Perfect One-Factorizations and Array Codes

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Joint work with Barbara Maenhaut.

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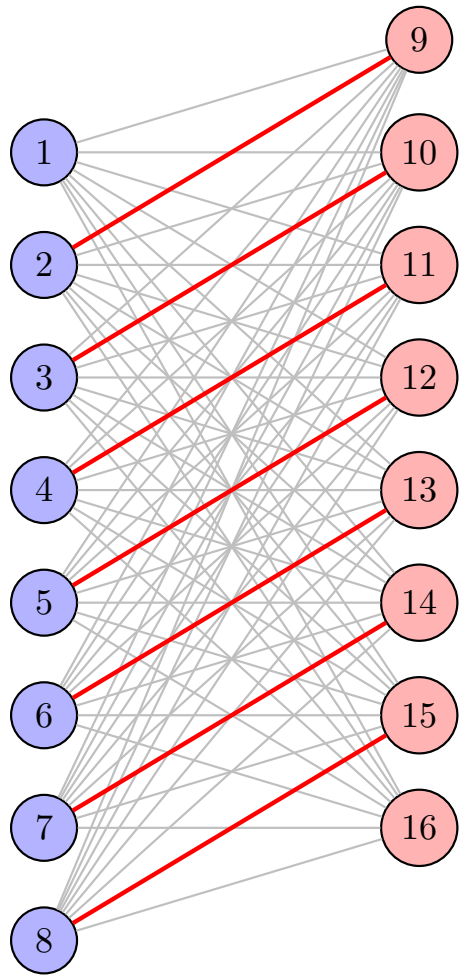


Perfect One-Factorizations

Definition

- A *1-factor* of a graph $G = (V, E)$ is a spanning subgraph in which every vertex has degree 1; equivalently, it is a perfect matching.
- A *1-factorization* of a k-regular graph G is a decomposition of the edge set into disjoint 1-factors

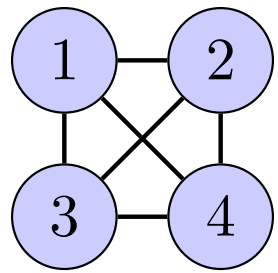
$$E = F_1 \dot\cup F_2 \dot\cup \cdots \dot\cup F_k.$$

- A 1-factorization is called a *perfect 1-factorization (P1F)* if for every pair F_i, F_j with $i \neq j$, the union

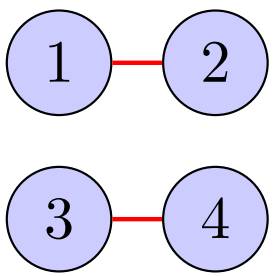
$$F_i \cup F_j$$

forms a Hamilton cycle of G.

P1F of K_4



F_1:



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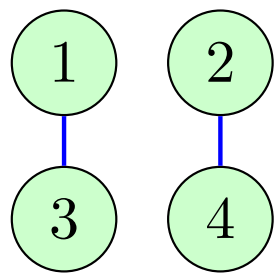

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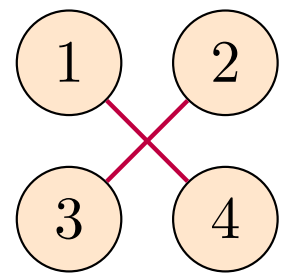


Known Existence Results

- A P1F of K_m can exist only if m is even.
- Conjecture: every K_{2n} admits a perfect 1-factorization.
- A P1F of $K_{n,n}$ can exist only if n is odd.

Theorem

1. *There are infinitely many odd primes p for which K_{2p} admits a P1F.*
2. *If p is an odd prime, then K_{p+1} admits a P1F.*
3. *If p is an odd prime, then $K_{p,p}$ and K_{p^2,p^2} admit a P1F.*

Array Codes and B-Codes

- Array codes $\cong$
  $\mathbb{F}_q$-linear codes in $\mathbb{F}_{q^m}^n$ (represented as $m \times n$ matrices)
- For generator/parity check matrices, we represent them in $\mathbb{F}_q^{mn}$:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \mapsto (10 \mid 11 \mid 01)$$

Definition (B-code)

A *B-code* is a binary linear array code whose parity-check matrix has constant (small) row weights with respect to the array structure.

This regular sparsity ensures that updating one information symbol changes only a bounded number of symbols in the encoded array (low update complexity).

Known Correspondence P1F$\leftrightarrow K_m$

Theorem

There is a 1-to-1 correspondence between

- *P1Fs of* K_{2n+2} *and*
- $n \times (2n+1)$ *B-codes of dimension* $2n-1$ *and minimum (array) distance* 3 *(MDS array codes) with update complexity* $2n$*.*

New Correspondence P1F$\leftrightarrow K_{m,m}$

Theorem

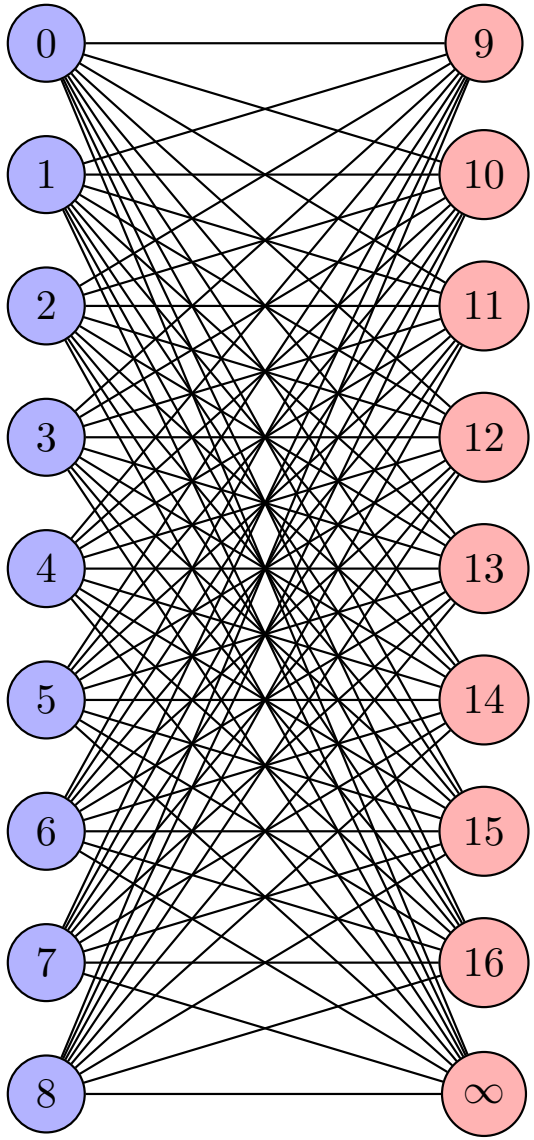
There is a 1-to-1 correspondence between

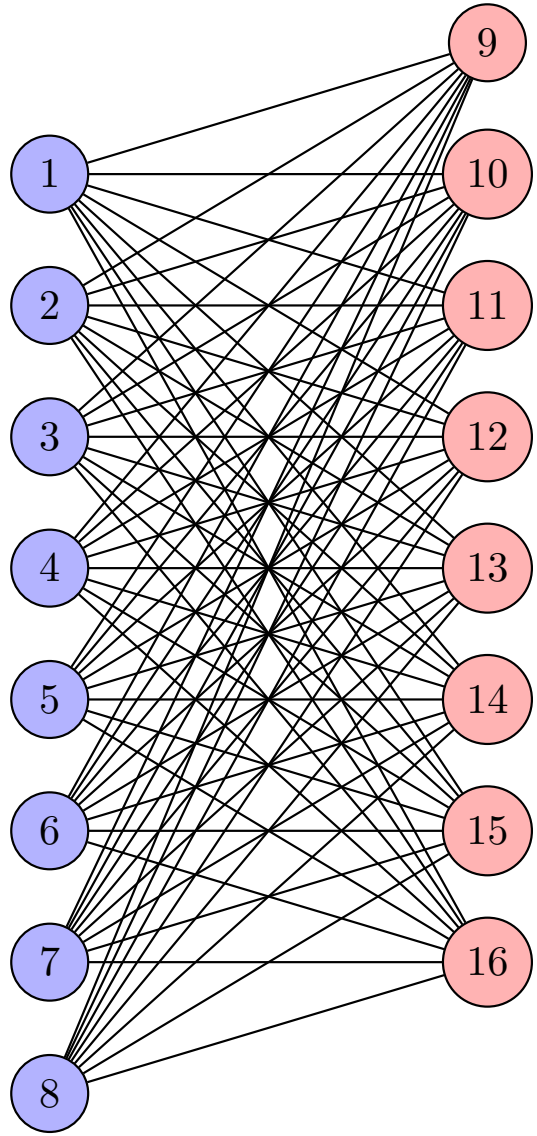
- *P1Fs of* $K_{n+1,n+1}$ *and*
- $n \times (n+2)$ *B-codes of dimension* $2n$ *and minimum (array) distance* 3 *(MDS array codes) with update complexity* $n+1$*, if there exists a parity check matrix such that every column of weight* 2 *has exactly one* 1 *in the top and one in the bottom half.*

Constructing $n \times (n+2)$ B-codes from P1F of $K_{n+1,n+1}$

1. Label nodes on the left with $0, 1, \ldots, n$ and on the right with $n+1, \ldots, 2n, \infty$
2. Let F_i denote the one-factor that contains the edge $\{\infty, i\}$.
3. Delete the two vertices 0 and ∞ and all the edges that are incident to these two vertices.
4. Construct a parity check matrix with $n+1$ blocks, where each block corresponds to a factor F_i. In the ith block write the first column vectors as the ith unit vector, for $i = 1, \ldots, n$. The other column vectors are the incidence vectors of the remaining edges in F_i.
5. Extend H by an additional block, consisting of $(0_{n \times n} \mid I_n)^\top$.

Example $K_{9,9}$

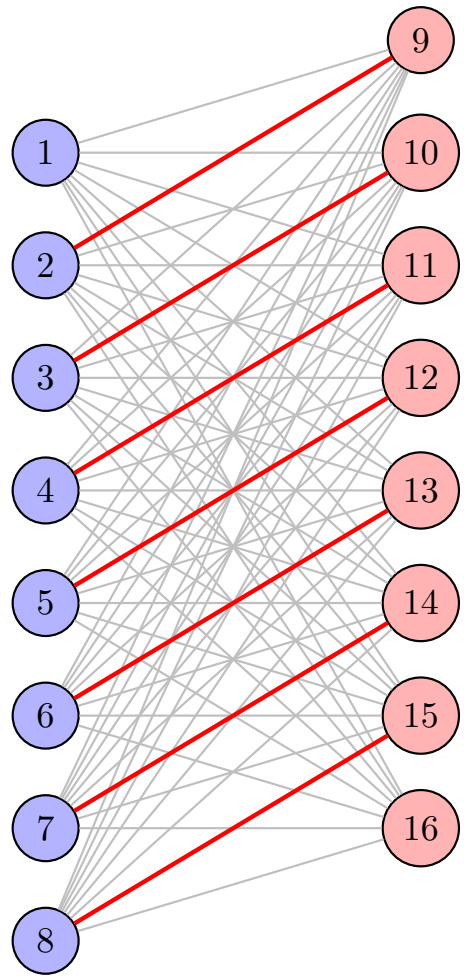




Example $K_{9,9}$

$$F_1 = \{\{2,9\},\{3,10\},\{4,11\},\{5,12\},\{6,13\},\{7,14\},\{8,15\}\} \cup \underbrace{\{\{1,\infty\},\{0,16\}\}}_{\text{erased}}$$

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



Constructing a P1F of $K_{n+1,n+1}$ from a B-code

Let H be a parity check matrix of the code fulfilling the condition from before.

1. Let F_i denote the set of edges with the label $i = 1, 2, ..., n, \infty$.
2. Add vertex 0 to the first part (the vertices $1, 2, \ldots, n$) and vertex ∞ to the second part (the vertices $n+1, \ldots, 2n$) of the graph.
3. For $i = 1, 2, ..., n$, add the two edges $\{0, i\}$ and $\{k_i, \infty\}$ to F_i, where k_i is an integer from 1 to n such that the expanded set F_i is a one-factor of the bipartite graph. Add $\{0, \infty\}$ to F_0.

Theorem

The graph constructed in Algorithm 2 is a bipartite graph $K_{n+1,n+1}$ and the sets F_i form a P1F of it.

Comparison of Parameters

Parameter	K_{2n+2}	$K_{n+1,n+1}$
Array size	$n \times (2n+1)$	$n \times (n+2)$
Block length (array height)	n	n
Number of blocks	$2n+1$	$n+2$
Binary length of H	$2n^2+n$	n^2+2n
Co-dimension (height of H)	$2n$	$2n$
Fixed row weight of H	$2n$	$n+1$
Minimum distance	3	3
Cardinality	$2^{n(2n-1)}$	2^{n^2}

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- Some parameters on the right can also be achieved by puncturing the codes from K_{2n+2}.
- However, P1Fs for $K_{n+1,n+1}$ are known, that lead to codes that are not achievable from known P1Fs of K_{2n+2}.
- This corresponds to P1Fs of K_{2n+2} that contain P1Fs of $K_{n+1,n+1}$.

Length Optimality

- Since every block of H contains only weight-1 or all-zero vectors, the array code can be represented by a normal linear (vector) code over an alphabet of size $n+1$ (if this size allows for the concept of linearity[1]).
- The resulting vector code's length is then the maximal possible length given by the MDS conjecture, i.e., the length is one more than the underlying alphabet size.

$$\implies \text{ our codes are length-optimal}$$

[1] which is the case for $n+1 = p^2$

Open Questions

- Extend the B-codes to get new P1Fs?
- Non-existence results?
- Generalization to $d > 3$?

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Thank you for your attention!
Questions? – Ideas?

