School of Finance

University of St.Gallen

Estimation and Application of Fully Parametric Multifactor Quantile Regression with Dynamic Coefficients

Florentina Paraschiv
Derek W. Bunn
Sjur Westgaard

Working Papers on Finance No. 2016/07

Institute of Operations Research and Computational Finance (ior/cf – HSG)

March 2016

Estimation and application of fully parametric multifactor quantile regression with dynamic coefficients

Florentina Paraschiv* Derek Bunn† Sjur Westgaard‡

March 3, 2016

Abstract

This paper develops and applies a novel estimation procedure for quantile regressions with time-varying coefficients based on a fully parametric, multifactor specification. The algorithm recursively filters the multifactor dynamic coefficients with a Kalman filter and parameters are estimated by maximum likelihood. The likelihood function is built on the Skewed-Laplace assumption. In order to eliminate the non-differentiability of the likelihood function, it is reformulated into a non-linear optimisation problem with constraints. A relaxed problem is obtained by moving the constraints into the objective, which is then solved numerically with the Augmented Lagrangian Method. In the context of an application to electricity prices, the results show the importance of modelling the time-varying features and the explicit multi-factor representation of the latent coefficients is consistent with an intuitive understanding of the complex price formation processes involving fundamentals, policy instruments and participant conduct.

JEL Classification: *C01*; *C13*; *C22*
Keywords: *Quantile Regression*; *Dynamic Coefficients*; *Parametric Estimation*; *Electricity Prices*

*University of St.Gallen, Institute for Operations Research and Computational Finance, Bodanstrasse 6, CH-9000 St. Gallen, Switzerland, florentina.paraschiv@unisg.ch, funded by the Swiss Federal Office of Energy SFOE, Research Grant Energy-Economy-Society (EWG).

†London Business School, Department of Management Science and Operations, Sussex Place, Regent's Park, NW1 4SA, London, UK.

‡Department of Industrial Economics and Technology Management, Norwegian University of Science and Technology, Alfred Getz vei 3, 7041 Trondheim, Norway.

1 Introduction

The estimation of price quantiles in financial, commodity and product markets has attracted substantial and increasing methodological research not only because of its technical challenges but also as a practical consequence of risk management and regulatory compliance procedures that require explicit controls at specific probability levels. Whilst tail probabilities in time series densities typically pose problems of robust estimation and therefore require substantial empirical histories, the underlying price formations in many markets often have evolutionary properties as technology, market structure and participant conduct change over time, and so a crucial technical task is the estimation of extreme quantiles that have dynamic dependences upon fundamental factors. In its fully specified form, this dynamic representation is an under-researched topic.

Various researchers have developed quantile models with, to some extent, dynamic characteristics. These include the use of a lagged dependent variable (ie quantile) to model dynamic adaptation, or a dependence upon exogenous factors which induce dynamic quantiles through the evolution of the factors, or via a stochastic representation of the time series heteroscedasticity, eg by means of a GARCH feature. None of these approaches estimate time varying quantile coefficients upon fundamental factors, and the contribution of this paper is therefore motivated by these requirements. We suggest that this context represents an important class of problems in practice and provide a detailed application to electricity prices in which the quantile coefficients change dynamically within the day as demand, wind and solar variables fluctuate, as well as monthly as gas and coal prices change, and annually as underlying demand and supply characteristics evolve. The Maximum Likelihood estimation of these multi-factor quantile coefficients is awkward, being nonlinear and nondifferentiable, and we develop a new ML estimation process for this purpose based upon nonlinear optimisation solved by the augmented Lagrangian method.

The conventional quantile regression formulation usually proceeds as follows. Given a set of T observations, $y_t, t = 1, ..., T$, define $q(\alpha)$ as the α-th quantile. The probability that an observation is less than $q(\alpha)$ is α, where $0 < \alpha < 1$. In quantile regression, the quantile $q_t(\alpha)$ corresponding to the t-th observation is a linear function of explanatory variables, x_t, that is $q_t = x_t'\beta$. The quantile regression estimates are obtained by minimising $\sum_t \rho_\alpha(y_t - x_t'\beta)$ with respect to the parameter vector β (see Rossi & Harvey (2006)). Different weights ρ_α are applied, depending on whether y_t is to the left or to the right of $q_t(\alpha)$. Estimates computed by linear programming, following Koenker & Bassett (1978), provide a semi-parametric approach as no distributional assumptions are required. However parametric methods that make distributional assumptions at each quantile level can provide alternative estimators, are analytically attractive and facilitate extended specifications to include, for example, dynamic characteristics.

Thus, Rossi & Harvey (2006) formulate a dynamic model with autoregressive quantiles Q_t where the likelihood function at each quantile is built on the assumption of an asymmetric double exponential distribution for the observations. The authors show that maximising the log-likelihood function is equivalent to minimising the criterion function in the linear programming approach. Other parametric formulations have used a Skewed-Laplace (SL) distribution for the observations to achieve a similar equivalence, eg Yu & Moyeed (2001), Tsionas (2003) and Gerlach et al. (2011). In the latter case the dynamic quantiles result from the inclusion of time-dependent volatility. Where exogenous fundamental factors are known to influence price formation, it is often the case that the autocorrelation apparently present in the time series is better reflected in the price adaption process to market fundamentals (electricity prices are a good example, as in Karakatsani & Bunn (2010) and Paraschiv et al. (2014)). Hence there are benefits in formulating dynamic quantiles with time-varying coefficients on multiple factors, and an intuitive attraction to modelling explicitly the factors that might influence market participant behaviour prior to market clearing.

Variable coefficient models for conditional quantiles applied to independent data were developed in non time series contexts by Honda (2004) and Kim (2007), using local polynomials and polynomial-splines, respectively. In our time series context, instead of predefining polynomials to describe the variation in coefficients, we model them as unobserved latent variables in a state-space formulation. Furthermore, in a multi-factor Kalman filter (Kalman (1960)) representation both the observations and latent states can be expressed as functions of exogenous fundamental variables. Quantiles can be formulated using the parametric Skewed-Laplace assumption, but maximum likelihood estimation is awkward. Facing a similar problem in estimating a T-CAViaR model, Gerlach et al. (2011) emphasize the difficulty of solving the non-linear, stepwise likelihood function based on the SL-density of observations and they used a pragmatic Bayesian specification. In order to eliminate the non-differentiability of the likelihood function caused by the indicator function, we reformulated the problem as a non-linear optimisation with constraints. We obtained a relaxed problem by moving the constraints into the objective, which is then numerically solved with the Augmented Lagrangian Method. The application to electricity prices demonstrates the value of this approach in terms of the detailed specification and the explicit, plausible transparency of the time-varying quantile drivers.

In the next section we describe the model specification and estimation process in general. Then we present a fundamental description of price formation in the German wholesale electricity market, which is the main reference for electricity trading in Europe. The results follow, with some new interpretations that reflect the complexity of fundamental interactions and advance our understanding of electricity price risk drivers, as revealed by this methodology. Section four concludes the paper.

2 Model specification

2.1 Preliminaries

The general specification of a dynamic time series quantile model is:

$$y_t = f_t(\beta, x_t) + u_t \tag{1}$$

with $t = 1, ..., n$ the time dimension; $y_t \in \mathbb{R}$ is the dependent variable; $x_t = (x_{1t}, ..., x_{pt})' \in \mathbb{R}^p$ are the explanatory variables; β is a p-vector of unknown parameters and u_t is an error term.

The conditional $\alpha \in (0, 1)$ level quantile is then

$$q_\alpha(y_t|\beta^\alpha, x_t) = f_t(\beta^\alpha, x_t) \tag{2}$$

For any $\alpha \in (0, 1)$ the distance from y_t to a given quantile level q_α is measured by the absolute distance, but different weights are applied depending on whether y_t is to the left or to the right of q_α (see Hao & Naiman (2007)). Thus, the distance from y_t to a given q_α is defined:

$$d_\alpha(y_t, q_\alpha) = \begin{cases} (\alpha - 1)|y_t - q_\alpha|, & y_t < q_\alpha \\ \alpha|y_t - q_\alpha|, & y_t \geq q_\alpha \end{cases} \tag{3}$$

We look for the value q_α that minimises the mean distance from y_t: $\mathbb{E}[d_\alpha(y_t, q_\alpha)]$. As shown in Hao & Naiman (2007) and Gerlach et al. (2011) the minimum occurs when q_α is the α quantile of y_t. q_α depends on β^α which is the solution to:

$$\min_\beta \sum_t \rho_\alpha(y_t - f_t(\beta, x_t)), \tag{4}$$

where $\rho(\cdot)$ is a loss function specified as:

$$\rho_\alpha(u_t) = u_t \cdot (\alpha - \mathbf{1}(u_t < 0)) \tag{5}$$

with $u_t = y_t - f_t(\beta, x_t)$.

Below we show the link between the minimisation problem given in Equation (4) and the likelihood function derived based on the assumption of Skewed-Laplace (SL) distributed residuals u_t. Based on the density of the Skewed Laplace distribution, the likelihood function is derived as:

$$\text{Ł}_{\alpha}(\beta, \tau; y, x) \propto \tau^{-n} \exp\left\{-\tau^{-1}\sum_{t=1}^{n}(y_t - f_t(\beta, x_t)) \times [\alpha - \mathbf{1}_{(-\infty,0)}(y_t - f_t(\beta, x_t))]\right\}. \quad (6)$$

We observe that the summation to be minimised in Equation (4) is contained in the exponent of the likelihood. Thus, the maximum likelihood estimation for β is equivalent to the quantile estimator in (4) (see Gerlach et al. (2011)).

2.2 Extension to quantile regression with time-varying coefficients

We extend the model version in Equation (1) by introducing time-varying coefficients. We therefore formulate a state space model with time-dependent coefficients recursively filtered with a Kalman filter (Kalman (1960)) and parameters estimated by maximum likelihood. The likelihood function is similar to the specification in (6) but we allow for time-varying β_t. The assumption that u_t follow a $SL(\mu, \tau, \alpha)$ distribution is used here to estimate parametrically the quantiles of the dependent variables y_t. The SL assumption is a realistic choice for distributions with fat tails.

The state space formulation reads:

$$y_t^{\alpha} = (\beta_t^{\alpha})' x_t + u_t \quad (7)$$

$$\beta_t^{\alpha} = c + D\beta_{t-1}^{\alpha} + w_t \quad (8)$$

where $u_t \sim \mathrm{SL}(0, \tau, \alpha)$, Ξ is the variance of residuals u_t and $w_t \sim \mathrm{N}(0, \Omega)$. Equation (7) is the measurement equation, which relates a known quantity (vector of exogenous variables) x_t to an observed variable y_t, while Equation (8) describes the process governing the unobserved state β_t^α and is a latent transition equation. In our model specification, we do not predefine the process driving the coefficients β_t^α as random walk, but we formulate a more flexible model to allow for mean reversion and subsequently exogenous influences on the latent states as well.

In Figure 1 an overview of the recursive algorithm for filtering the coefficients is provided, as pre-requisite for the maximisation of the likelihood function. Essentially it consists of a sequence of equations that implement a predictor-corrector type estimator for the unobserved state variables with two alternating steps: In the first *prediction step*, the Kalman filter forms an optimal predictor of the unobserved state β_t given all the information available up to time $t-1$. To this end, the state vector is extrapolated and prior estimates for time t are obtained. In the subsequent *updating step*, new information that becomes available at time t is used to update the prior estimates for β_t to obtain the posterior estimates. Below, we denote by $\beta_{t|t-1}$ the prior estimate and by $\beta_{t|t}$ the posterior estimate for the state β_t (here the specific quantile level α is skipped in the notation for simplicity).

The likelihood function is built on the prediction errors $u_t := y_t - f_t(\beta_t, x_t)$, where $f_t(\beta_t, x_t) := (\beta_{t|t-1})' x_t$ due to the linear model structure and $\beta_{t|t-1}$ are the prior estimates of the time-varying coefficients. There are several challenges for the maximisation procedure to obtain the optimal likelihood value and parameter estimates. Firstly, in each iteration of the optimisation procedure the likelihood function is updated with estimates of the time dependent state β_t which requires a new run through the Kalman filter and enhances the complexity of the whole procedure. Another difficulty comes from the shape of the likelihood function which is *piecewise* (V-shape) and *nonlinear* due to the indicator function.

The advantage of solving quantile regressions by a fully parametric approach is that by filtering the coefficients with the Kalman Filter, this is generally considered to be robust with respect to the variance and distribution of residuals. However, one important consideration is that in this approach, each quantile is independently estimated. Whilst it might be thought there should be a monotonic ordering constraint across quantiles, it is not necessarily the case that there should be a monotonic relationship across the quantile coefficients for each factor. We did not impose any ordering constraints and return to this point when discussing the results of the application to the electricity prices.

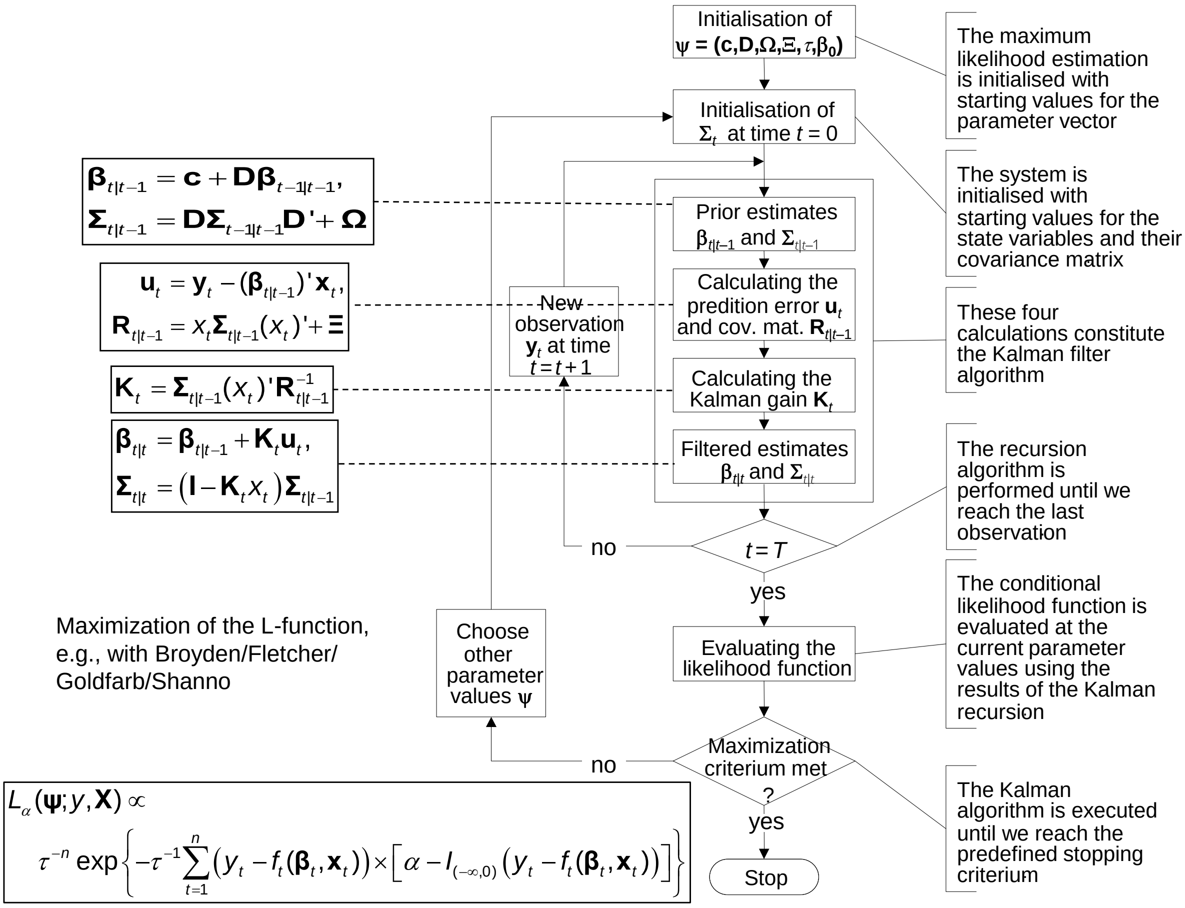
2.3 Estimation procedure

As outlined in the above subsection, solving the non-linear, stepwise likelihood function based on the SL-density of observations poses several challenges for the optimisation. Since the function is piecewise, we proceed with the elimination of the non-differentiability of the likelihood function caused by the indicator function. We further reformulate the problem as a non-linear optimisation with constraints and obtain a relaxed problem by moving the constraints into the objective, which is then solved numerically with the Augmented Lagrangian Method.

Recall from the Kalman filter that the definition of the prediction error based on the prior estimates is:

$$u_t := y_t - (\beta_{t|t-1})' x_t \tag{9}$$

Given the vector of unknown parameters $\Psi = (c, D, \Omega, \Xi, \tau, \beta_0)$, the logarithm of the likelihood function can be written as:



$$\boldsymbol{\beta}_{t|t-1} = \mathbf{c} + \mathbf{D}\boldsymbol{\beta}_{t-1|t-1}, \quad \boldsymbol{\Sigma}_{t|t-1} = \mathbf{D}\boldsymbol{\Sigma}_{t-1|t-1}\mathbf{D}' + \boldsymbol{\Omega}$$

$$\mathbf{u}_t = \mathbf{y}_t - (\boldsymbol{\beta}_{t|t-1})'\mathbf{x}_t, \quad \mathbf{R}_{t|t-1} = x_t\boldsymbol{\Sigma}_{t|t-1}(x_t)' + \boldsymbol{\Xi}$$

$$\mathbf{K}_t = \boldsymbol{\Sigma}_{t|t-1}(x_t)'\mathbf{R}_{t|t-1}^{-1}$$

$$\boldsymbol{\beta}_{t|t} = \boldsymbol{\beta}_{t|t-1} + \mathbf{K}_t\mathbf{u}_t, \quad \boldsymbol{\Sigma}_{t|t} = \left(\mathbf{I} - \mathbf{K}_t x_t\right)\boldsymbol{\Sigma}_{t|t-1}$$

$$L_\alpha(\boldsymbol{\psi}; y, \mathbf{X}) \propto \tau^{-n}\exp\left\{-\tau^{-1}\sum_{t=1}^{n}\left(y_t - f_t(\boldsymbol{\beta}_t, \mathbf{x}_t)\right)\times\left[\alpha - I_{(-\infty,0)}\left(y_t - f_t(\boldsymbol{\beta}_t, \mathbf{x}_t)\right)\right]\right\}$$

Figure 1: Algorithm estimation procedure

$$
\begin{aligned}
\ln \mathrm{L}_\alpha(\Psi; y, x) &\propto -n \ln \tau - \frac{1}{\tau} \sum_{t=1}^{n} u_t \times \left[\alpha - \mathbf{1}_{(-\infty,0)}(u_t)\right] \\
&= -n \ln \tau - \frac{1}{\tau} \left[\alpha \sum_{t=1}^{n} \{u_t | u_t \geq 0\} + (\alpha - 1) \sum_{t=1}^{n} \{u_t | u_t < 0\}\right] \\
&= -n \ln \tau - \frac{1}{\tau} \left[\alpha \sum_{t=1}^{n} \{u_t | u_t \geq 0\} + (1 - \alpha) \sum_{t=1}^{n} \{-u_t | u_t < 0\}\right] \\
&= -n \ln \tau - \frac{\alpha}{\tau} \sum_{t=1}^{n} v_t^1 - \frac{(1-\alpha)}{\tau} \sum_{t=1}^{n} v_t^2 \qquad (10)
\end{aligned}
$$

where $v_t^1 := \max\{u_t, 0\}$, $v_t^2 := \max\{-u_t, 0\}$ are introduced as linear constraints in the optimisation problem. Thus, the maximisation of the log-likelihood function (LF) is equivalent to a non-linear constrained problem (due to the indicator function in the original formulation of LF). Writing this optimisation problem in the standard formulation as non-linear minimisation problem with linear constraints leads to:

$$
\min n \ln \tau + \frac{\alpha}{\tau} \sum_{t=1}^{n} v_t^1 + \frac{(1-\alpha)}{\tau} \sum_{t=1}^{n} v_t^2 \qquad (11)
$$

$$
\begin{aligned}
\text{s.t.} \quad & v_t^1 \geq y_t - (\beta_{t|t-1})' x_t, && t = 1, ..., n \\
& v_t^2 \geq (\beta_{t|t-1})' x_t - y_t, && t = 1, ..., n \\
& v_t^1, v_t^2 \geq 0, && t = 1, ..., n
\end{aligned}
$$

However, the formulation as defined in (11) is not ready to be passed to a solver, since the coefficients $\beta_{t|t-1}$ are not constant. Instead, the time-varying coefficients are functions of the parameter vector Ψ that should be estimated, i.e., $\beta_{t|t-1} := \beta_{t|t-1}(\Psi)$. As a consequence, the estimates $\beta_{t|t-1}$ for $t = 1, ..., n$, which are now part of the constraints, must always be updated with the Kalman filter when the parameter vector changes in one iteration of the optimisation algorithm before the log-likelihood function (i.e., the objective) is evaluated. For that reason, we integrate the constraints into the objective function. In a first step, we rewrite the optimisation problem in 11 as one with equality constraints:

$$\min n \ln \tau + \frac{\alpha}{\tau} \sum_{t=1}^{n} v_t^1 + \frac{(1-\alpha)}{\tau} \sum_{t=1}^{n} v_t^2 \tag{12}$$

$$\begin{aligned} \text{s.t.} \quad & -v_t^1 + v_t^2 + y_t - (\beta_{t|t-1})' x_t = 0, && t = 1, ..., n \\ & v_t^1, v_t^2 \geq 0, && t = 1, ..., n \end{aligned}$$

Now we define $\widetilde{\Psi}$ as a new parameter vector into which also the auxiliary variables $v_1^i, ..., v_n^i$, $i \in \{1, 2\}$, are integrated and

$$\begin{aligned} f(\widetilde{\Psi}) &:= n \ln \tau + \frac{\alpha}{\tau} \sum_{t=1}^{n} v_t^1 + \frac{(1-\alpha)}{\tau} \sum_{t=1}^{n} v_t^2 \\ c_t(\widetilde{\Psi}) &:= -v_t^1 + v_2^1 + y_t - \left(\beta_{t|t}(\widetilde{\Psi})\right)' x_t \end{aligned}$$

so that the problem can be rewritten shortly as

$$\min_{\widetilde{\Psi}} f(\widetilde{\Psi}) \quad \text{s.t.} \quad c_t(\widetilde{\Psi}) = 0; \quad v_t^1, v_t^2 \geq 0, \quad t = 1, ..., n. \tag{13}$$

By moving the equality constraints into the objective as a penalty term, we can solve the problem with the Augmented Lagrangian Method (e.g., see Noecdal/Wright, 2006, ch 17). The objective of the relaxed problem reads as:

$$\Phi(\widetilde{\Psi}, \Lambda; \mu) := f(\widetilde{\Psi}) + \frac{\mu_k}{2} \sum_{t=1}^{n} \left(c_t(\widetilde{\Psi})\right)^2 - \sum_{t=1}^{n} \lambda_t c_t(\widetilde{\Psi}) \tag{14}$$

where μ_k is a penalty coefficient and $\lambda_1, \ldots, \lambda_n$ are Lagrange multipliers. The relaxed problem (14), together with the bounds $v_t^1, v_t^2 \geq 0$, $t = 1, \ldots, n$, is solved first with a small penalty μ_0 (we used $\mu_0 = 1$) and initial values for the model parameters Ψ_0. The latter imply a sequence of prior estimates for the state variables $\beta_{t|t-1}$ from a cycle of the Kalman filter, then initial values for the auxiliary variables v_t^1, v_t^2 in the extended parameter vector $\widetilde{\Psi}_0$ can be set to $v_t^1 := \max\{u_t, 0\}$, $v_t^2 := \max\{-u_t, 0\}$, where u_t is defined in (9).

Efficient algorithms exist for the numerical optimisation of bound-constrained non-linear problems. Thus, we used the L-BFGS-B algorithm (see Zhu et al. (1997)). After a solution $\widetilde{\Psi}_1$ of the problem was found, the penalty parameter μ_k is increased and the corresponding problem is solved in subsequent iterations $k = 1, 2, \ldots$. In each iteration, the resulting solution $\widetilde{\Psi}_{k+1}$ is used as initial solution for the next iteration $k + 1$ and the variables λ_t are updated according to

$$\lambda_t = \lambda_t - \mu_{k+1} c_t(\widetilde{\Psi}_{k+1}).$$

This is repeated until in some iteration k' the absolute value of the constraint functions $c_t(\widetilde{\Psi}_{k'+1})$ than some tolerance level for all $t = 1, \ldots, n$.

We start the estimation for the 50%-quantile ($\alpha = 0.5$) on the basis that this is the most robust. The initial values for the parameters stacked in the vector Ψ are obtained from the estimation of a conventional mean regression model with time-varying coefficients as used in Paraschiv et al. (2014). The resulting parameters are then used to construct the starting values for the solution of the quantile regression model for 40% and 60%, the solution of the 40% (60%) quantile regression provides the input for the 30% (70%) model, and the algorithm proceeds iteratively in this manner.

3 Dynamic MultiFactor Electricity Price Quantiles

3.1 Price Formation Fundamentals

The European Power Exchange provides the main trading platform for electricity prices in Europe and, amongst a wide range of products, the day ahead, hourly prices for

Germany ("Phelix") are the major reference prices and the most actively researched (https://www.epexspot.com/en/market-data/dayaheadauction). Each day, an auction clears offers from generators and bids from retailers (and large users) at midday and sets separate prices for delivery in each hour of the subsequent day. Figure 2 displays a typical price clearing result of the auction for a specific hour in 2014. The supply function, representing the stack of offers for production quantities in ascending prices, distinctively increases through concave, flat and convex regions. Since electricity is produced to meet demand instantaneously, with very little storage by end-users, hourly variations in price are due to fluctuations in demand being mapped through the nonlinear supply function into prices and also through changes in the shape of the supply function itself due to availabilities of wind solar and other sources of power, as well as the pricing strategies of generators. Weather and patterns of consumer behaviour are the main drivers of demand and their periodic nature thereby maps into prices, whilst transient episodes at the more steeply convex or concave regions of the supply function induce price volatility.

Thus, both periodic and heteroscedastic components feature in statistical specifications, (eg Koopman & Cornero (2007)), as well as non Gaussian price densities and hourly co-movements (eg Panagiotelis & Smith (2008)). The distinctive bi-inflexioned shape of the supply function is a result of various generating technologies, policy support and participant conduct. The lower concave region, which often extends to negative prices, is becoming common in power markets where there is a substantial amount of renewable technologies (wind, solar, biomass) incentivised by policy support. In Germany at the time of this analysis, renewable technologies were given fixed feed-in tariffs and priority dispatch which effectively meant that whatever could be produced could be sold at a fixed price. At times of high supply has the effect of shifting the supply function to the right, and if demand is low as well, this presents difficulties to other generating facilities that might be inflexible and expect to run continuously (nuclear, district heating and industrial co-generation facilities, as well a some large coal power stations). The inflexible facilities have high shut-down and start-up costs and will therefore pay in order to generate continuously. Hence the deeply discounted and even negative offer prices at the low end of the supply function.

In the mid-region of the supply function, where demand tends to be most of the time, the supply function is flat and price volatility relatively low. The generating technologies here are a mixture of coal and gas. Both are subject to carbon emission supplements, with coal using roughly twice the number of emission certificates compared to gas, per unit of power generated. Furthermore, the operational efficiencies of the various coal and gas plants vary, so that the marginal cost ordering tends to be a mixture of the technologies. It is not the case that all the coal plants are located below all the gas plant in the supply function. Thus, as commodity prices for coal and gas fluctuate, the ordering of the various gas and coal facilities in this section of the supply function also interchange. This leads to a competitive and intricate relationship of power prices to gas and coal commodity prices. The upper convex region of the supply function is characteristic of power markets at times of high demand and increasingly scarce supply. The technologies in this section tend to be low capital cost, high marginal cost, gas and oil (diesel). Not only do these facilities have higher running costs, they need to recover fixed costs over fewer running hours in the year. Furthermore, they tend to be owned by the larger generators and with imperfect competition at times of relative scarcity, offers substantially above marginal costs frequently emerge ("price spikes").

3.2 Model Specification

With the above considerations in mind, we specify a regression model for price quantiles which recognises that the above factors are likely to have varying effects over time as market structure, technologies, polices and commodity prices emerge. The key exogenous variables are: demand, reserve margin (being defined as the difference between the demand for electricity in a particular hour and the power plant availability, "PPA", for that day), coal price, gas price, oil price, carbon allowance prices (CO2), wind production and solar production (PV). Since the market prices are set from a single day ahead auction, and all exogenous variables are taken either as forecasts or day-ahead prices known ahead of the auction to the market participants, it is sufficient to model the price quantiles in reduced form as

$$
\begin{aligned}
P_t^\alpha &= \beta_{0t}^\alpha + \beta_{1t}^\alpha P_{t-1} + \beta_{2t}^\alpha Wind_t + \beta_{3t}^\alpha PV_t + \beta_{4t}^\alpha Demand_t + \\
&\quad + \beta_{5t}^\alpha Coal_t + \beta_{6t}^\alpha Gas_t + \beta_{7t}^\alpha Oil_t + \beta_{8t}^\alpha CO2_t + \beta_{9t}^\alpha Reserve_t + u_t \qquad (15)\\
\beta_{jt}^\alpha &= c + D\beta_{j(t-1)}^\alpha + w_t \quad \text{for} \quad j = 1, ..., 9. \qquad (16)
\end{aligned}
$$

We first estimated a simpler version of the transition equation by modelling the time-varying coefficients as a random walk and this allows a stochastic representation of the coefficients to emerge. A similar time-varying approach applied to expected electricity prices was used in Karakatsani & Bunn (2010) and Paraschiv et al. (2014). Subsequently, we considered mean-reversion and exogenous variables in the transition equation.

For analysis, 24 time series data sets have been constructed for each hour of a day between 01/01/2010–31/05/2014 to apply this model to the German electricity prices. Tables 2 and 1 show the sources of data employed and descriptive statistics of price indexes. In Table 3 we show the data granularity.

In the above formulation, the renewable energies are assumed to have a direct impact on electricity prices; more (less) renewable output will shift the supply curve to the right (left) and thereby lower (increase) prices. Thus we included wind and PV in the observation equation. However, we also envisage that renewable output will have an indirect effect on prices at various quantiles through their influence on the coefficients of gas and coal. The intuition is that as the supply function shifts, the relative influences of the gas and coal technologies will change.

	Phelix spot	Demand forecast	Expected PPA	Coal	Gas	Oil	CO2	PV forecast	Wind forecast
Mean	46	42448	54764	76	22	76	11	3674	5043
Median	46	42801	55249	74	23	76	12	1985	3691
Maximum	210	57625	63981	99	38	91	17	21862	24690
Minimum	-222	24818	40016	52	11	59	3	0	229
Std. Dev.	16	7633	5008	10	4	8	4	4227	4228
Skewness	-2	0	0	0	-1	0	0	1	2
Kurtosis	30	2	2	2	3	2	2	4	5
Jarque-Bera	837967	1406	42	22	120	59	112	4699	18027
Probability	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Observations	27720	27720	1155	1155	1155	1155	1155	13860	27720

Table 1: Descriptive statistics for the level of the input variables, between 1 January 2010 and 28 February 2013. Data granularity is consistent with Table 3.

To test for both the direct and indirect effects, we estimated a version in which the transition equations for the coefficients of coal and gas contain, as exogenous variables, wind and PV[1].

$$P_t^\alpha = \beta_{0t}^\alpha + \beta_{1t}^\alpha P_{t-1} + \beta_{2t}^\alpha Wind_t + \beta_{3t}^\alpha PV_t + \beta_{4t}^\alpha Demand_t + \\ + \beta_{5t}^\alpha Coal_t + \beta_{6t}^\alpha Gas_t + \beta_{7t}^\alpha Oil_t + \beta_{8t}^\alpha CO2_t + \beta_{9t}^\alpha Reserve_t + u_t \tag{17}$$

$$\beta_{jt}^\alpha = c + D\beta_{j(t-1)}^\alpha + \varepsilon_j^\alpha Wind_t + \gamma_j^\alpha PV_t + \phi_j^\alpha Reserve_t + w_t \quad \text{for} \quad j = \{1, ..., 9\}; \tag{18}$$

where for $j = \{1, ..., 5; 8, 9\}$ ε_j and γ_j take the value 0.

[1]We tested also an additional model in which wind and PV were used only as exogenous variables in the transition equation for the coefficients of coal and gas and excluding them from the observation equation. However, the estimation procedure did not yield significant coefficients. This revealed that it is important to account for a direct effect of renewables on the electricity prices.

Variable units	**Description**	**Data Source**
Spot Price EUR/MWh	Market clearing price for the same hour of the last relevant delivery day	European Energy Exchange: `http://www.eex.com`
Coal Price EUR/12'000 t	Latest available price (daily auctioned) of the front-month Amsterdam-Rotterdam-Antwerp (ARA) futures contract before the electricity price auction takes place	European Energy Exchange: `http://www.eex.com`
Gas Price EUR/MWh	Last price of the NCG Day Ahead Natural Gas Spot Price on the day before the electricity price auction takes place	Bloomberg, Ticker: GTHDAHD Index
Oil Price EUR/bbl	Last price of the active ICE Brent Crude futures contract on the day before the electricity price auction takes place	Bloomberg, Ticker: COA Comdty
Price for EUA EUR 0.01/EUA 1'000 t CO2	Latest available price of the EEX Carbon Index (Carbix), daily auctioned at 10:30 am	European Energy Exchange: `http://www.eex.com`
Expected Wind and PV Infeed MWh	Sum of expected infeed of wind- and PV electricity into the grid, published by German transmission system operators in the late afternoon following the electricity price auction	Transmission system operators: `http://www.50Hertz.com`, `http://www.amprion.de`, `http://www.transnetbw.de`, `http://www.tennettso.de`
Expected Power Plant Availability MWh	Ex ante expected power plant availability for electricity production (voluntary publication) on the delivery day (daily granularity), daily published at 10:00 am	European Energy Exchange & transmission system operators: `ftp://infoproducts.eex.com`
Expected Demand MWh	Demand forecast for the relevant hour on the delivery day	European Network of Transmission System Operators (ENTSOE): `https://transparency.entsoe.eu/`

Table 2: Overview of fundamental variables used in the analysis. Note: For oil, the price of this contract is typically used as a reference for derivatives contracts and thus most relevant for trading. For coal, gas, and carbon, the prices result from a daily auction which does not exist for oil.

Variable	**Daily**	**Hourly**
Spot Price		×
Coal Price	×	
Gas Price	×	
Oil Price	×	
Price for EU Emission Allowances	×	
Expected Wind and PV Infeed		×
Expected Power Plant Availability	×	
Expected Demand		×

Table 3: Data granularity of fundamental variables

3.3 Results

From the 24 separate hourly models estimated, we comment in detail on the salient features one hour[2], hour 13 (12:00–13:00), in order to justify the value of the modelling framework presented, the estimation process and the underlying concept of factor influences from consideration of the bi-inflexioned shape of the supply function. The inclusion of demand in our formulation encompasses weather and seasonal effects (as in cite Karakatsani). Estimation results and optimised initial values for the random walk transition equations for all coefficients are shown in Table 4.

In Figure 3 we show the time-varying coefficients of gas at a range of quantiles. We observe that gas prices have a positive effect on electricity prices at all quantiles and more so in consideration of the risks of high (P90, P80, P70) prices. These events occur in the upper region of the supply function where gas is the marginal technology. These three quantile coefficients have shown a smooth decline over time reflecting the increasingly competitive conditions in the German market. In contrast, the middle quantiles (P40,P50, P60) show considerable volatility and this reflects the middle part of the supply function where gas and coal facilities intermingle and the marginal technologies will switch according to gas and coal price spreads as well as the overall supply function shifts induced by wind and solar generation.

For comparison, Figure 4 shows the quantile coefficients for wind, all of which are negative as expected. The smoothest effect is at P10 and reveals the influence of wind in driving lower electricity prices when high wind and low demand induce price formation in the lower concave region of the supply function. For the mid quantiles we again see a more volatile sequence of coefficient estimates as price formation emerges from the flat mid region of the supply function, and in Figure 5 the highly sensitive interaction of the estimates for wind and gas at the P50 quantile is clearly illustrated. When the wind production is high (low), it moves the supply function to the right (left) and the higher

[2]Results for other hours of the day are available upon request.

cost gas facilities are pushed out of (into) action. In this Figure, the correlation coefficient between gas and wind coefficients at the P50 quantile is accordingly -0.256. We found a similar, though weaker negative correlation for PV and gas coefficients.

Table 5 provides further support for this intricate explanation of the quantile coefficient drivers as we introduce exogenous variables into explaining the transition equation for the gas coefficients. For all electricity price quantiles except the lowest, we see that increased wind and solar production lowers the effect of gas prices. We also see that for price spikes (P90) in the upper convex region of the supply function, the high offers from gas generators are moderated by a higher reserve margin; conduct we would expect from reduced scarcity.

With regard to conduct at the extremes, in Figure 6 we observe that for hour 13 the coefficients of the lagged spot prices have a positive sign at all quantiles, but only at P10 and P90 do they appear to be substantial. This is an intuitive corroboration of the observations advanced earlier that at the lower and upper regions of the supply function, conduct tends to be a more important feature of price formation than demand, supply or fuel price fundamentals. Very low prices tend to reflect the discounting of inflexible facilities and high price spikes reflect the exercise of market power. As a behavioural feature we would expect to see some adaptive behaviour and the results that the coefficients for lagged spot only stand out at P10 and P90 are a reinforcement of this concept. Furthermore, the P10 coefficient is larger and increasing, illustrative of the increasing difficulty, and as a consequence conduct co-ordination, in dealing with low extreme prices as more renewable capacity has been introduced into the power system.

The results from the other hours revealed similar intricate dynamics, the main differences being that in the night hours of low demand, coal featured more strongly than gas and of course solar PV was absent as a driving factor. One of the modelling areas of concern was the separate estimation of the quantiles and whether this might lead to incoherent estimation of the distribution as a whole. Whilst it is clear from the supply

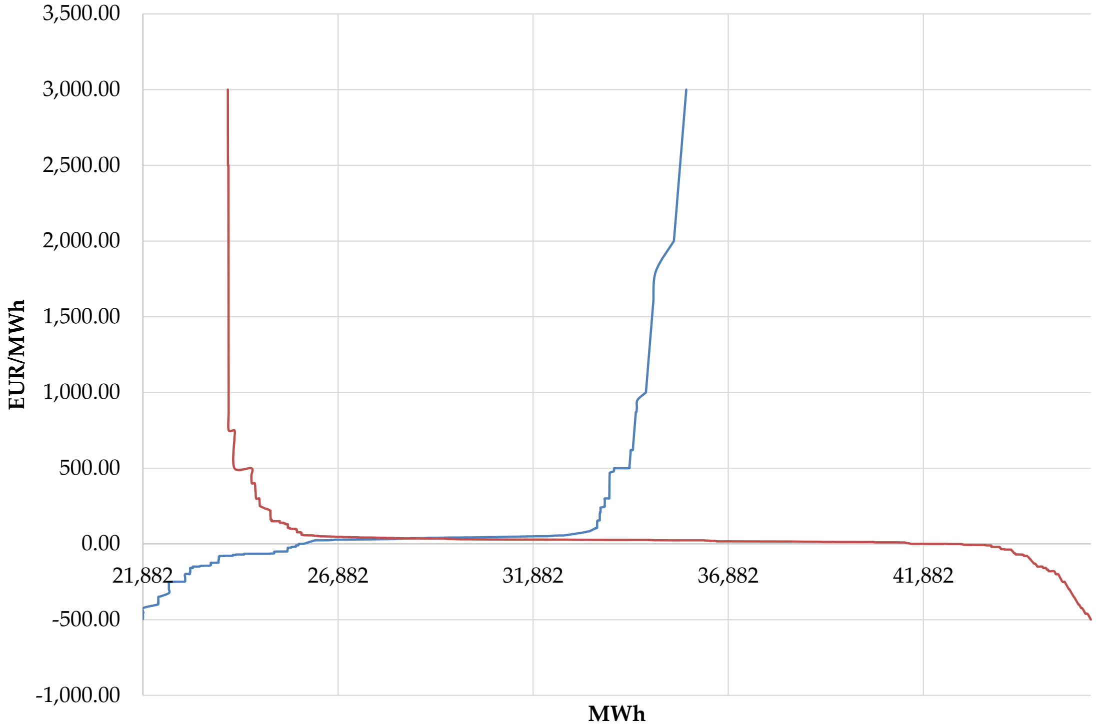


Figure 2: Actual supply and demand functions from the German market. Hour 11-12 on 22 Sept 2015. Source: EPEX Day ahead.

function concept and the discussion of the above results that we would not expect the factor coefficients to necessarily show a monotonic relationship with the levels of the quantiles, we would expect the model as a whole to produce a coherent predictive distribution for prices with a monotonic ordering of price quantile levels. Figure 7 displays the hourly quantile bands for a typical pair of days in the sample. Overall, in the data as a whole, less than 3% of the quantiles for the hourly price distributions were incoherent in the sense of violating a monotonic ordering. We take that to be reassuring for the model specification and a lack of overfitting.

4 Conclusion

We demonstrated the value of a well specified dynamic model for quantile estimation by means of an application to electricity price risk. Electricity prices are a commodity in which price formation is nonlinear in its relationship to fundamentals, dynamic in the relative influences of drivers, with further complications introduced by policy interventions for supporting specific technologies and opportunities for participant conduct to be influential at high and low prices. Despite these complications careful consideration of the shape of the supply function with its concave, flat and convex regions, together with the information that is available to market participants day ahead allows plausible expectations for the price dynamics to be considered, and these explain very well the signs and significance of the parameters in the estimated models. Nevertheless, the models need to have a detailed specification with the various quantiles being related to multiple factors through coefficients which have dynamic properties themselves related to some of the exogenous factors. This modelling requirement motivates the development of quantile models that need fully parametric specifications to capture dynamics through exogenous factors and time-varying coefficients.

A novel general methodology has therefore been developed in which time-varying multi factor coefficients are recursively estimated with a Kalman filter using maximum likelihood. Since the likelihood function is non-differentiable, the problem is re-formulated as a non-linear optimsation with constraints, and furthermore re-formulated again by moving the constraints into the objective function to solve an augmented Lagrangian method. With careful selection of starting values, maximum likelihood estimates were thereby acquired. As a general approach, we would expect this to be useful in many applications of risk management and quantile estimation where there is dynamic complexity in price formation and plausible exogenous price drivers.

References

Gerlach, R., Chen, C. W. S. & Chan, N. C. Y. (2011), 'Bayesian time-varying quantile forecasting for value-at-risk in financial markets', *Journal of Business & Economic Statistics* **29**, 481–492.

Hao, L. & Naiman, D. Q. (2007), *Quantile Regression*, Quantitative Applications in the Social Sciences, SAGE Publications.

Honda, T. (2004), 'Quantile regression in varying coefficient models', *Journal of Statistical Planning and Inference* **121**, 113–125.

Kalman, R. E. (1960), 'Transactions of the asme - journal of basic engineering', *Journal of Statistical Planning and Inference* **82 (Series D)**, 35–45.

Karakatsani, N. V. & Bunn, D. W. (2010), 'Fundamental and behavioural drivers of electricity price volatility', *Studies in Nonlinear Dynamics & Econometrics* **14(4)**(4), 1–42.

Kim, M. O. (2007), 'Quantile regression with varying coefficients', *The Annals of Statistics* **35**(1), 92–108.

Koenker, R. & Bassett, G. (1978), 'Regression quantiles', *Econometrica* **46**, 33–50.

Koopman, S.J., O. M. & Cornero, M. (2007), 'Periodic seasonal reg-arfimagarch models for daily electricity spot prices', *Journal of the American Statistical Association* **102(477)**, 16–27.

Panagiotelis, A. & Smith, M. (2008), 'Bayesian density forecasting of intraday electricity prices using multivariate skew t distributions', *International Journal of Forecasting* **24(4)**, 710–727.

Paraschiv, F., Erni, D. & Pietsch, R. (2014), 'The impact of renewable energies on eex day-ahead electricity prices', *Energy Policy* **73**, 196–210.

Rossi, G. D. & Harvey, A. (2006), 'Time-varying quantiles', *Working paper University of Cambridge* **CWPE 0649**.

Tsionas, E. G. (2003), 'Bayesian quantile inference', *Journal of Statistical Computation and Simulation* **9**, 659–674.

Yu, K. & Moyeed, R. A. (2001), 'Bayesian quantile regression', *Statistics & Probability Letters* **54**, 437–447.

Zhu, C., Byrd, R. H. & Nocedal, J. (1997), 'L-bfgs-b: Algorithm 778: L-bfgs-b, fortran routines for large scale bound constrained optimisation', *ACM Transactions on Mathematical Software* **23(4)**, 550–560.

	Q 10%	Q 20%	Q 30%	Q 40%	Q 50%	Q 60%	Q 70%	Q 80%	Q 90%
$\Omega * 10^{-10}$	0.0004	0.0006	0.0011	0.0049	0.0005	0.0013	0.0001	0.0003	0.0053
	0.0000	2.9611	4.3414	3.8028	10.5005	0.0063	0.0001	0.0001	0.0001
	0.0000	2.5867	1.6459	1.4661	3.1369	0.0028	0.0006	0.0006	0.0001
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0178	0.0156	0.0308	0.0054	0.0151	0.0901	0.0003	0.0009	0.0072
	1.0510	29683.9052	2510432.0731	2953519.3948	8093839.7540	6061.2393	10.6293	31.1348	3.5574
	4.5723	1.8411	0.2306	0.0493	0.0062	0.0363	0.3076	0.0001	0.0006
	0.0001	0.0003	0.0005	0.0120	0.0301	0.1902	0.6769	0.6900	0.1968
	0.0000	3.9643	2.1434	2.7245	17.8127	0.0172	0.0459	0.0928	0.0000
β_0	-0.00404	-0.00508	-0.0067	-0.0048	0.0004	-0.0053	-0.0011	-0.0043	-0.0249
	0.00003	0.00014	0.0002	0.0004	0.0004	0.0002	0.0001	0.0001	0.0001
	0.00003	0.00028	0.0003	0.0002	0.0002	0.0001	0.0000	0.0000	0.0000
	-0.00001	-0.00005	-0.0001	-0.0002	-0.0001	-0.0001	0.0000	0.0000	-0.0001
	-0.00449	-0.01603	-0.0510	-0.0742	-0.0429	-0.0202	-0.0074	-0.0060	-0.0349
	-0.00008	-0.01774	-0.0596	-0.1820	-0.2000	-0.1573	-0.2064	-0.2025	-0.2828
	0.00034	0.01504	0.0505	0.0484	-0.0410	-0.0201	-0.0450	-0.0431	-0.0218
	-0.02572	-0.13382	-0.2397	-0.1619	-0.0449	0.0121	0.0263	0.0782	0.2058
	0.00001	0.00006	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	0.0001
Ξ	19.5589	16.9401	3.0117	7.2805	16.1200	0.0178	0.0263	0.0540	0.2401
τ	1.1148	2.0776	2.1053	2.1354	2.1344	2.1563	2.1621	1.9651	1.4764
Obj. Values	1819.9297	2777.0406	2816.3499	2827.7250	2835.4923	2834.2591	2788.7955	2692.8638	2130.9757

Table 4: Parameter estimates for hour 13, all quantiles. The coefficients shown in vectors Ω and β_0 follow the order indexed in Equation (15)

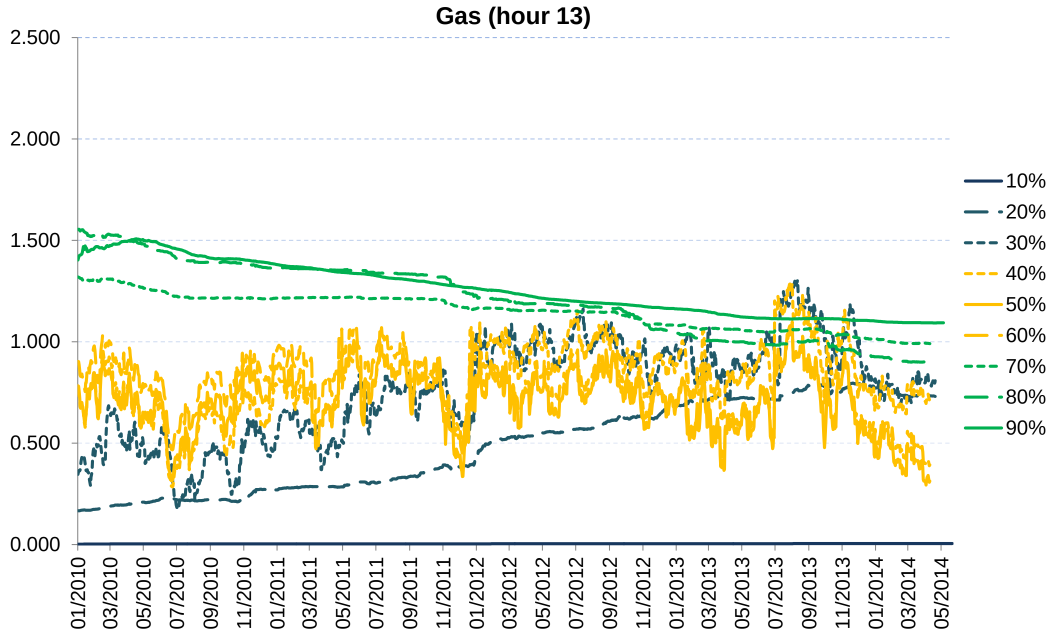


Figure 3: Gas Quantile Coefficients for Hour 13

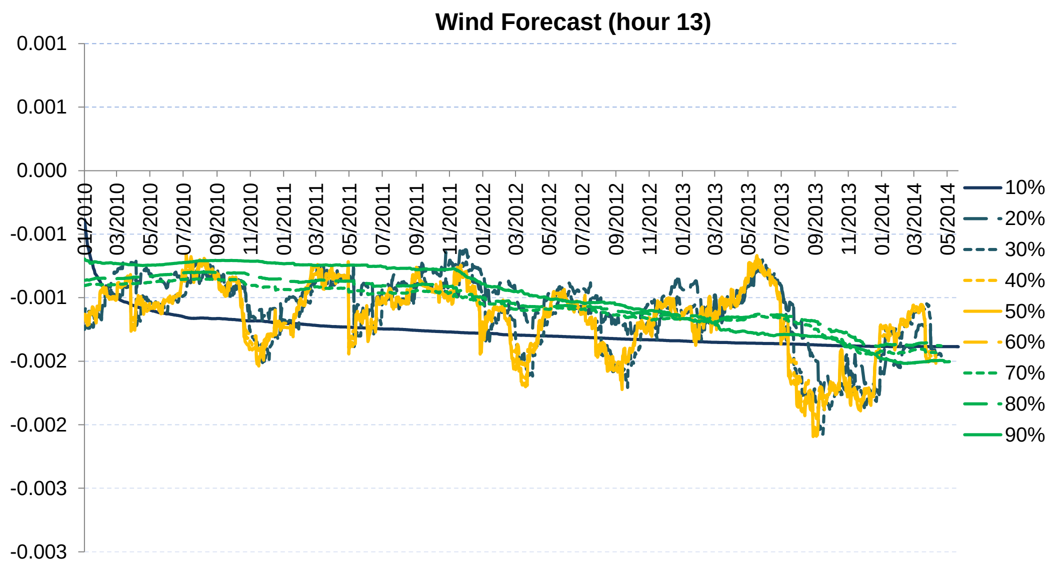


Figure 4: Wind Quantile Coefficients for Hour 13

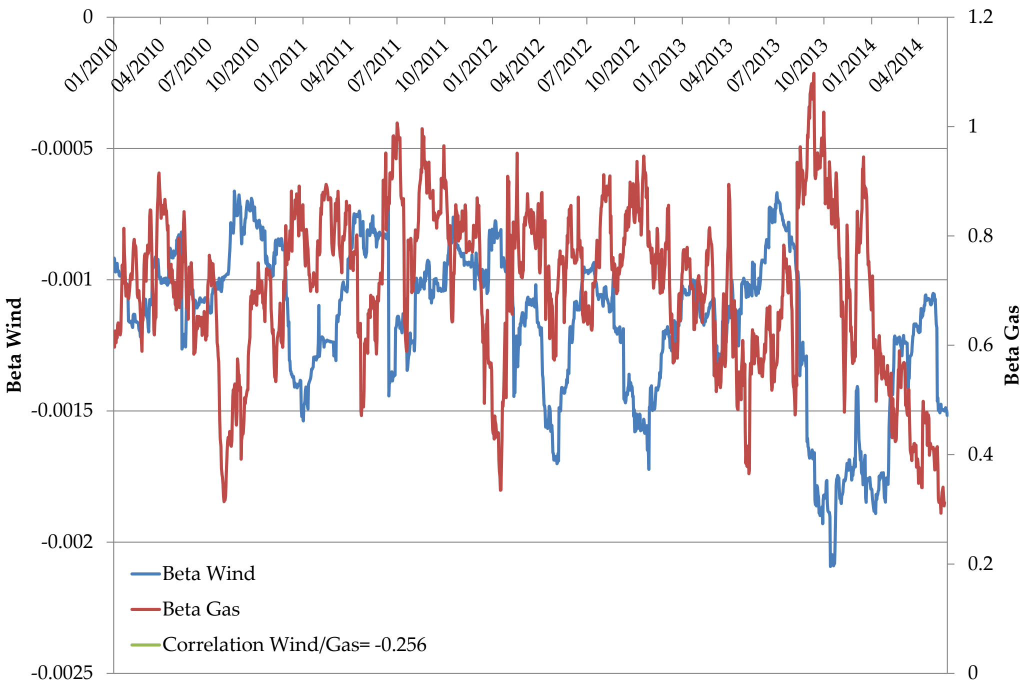


Figure 5: Correlation of Gas and Wind P50 Quantile Coefficients

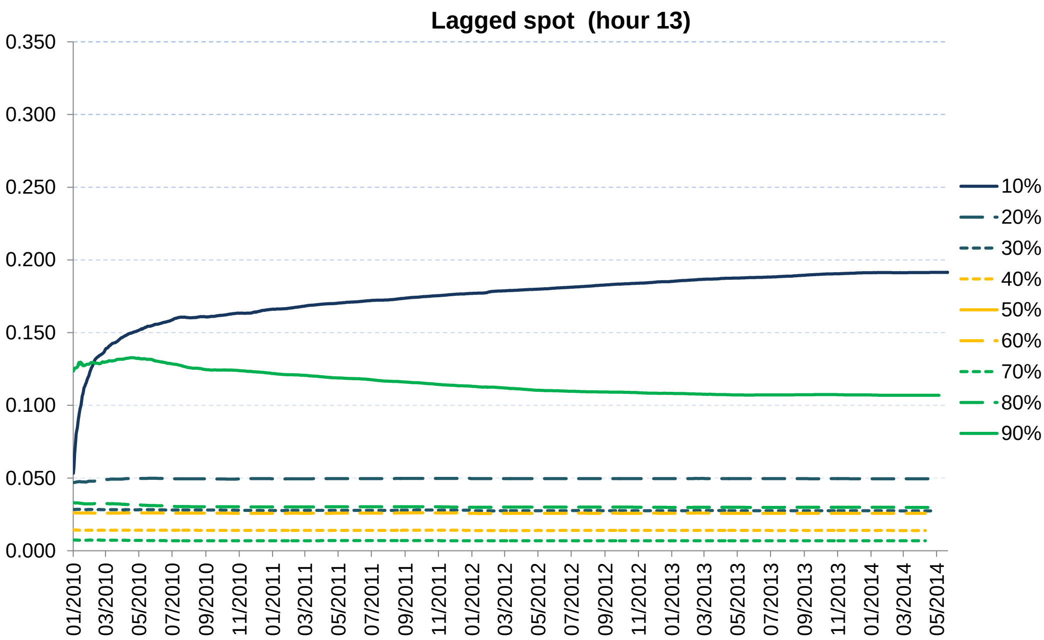


Figure 6: Lagged Price Quantile Coefficients for Hour 13

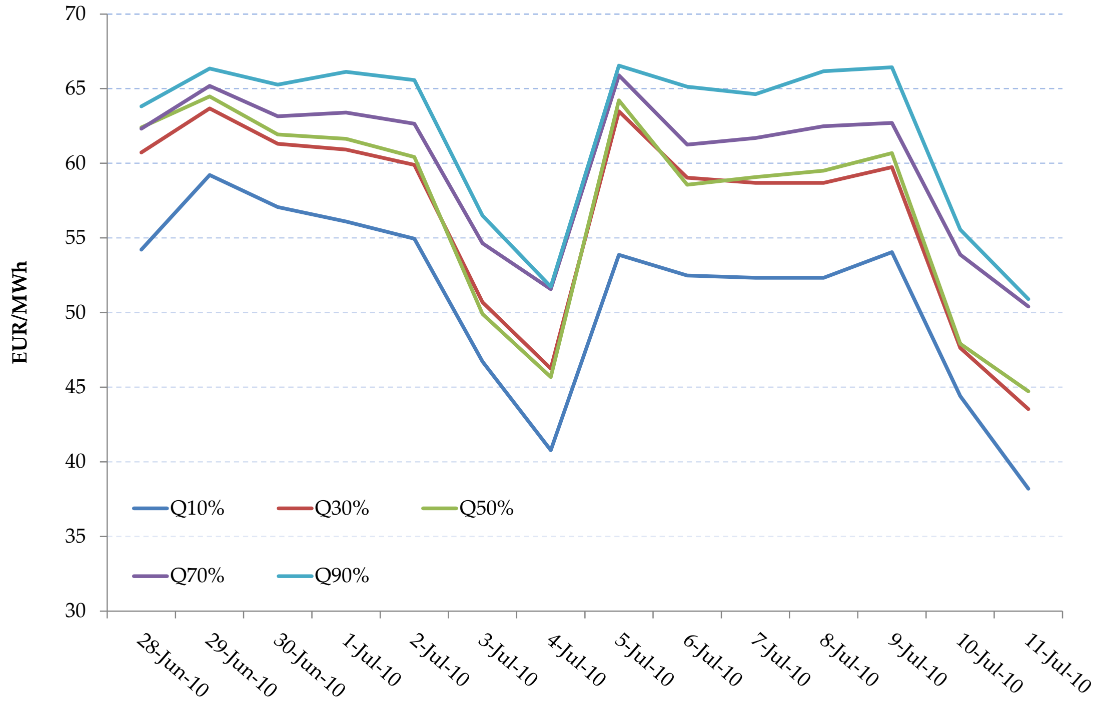


Figure 7: Predictive Hourly Price Quantiles for two days

Dependent variable: β_{gas}					
Method: Least Squares					
	Q 10%	**Q** 30%	**Q** 50%	**Q** 70%	**Q** 90%
C	0.0000	0.0103	0.0370	0.2507	0.2131
	(23.0451)	(4.1064)	(7.1436)	(21.0069)	(17.5405)
D_{gas}	0.9888	0.9872	0.9561	0.8012	0.8510
	(1795.4470)	(277.3021)	(149.5016)	(84.3599)	(99.6584)
$\varepsilon^{\alpha}_{Wind} * 10^8$	0.0018	-1.4500	-3.8600	-10.0000	-11.8000
	(2.8295)	(-0.9128)	(-1.8863)	(-6.1723)	(-5.9069)
$\gamma^{\alpha}_{PV} * 10^8$	0.0037	2.3300	-2.3200	-21.7000	-25.4000
	(5.6391)	(1.4540)	(-1.3332)	(-13.3793)	(-12.2649)
$\phi^{\alpha}_{Reserve} * 10^8$	0.0011	1.7500	2.4600	-3.6200	-3.1600
	(2.8067)	(1.7686)	(1.9042)	(-3.5875)	(-2.5796)
R-squared	0.9823	0.9863	0.9338	0.8889	0.9229
Test for omitted variables (wind, pv, reserves), p-values					
F-statistic	0.0000	0.1225	0.0279	0.0000	0.0000
Likelihood ratio	0.0000	0.1216	0.0276	0.0000	0.0000

Table 5: Parameter estimates for Equation (18), Hour 13, having as dependent variable the coefficient β for gas. To overcome autocorrelation, correlation, and heteroskedasticity in the error terms we employed the Newey-West estimator. t-statistics of coefficients are shown in paranthesis. We further performed a test for omitted variables and added the set of variables Wind, PV and Reserve Margin to Equation (16) to ask whether the set makes a significant contribution to explaining the variation in the dependent variable. In the lower panel of the table we show the p-values of the related F-statistic and Likelihood ratio.