggregation. One can now easily solve (20) for all γ^{a-1} except for the last one. Having computed γ^{A-1} , the survival rate γ^A of the last group follows directly from (20) on account of the restriction $\omega^A = 1$. Lines 6 and 7 of Table 1 list the resulting values of the exit probabilities. Having obtained ω^a and γ^a for all groups, the last line reports the marginal propensity to consume as it results from the stationary version of (9.iii-iv).

4.2 Other Parameters

The quantitative application is based on an extended model with endogenous labor supply which is outlined in Appendix B. The intensive margin of labor supply refers to hours worked of the active population while the extensive margin refers to the discrete retirement choice of people in their sixties. The model is calibrated to stylized data of a representative economy. Table 2 states key parameters that characterize preferences and technology. These are commonly used values as in Gertler (1999), Altig et. al (2001), Blundell et al. (2003), and Krueger and Kubler (2006), for example, or in the real business cycle literature such as in Rios-Rull (1996), Krusell and Smith (1998) and Gomme et al. (2005). To keep the discussion short, we refer to these papers for a review of the econometric evidence. The replacement rate gives the size of the pension compared to the last net wage income. Tax revenue is used to finance pensions and public spending as in (16) and requires a wage tax rate of roughly 30% to balance the budget. According to Gruber and Wise (2005) and Boersch-Supan (2000), the retirement decision is arguably the most important behavioral response to pension reform. The optimality condition in (B.2) means that people in age group a^R postpone retirement, i.e. a larger share δ of agents in this group remain working, if work becomes more attractive relative to retirement. The retirement elasticity is taken from empirical studies. Börsch-Supan (2000) estimates that a decrease in benefits by 12% would reduce the retirement probability of the 60 years old from 39.3% to 28.1%. This amounts to a semi-elasticity of the retirement decision of $\varepsilon^R = 1$.

Table 2: Taste and Technology Parameters

Real interest rate r	0.050
Depreciation rate δ^K	0.100
Output elasticity of capital α	0.350
Subjective discount factor β	0.978
Intertemporal elasticity of substitution σ	0.400
Wage elasticity of labor supply ε^L	0.300
Retirement semielasticity ε^R	1.000
Proportional wage tax rate τ	0.308
Pension replacement rate ζ	0.381

Notes: $Q = XK^\alpha(ZL)^{1-\alpha}$ is the production technology, $\zeta = p/[(1 - \tau)w\theta^{a^R}]$ is the pension replacement rate and $\varepsilon^L \equiv \varphi'/(l\varphi'')$ is the labor supply elasticity, see Appendix B.

4.3 A Quantitative Application

Table 3 reports the long-run consequences when pension spending of the pay as you go type is raised from 5 to 7 percent of GDP. Initially, net foreign assets of the model economy are zero. The first column refers to the small open economy case where real interest is exogenously given on world capital markets. Any imbalance of domestic savings and investment is then reflected in the current account. The immediate implication of the scenario is that higher pension spending necessitates an increase in the labor tax. The scenario thus tilts the life-cycle profile of disposable income towards the future and thereby undermines the income smoothing motive as a prime determinant of life-cycle savings. In consequence, accumulated savings and household asset wealth decline substantially. Furthermore, the scenario importantly discourages aggregate labor supply. First, the higher tax rate discourages hours worked and labor supply on the intensive margin. In equilibrium, per capita labor supply of active workers shrinks by 2.2%. Second, the scenario strongly induces early retirement since pension income upon retirement becomes much more generous compared to net of tax wages from continued employment. The

replacement rate, defined by the pension as a fraction of the last net of tax wage, increases by 7 percentage points from 38 to 45%. The incentives for early retirement reduce the share of the active population in age group five, corresponding to people aged 60 to 70 in the cohort model, from 45 to 14%. Early retirement shrinks the workforce and inflates the number of retirees. The dependency ratio thus increases by almost 9 percentage points to 39%. The induced early retirement reinforces the necessary adjustments of taxes and benefits to keep the system sustainable.

Table 3: Long-Run Impact of Pension Increase

Key Macro Variables in %		Open	Open [†]	Closed
r	real interest* (0.050)	0.050	0.050	0.053
D^f/Y	debt GDP ratio* (0.000)	-0.189	-0.156	0.000
τ	contribution rate* (0.308)	0.358	0.354	0.363
ζ	replacement rate * (0.381)	0.454	0.452	0.452
δ	retirement* (0.450)	0.139	0.144	0.121
N^R/N^W	dependency ratio* (0.307)	0.391	0.390	0.396
$(1 - \tau)w$	net of tax wage	-7.271	-6.673	-9.010
l	labor supply per capita	-2.239	-0.688	-2.793
L^D	employment, efficiency units	-7.816	-6.266	-8.658
K	capital stock	-7.816	-6.266	-11.477
Y	gross domestic product	-7.816	-6.266	-9.654
C	aggregate consumption	-11.060	-8.918	-11.313
A	aggregate assets	-14.917	-12.226	-11.216

Note: %) Percent changes relative to ISS. *) Absolute values, initial values in brackets. †) Simulation with $\varepsilon^L = 0.1$.

The combined effect of early retirement and hours worked leads to a decline in aggregate employment by about 7.8%. Since the real interest is exogenously fixed in a small open economy, the capital stock falls in parallel with the reduction in aggregate employment to keep the capital labor ratio constant. Output thus declines by exactly the same

amount. Since the scenario shifts the individual income profile from work to retirement, individual savings incentives to provide for old age income are much reduced. Aggregate assets fall by almost 15%. Since the decline in asset wealth exceeds the reduction in the value of capital, capital market equilibrium $A = V^K + D$ as noted in section 3.4 implies an increase in net foreign debt which eventually amounts to almost 19% of GDP in Table 3. The long-run adjustment of the savings investment balance is also demonstrated in Figure 3. However, the figure points to a potential short-run non-monotonicity which results from the fact that the adjustment dynamics of aggregate savings in an OLG model is much slower than the adjustment of investment. Note that net foreign debt is zero initially and historically predetermined while the value of capital is a jump variable reflecting the future returns to capital. Consequently, the revaluation of equity implies a revaluation of household sector assets by exactly the same percentage amount. However, since the subsequent decline in accumulated savings occurs with rather slow speed, the reduction in the value of capital exceeds the decline in aggregate savings in a first adjustment phase. Consequently, the economy first builds up a net asset position before it starts to incur net foreign debt in the long-run.

The last two columns of Table 3 are for sensitivity analysis. The second column simulates the effects of the same policy shock subject to a lower wage elasticity of hours worked. Labor supply per capita of the active population falls by only 0.7 instead of 2.2% while the tendency for early retirement is largely of the same magnitude. The macroeconomic response is scaled down on account of a smaller reduction in aggregate employment which predominantly stems from the less elastic labor supply response on the intensive margin. The last column reports the long-run effects in a closed economy where net foreign assets are restricted to be zero and the domestic real interest rate must adjust to balance savings and investment. Since the immediate impact of the pension reform is a reduction in life-cycle savings, the interest rate must increase to bring accumulated savings and investment into balance again. The higher cost of capital leads firms to choose a lower capital intensity which magnifies the reduction of the capital stock (-11.5%) compared to aggregate employment (-8.6%). Consequently, the reduction of the

net of tax wage becomes larger since it reflects not only the higher tax rate but also a reduced gross wage rate on account of a lower capital intensity. The quantitatively larger decline in net wages exacerbates not only the reduction of hours worked but also reinforces the trend to early retirement. Aggregate employment thus falls by 8.6 compared to 7.8% in the base case scenario. The loss in aggregate output turns out much larger on account of reduced factor inputs.

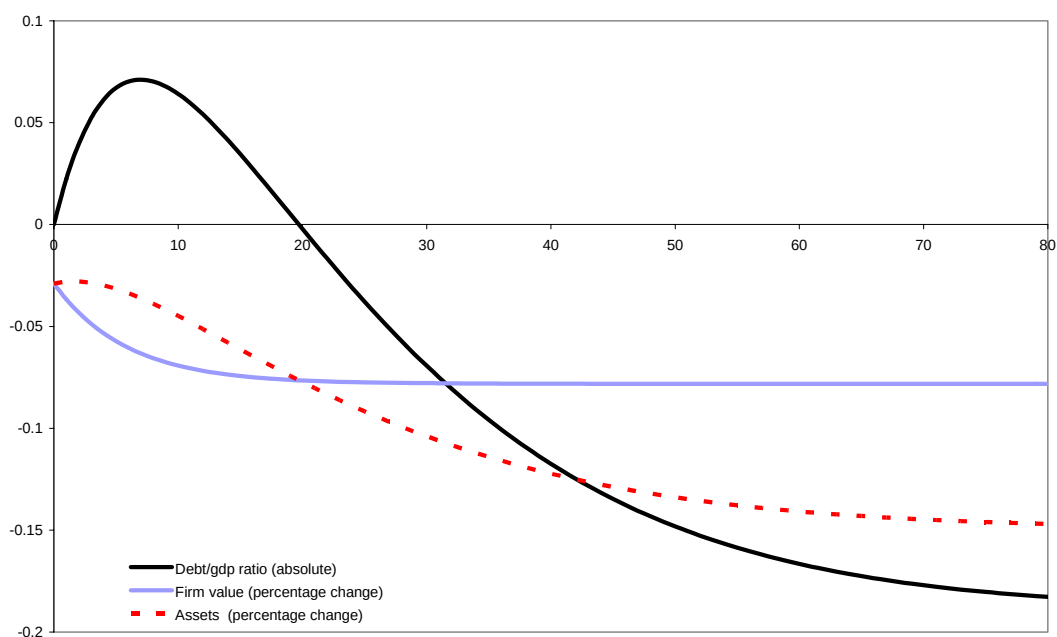


Fig. 2: Savings, Investment and Net Foreign Assets

5 Conclusions

The theory of probabilistic aging separates the notion of age from time since birth. Economic age is a set of personal characteristics such as labor productivity, tastes, mortality etc. that change less frequently over the life-cycle than periods since birth. People can retain their characteristics over several periods before they change as a result of aging. Aging occurs stochastically. Identifying a person with her biography which is the sequence of discrete aging events since birth, allows to analytically aggregate a multitude of different generations into a low number of age groups. The PA model is thus a generalized

OLG model. Existing models follow as special cases which reflect specific assumptions with respect to aging frequency, mortality and period length.

In standard OLG models, aging occurs every period and the probability of dying is unity in the last period of life. The number of life-cycle periods corresponds to the number of age groups or cohorts. As Laitner (1990) has shown, typical models with 55 generations operate in a state space of 108 dimensions and are thus very expensive to implement. Results are difficult to replicate. If aging occurs less frequently, the number of age groups becomes smaller than the number of cohorts. We found that allowing for eight groups already leads to a quite accurate approximation of life-cycle properties. Counting as in Laitner (1990), the state space would be 14 instead of 108 dimensions. This low dimensionality greatly facilitates empirical implementation and numerical solution. It becomes much easier to replicate results from existing studies if necessary. An illustrative application relating to pension reform has demonstrated that the PA model is a much more powerful tool of policy analysis compared to the perpetual youth model due to Blanchard (1985) and Yaari (1965) as well as the recent extension by Gertler (1999) with two age groups. These models are popular because of analytical tractability and their simple empirical implementation. Yet they are quite limited when it comes to trace out the life-cycle implications of certain policy shocks. The PA model is in between the Auerbach-Kotlikoff and Blanchard-Yaari models, allowing for analytical aggregation to a low number of age groups combined with a realistic modeling of the life-cycle.

Appendix

A Proofs

Proposition 1 As argued in (2), the stock of people in age group $a > 1$ in period $t + 1$ consists of those who were already in group a in period t and survived in this group, plus those agents from group $a - 1$ who aged and are thus newly arriving in group a . Summing

over people by their biography α as in (4),

$$\begin{aligned}\sum_{\alpha \in \mathcal{N}_{t+1}^a} N_{\alpha,t+1}^a &= \sum_{\alpha \in \mathcal{N}_t^a} N_{\alpha,t+1}^a + \sum_{\alpha' \in \mathcal{N}_t^{a-1} \times (t+1)} N_{\alpha',t+1}^a, \\ N_{t+1}^a &= \gamma^a \omega^a N_t^a + \gamma^{a-1} (1 - \omega^{a-1}) N_t^{a-1}.\end{aligned}\tag{A.1}$$

The second line results upon substituting (3.ii-iii) on the right hand side. Since the transition probabilities are the same for all members in a given age group, and thus independent of past history, the probabilities can be moved in front of the sum operators. Applying the aggregation operator in (4) then results in the second line which is listed in (5.a). Since the last group cannot age further, the event in (3.iii) is precluded, implying the restriction $\omega^A = 1$ in (5.a).

The aggregation of group 1 in (5.b) is analogous. Replacing a by 1 in the last term of (A.1) gives $\sum_{\alpha' \in \mathcal{N}_t^0 \times (t+1)} N_{\alpha',t+1}^1$. The set of biographies $\mathcal{N}_t^0$ referring to unborns consists of an empty vector $()$ since no aging event has occurred. The Cartesian product yields a set with a single element $\alpha' = (t+1)$. Hence, the last term gives the number of newborns $N_{(t+1),t+1}^1$ flowing into group 1 which yields (5.b). Adding up (5.a) and (5.b) over all age groups and noting the restriction $\omega^A = 1$ yields (5.c).

Proposition 2 To solve the dynamic programming problem (8) subject to (6), define shadow prices of assets

$$\eta_{\alpha,t}^a \equiv \frac{dV_{\alpha,t}^a}{dA_{\alpha,t}^a}, \quad \bar{\eta}_{\alpha,t+1}^a \equiv \omega^a \cdot \frac{dV_{\alpha,t+1}^a}{dA_{\alpha,t+1}^a} + (1 - \omega^a) \cdot \frac{dV_{\alpha',t+1}^{a+1}}{dA_{\alpha',t+1}^{a+1}}.\tag{A.2}$$

The expected shadow price of next period's assets, $\bar{\eta}_{\alpha,t+1}^a = \omega^a \eta_{\alpha,t+1}^a + (1 - \omega^a) \eta_{\alpha',t+1}^{a+1}$ reflects the possibility of aging. The optimality condition for consumption is $(C_{\alpha,t}^a)^{\rho-1} = (\bar{V}_{\alpha,t+1}^a)^{\rho-1} \beta R_{t+1} \bar{\eta}_{\alpha,t+1}^a$. Calculating the envelope condition and using it to rewrite the optimality condition yields

$$\eta_{\alpha,t}^a = (V_{\alpha,t}^a / C_{\alpha,t}^a)^{1-\rho}, \quad \eta_{\alpha,t}^a = (V_{\alpha,t}^a / \bar{V}_{\alpha,t+1}^a)^{1-\rho} \cdot \beta R_{t+1} \cdot \bar{\eta}_{\alpha,t+1}^a.\tag{A.3}$$

The shadow price measures the marginal utility of current consumption, $dV_{\alpha,t}^a / dC_{\alpha,t}^a = (V_{\alpha,t}^a / C_{\alpha,t}^a)^{1-\rho} = \eta_{\alpha,t}^a$, which follows from taking the derivative of (8). The marginal

utility gain of one Euro tomorrow, in units of current consumption, is $dV_{\alpha,t}^a/dA_{\alpha,t+1}^a = (V_{\alpha,t}^a/\bar{V}_{\alpha,t+1}^a)^{1-\rho} \gamma^a \beta \bar{\eta}_{\alpha,t+1}^a$. An Euro saved today yields R_{t+1}/γ^a Euros tomorrow. The consumer balances the marginal welfare gains and losses from postponing consumption such that $dV_{\alpha,t}^a/dC_{\alpha,t}^a = (R_{t+1}/\gamma^a) dV_{\alpha,t}^a/dA_{\alpha,t+1}^a$. Substituting the derivatives established above yields (A.3).

The necessary conditions in (A.3) lead to a modified Euler equation of consumption growth. When the consumer saves today, she disposes of the same amount tomorrow, irrespective of whether she remains in the current age state or switches to the next one, see (6). However, since life-cycle characteristics change when an agent moves from age state a to $a+1$, an extra Euro will be valued differently in the two states. The consumer's relative welfare gain from an extra Euro in the two alternative states is expressed by the marginal rate of substitution,¹⁰ $(\partial V^{a+1}/\partial A^{a+1})/(\partial V^a/\partial A^a) = \eta^{a+1}/\eta^a$. The expected shadow price of assets next period thus emerges as $\bar{\eta}^a = \Omega^a \eta^a$ where $\Omega^a = [\omega^a + (1 - \omega^a)(\eta^{a+1}/\eta^a)] \eta^a$. Using (A.3) yields $\eta^{a+1}/\eta^a = (\Lambda^a)^{1-\rho}$ and

$$\Omega^a \equiv \omega^a + (1 - \omega^a)(\Lambda^a)^{1-\rho}, \quad \Lambda^a \equiv \frac{V^{a+1}/C^{a+1}}{V^a/C^a}. \quad (\text{A.4})$$

Replacing η^a by (A.3) again gives an expected marginal welfare gain next period equal to $\bar{\eta}^a = \Omega^a \eta^a = \Omega^a (V^a/C^a)^{1-\rho}$. Next, rewrite expected utility in (7), using the definition of Λ^a , and get $\bar{V}^a = [\omega^a C^a + (1 - \omega^a) \Lambda^a C^{a+1}] (V^a/C^a)$. The envelope condition thus results in a modified Euler equation of consumption growth where $\sigma = 1/(1 - \rho)$:

$$\omega^a C_{\alpha,t+1}^a + (1 - \omega^a) \Lambda_{\alpha,t+1}^a C_{\alpha',t+1}^{a+1} = (\beta R_{t+1} \Omega_{\alpha,t+1}^a)^{\sigma} \cdot C_{\alpha,t}^a. \quad (\text{A.5})$$

The next step is to show that the consumption function stated in (9) fulfills the Euler equation and is thus the optimal policy. Note that (9.ii) and (A.3) imply Λ^a as in (9.iv). Since (9.iii-iv) is a forward looking system that depends only on the interest rate and other factors independent of α , the terms Δ_t^a and by implication Λ_t^a and Ω_t^a are independent of α . Insert (9.i) into the left hand side of (A.5), use $A_{\alpha,t+1}^a = A_{\alpha',t+1}^{a+1}$, collect terms, note the

¹⁰To save notation, we sometimes suppress the lower indices α and t in the text.

definitions of Ω_{t+1}^a and $\bar{H}_{\alpha,t+1}^a$, and get $[\Omega_{t+1}^a A_{\alpha,t+1}^a + \bar{H}_{\alpha,t+1}^a] / \Delta_{t+1}^a = (\beta \Omega_{t+1}^a R_{t+1})^\sigma \cdot C_{\alpha,t}^a$. Multiply by $\gamma^a \Delta_{t+1}^a / (\Omega_{t+1}^a R_{t+1})$ and use (9.v) and (6) on the left side: $A_{\alpha,t}^a + H_{\alpha,t}^a - C_{\alpha,t}^a = \gamma^a \beta^\sigma (\Omega_{t+1}^a R_{t+1})^{\sigma-1} \Delta_{t+1}^a \cdot C_{\alpha,t}^a$. Substitute again (9.i) on the left side and cancel $C_{\alpha,t}^a$. The result corresponds to (9.iii). Hence, the stated policy is optimal since it satisfies (A.5) which is merely a reformulation of the necessary conditions in (A.3).

The last step shows that (9.ii) gives indirect utility as it identically fulfills the Bellmann equation. Again, start with the Euler equation and replace all consumption terms by the appropriate version of (9.ii). Multiply the result by $(\Delta_{t+1}^a)^{1/\rho}$, use the definitions of Λ_{t+1}^a and $\bar{V}_{\alpha,t+1}^a$ and get $\bar{V}_{\alpha,t+1}^a = (\beta \Omega_{t+1}^a R_{t+1})^\sigma \cdot V_{\alpha,t}^a (\Delta_{t+1}^a / \Delta_t^a)^{1/\rho}$. Next, take the power of ρ , write $\rho\sigma = \sigma - 1$, multiply by $\gamma^a \beta$, and use (9.iii) to substitute $\gamma^a \beta^\sigma (\Omega_{t+1}^a R_{t+1})^{\sigma-1} \Delta_{t+1}^a = \Delta_t^a - 1$. The result is $\gamma^a \beta (\bar{V}_{\alpha,t+1}^a)^\rho = (1 - 1/\Delta_t^a) \cdot (V_{\alpha,t}^a)^\rho$. By (9.ii), $(V_{\alpha,t}^a)^\rho / \Delta_t^a = (C_{\alpha,t}^a)^\rho$, which is used on the right hand side. A minor rearrangement shows that the Bellmann equation (8) is identically fulfilled.

Proposition 3 Multiply (6) by $N_{\alpha,t}^a$ and sum over all biographies α ,

$$\sum_{\alpha \in \mathcal{N}_t^a} A_{\alpha,t+1}^a \gamma^a N_{\alpha,t}^a = S_t^a, \quad S_t^a \equiv R_{t+1} \cdot [A_t^a + Y_t^a - C_t^a]. \quad (\text{A.6})$$

Multiplying by ω^a leads to

$$\begin{aligned} \omega^a S_t^a &= \sum_{\alpha \in \mathcal{N}_t^a} A_{\alpha,t+1}^a N_{\alpha,t+1}^a = A_{t+1}^a - \sum_{\alpha \in \mathcal{N}_t^{a-1} \times (t+1)} A_{\alpha',t+1}^a N_{\alpha',t+1}^a \\ &= A_{t+1}^a - \gamma^{a-1} (1 - \omega^{a-1}) \sum_{\alpha \in \mathcal{N}_t^{a-1}} A_{\alpha,t+1}^{a-1} N_{\alpha,t}^{a-1}. \end{aligned} \quad (\text{A.7})$$

The first equality uses (3.ii). From all $N_{\alpha,t}^a$ people in age group a at date t , only a fraction $\gamma^a \omega^a$ survives and remains in the same group. Aggregation importantly rests on the assumption that the transition probabilities are independent of past aging history and are thus identical of all agents within a given age group. The next equality in (A.7) expresses the fact, in analogy to (2), that aggregate assets A_{t+1}^a of group a next period consist of two parts. The first part stems from all those who were already in group a in period t . The second part consists of the inflowing assets of all those who were in age group $a-1$ in period t , were hit by an aging event and are now part of group a . The subset flowing into age

group a is the set of biographies which figure $t+1$ in the last entry, $\mathcal{N}_t^{a-1} \times (t+1) \subset \mathcal{N}_{t+1}^a$. The last equality looks more precisely at the newcomers in group a . Corresponding to case (3.iii), the mass of newcomers is $N_{\alpha', t+1}^a = \gamma^{a-1} (1 - \omega^{a-1}) N_{\alpha, t}^{a-1}$. Since all agents in group $a-1$ have a chance of moving to state a , the second line sums over the entire set $\mathcal{N}_t^{a-1}$. Each of the movers has assets $A_{\alpha', t+1}^a = A_{\alpha, t+1}^{a-1}$. Substituting $A_{\alpha, t+1}^{a-1} = S_{\alpha, t}^{a-1} / \gamma^{a-1}$ in the second line and performing the aggregation, one finds $\sum_{\alpha \in \mathcal{N}_t^{a-1}} A_{\alpha, t+1}^{a-1} N_{\alpha, t}^{a-1} = S_t^{a-1} / \gamma^{a-1}$. The aggregation in (A.6) finally yields $\omega^a S_t^a = A_{t+1}^a - (1 - \omega^{a-1}) S_t^{a-1}$ which corresponds to (12.b). The newborns entering the first group are bare of any assets since unborn agents in group 0 cannot save by definition, leading to (12.a). Finally, take the sum of (12.b) over all groups and use the restriction $\omega^A = 1$ to arrive at (12.c).

Proposition 4 To compare with the PA model, we solve for the consumption function in the Auerbach Kotlikoff model where aging occurs every period. The upper index a may thus be suppressed. Age $a = t+1-s$ is equal to time since birth. Given maximum life-time A , the last period of life occurs at date $T = s+A-1$. An agent born at date $s \leq t$ saves and accumulates assets $A_{s, t+1} = R_{t+1} [A_{s, t} + y_{s, t} - C_{s, t}]$. Welfare in recursive form is $V_{s, t} = u(C_{s, t}) + \beta V_{s, t+1}$. Solving forward yields life-time utility and the intertemporal budget constraints. Defining human wealth by $H_{s, t} \equiv \sum_{\tau=t}^T y_{s, \tau} \prod_{u=t+1}^{\tau} \frac{1}{R_u}$, the optimization problem for choosing consumption over the remaining life-time is

$$V_{s, t} = \max_{C_{s, \tau}} \sum_{\tau=t}^T u(C_{s, \tau}) \prod_{u=t+1}^{\tau} \beta + \eta_{s, t} \cdot \left[A_{s, t} + H_{s, t} - \sum_{\tau=t}^T C_{s, \tau} \prod_{u=t+1}^{\tau} \frac{1}{R_u} \right]. \quad (\text{A.8})$$

Using the iso-elastic specification $u(C) = C^\rho$ with $\rho = 1 - 1/\sigma$, the first order conditions yield the Euler equation for consumption growth, $C_{s, \tau+1} = (\beta R_{\tau+1})^\sigma C_{s, \tau}$. Solving forward results in $C_{s, \tau} = C_{s, t} \cdot \prod_{u=t+1}^{\tau} (\beta R_u)^\sigma$. Substituting this in the intertemporal budget constraint yields the consumption function $C_{s, t} = (A_{s, t} + H_{s, t}) / \Delta_{s, t}$, where the inverse of the marginal propensity to consume is

$$\Delta_{s, t} = \sum_{\tau=t}^T \prod_{u=t+1}^{\tau} \beta^\sigma (R_u)^{\sigma-1}, \quad \Delta_{s, t} = 1 + \beta^\sigma (R_{t+1})^{\sigma-1} \cdot \Delta_{s, t+1}. \quad (\text{A.9})$$

Since the product operator satisfies $\prod_{u=t+1}^t x_u \equiv 1$, we have $\Delta_{s, T} = 1$.

B Endogenous Labor Supply and Retirement

Consider age group $a = a_R$ with characteristics of people in their sixties. Assume that a fraction δ of this group is still active while the rest is retired. To retain symmetry, we assume that each agent receives a share δ of her labor income as wages and a share $1 - \delta$ as an old age pension. A higher value of δ means postponed retirement which reflects the extensive margin of labor supply. We refer to δ as the retirement date since postponed retirement raises the share of wages in average labor income in that period. If actively employed, agents may work a variable number of l^a hours, reflecting the intensive margin. They trade off income and consumption against disutility of work. A particularly simple and yet realistic approach is to exclude income effects.¹¹ We thus assume an additively separable subutility over consumption and foregone leisure, $\bar{C}^a \equiv C^a - \delta\varphi(l^a) - \phi(\delta)$, where $\varphi(l)$ and $\phi(\delta)$ stand for convex increasing disutility of work and postponed retirement.¹² Appropriately expanding (6), (8) and (13) yields

$$\begin{aligned}
(i) \quad V(A_{\alpha,t}^a) &= \max_{C_{\alpha,t}^a, l_{\alpha,t}^a, \delta_{\alpha,t}^a} [(\bar{C}_{\alpha,t}^a)^\rho + \gamma^a \beta (\bar{V}_{\alpha,t+1}^a)^\rho]^{1/\rho} \quad s.t. \\
(ii) \quad \gamma^a A_{\alpha,t+1}^a &= R_{t+1} [A_{\alpha,t}^a + \bar{y}_{\alpha,t}^a - \bar{C}_{\alpha,t}^a], \\
(iii) \quad \bar{C}_{\alpha,t}^a &= C_{\alpha,t}^a - \delta_{\alpha,t} \varphi(l_{\alpha,t}^a) - \phi(\delta_{\alpha,t}), \\
(iv) \quad \bar{y}_{\alpha,t}^a &= \max_{l_{\alpha,t}^a, \delta_{\alpha,t}} \delta_{\alpha,t} [w_t^a l_{\alpha,t}^a - \varphi(l_{\alpha,t}^a)] + (1 - \delta_{\alpha,t}) p_t - \phi(\delta_{\alpha,t}), \\
(v) \quad y_{\alpha,t}^a &= \delta_{\alpha,t} w_t^a l_{\alpha,t}^a + (1 - \delta_{\alpha,t}) e_t, \quad w_t^a \equiv (1 - \tau) w_t \theta_t^a.
\end{aligned} \tag{B.1}$$

A simple solution in three stages is possible. First, choose intensive and extensive labor supply to maximize effort adjusted wage income in (B.1.iv), yielding

$$w_t^a = \varphi'(l_t^a), \quad w_t^a l_t^a - \varphi(l_t^a) - p_t = \phi'(\delta_t). \tag{B.2}$$

Intensive labor supply exclusively depends on the current wage. On the extensive margin, the chosen retirement date reflects the difference between utility adjusted wages and

¹¹Excluding income effects is quite common in the literature on optimal income taxation or on real business cycles. See, for example, Heijdra (1998) and Greenwood et al. (1988) in intertemporal macroeconomics and Saez (2002) and Immervoll et al. (2004) on optimal income taxation.

¹²See Cremer and Pestieau (2003) for modeling postponed retirement this way.

alternative pension income. The more generous pensions are relative to net wages, the less people are inclined to incur the utility cost of postponed retirement, and the earlier they choose to retire. The share of wages in average labor income declines. Since wages and pensions are assumed independent of history, labor supply and the retirement date are symmetric, $l_{\alpha,t}^a = l_t^a$ and $\delta_{\alpha,t} = \delta_t$, implying symmetry of y_t^a and $\bar{y}_t^a$ as well.

The second step solves the intertemporal problem in (B.1.i-ii) of allocating subutility $\bar{C}_{\alpha,t}^a$ over time exactly as in Proposition 2, except that C is replaced by $\bar{C}$ and y by $\bar{y}$. The solution of the first two steps gives optimal values for $\bar{C}$, l and δ . As a last step, it remains to compute optimal commodity consumption $C_{\alpha,t}^a$ by inverting (B.1.iii).

For groups $a < a^R$, labor supply is reduced to the intensive dimension, i.e. $\delta = 1$ in (B.1) and ϕ is a constant that is set to zero. Finally, groups $a > a^R$ are fully retired. Endogenous retirement and hours worked yield aggregate labor supply and an endogenous decomposition of population into workers and retirees,

$$L_t^S = \theta^{a^R} l_t^{a^R} \cdot \delta_t N_t^{a^R} + \sum_{a=1}^{a^R-1} \theta^a l_t^a N_t^a, \quad N_t^R = (1 - \delta_t) N_t^{a^R} + \sum_{a=a^R+1}^A N_t^a. \quad (\text{B.3})$$

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