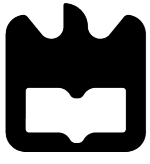
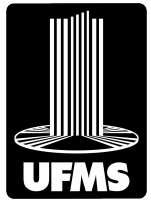
Decoding 2D convolutional codes over erasure channel

Raquel Pinto, Marcos Spreafico
and **Carlos Vela**

V Jornadas de la red MatSI
Alicante, 7 June 2024



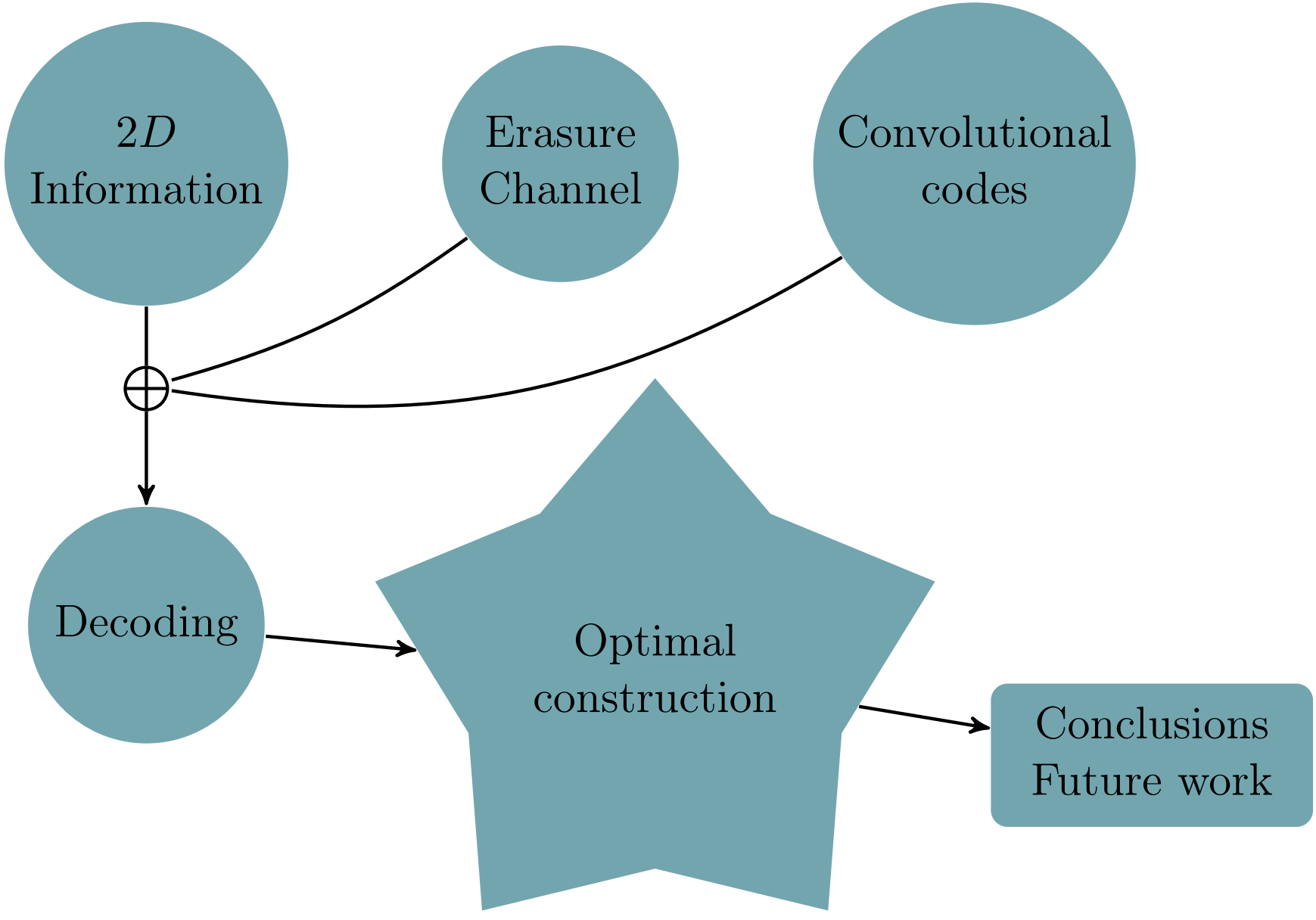
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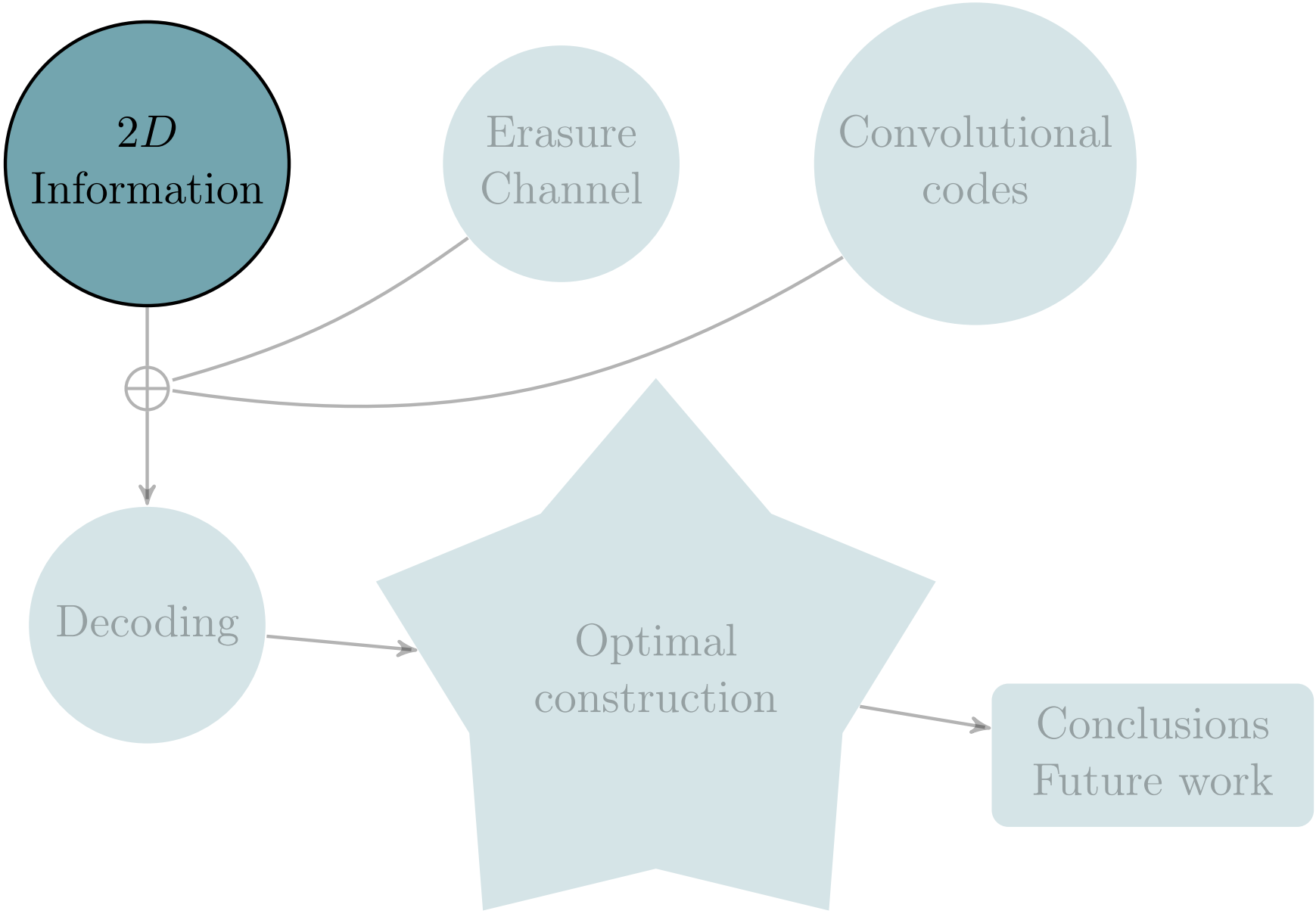


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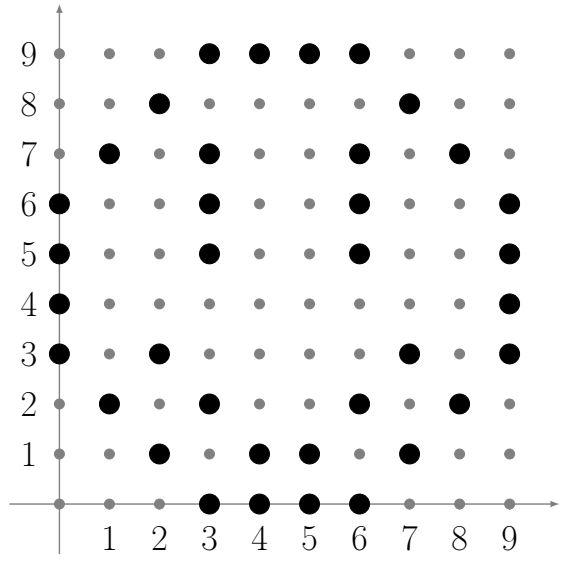


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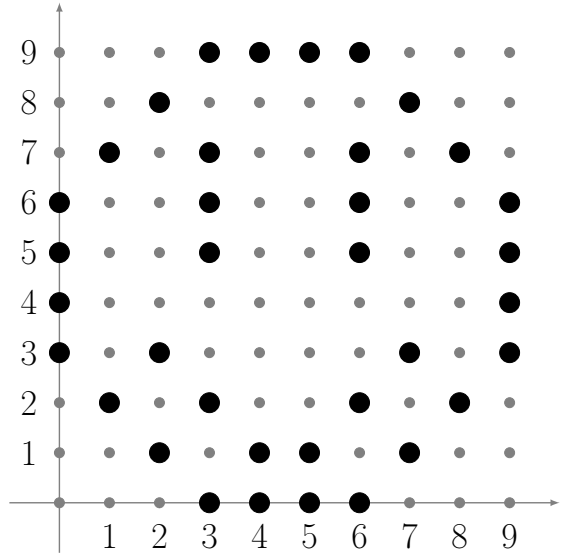

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2D information



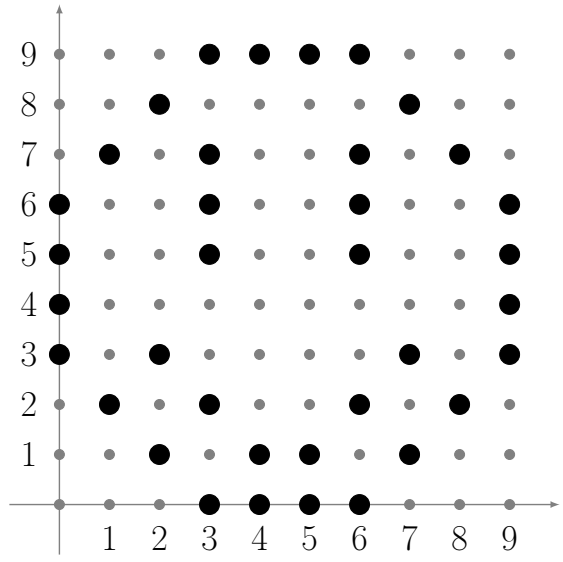
- Photography and video
- data storage.

2D information

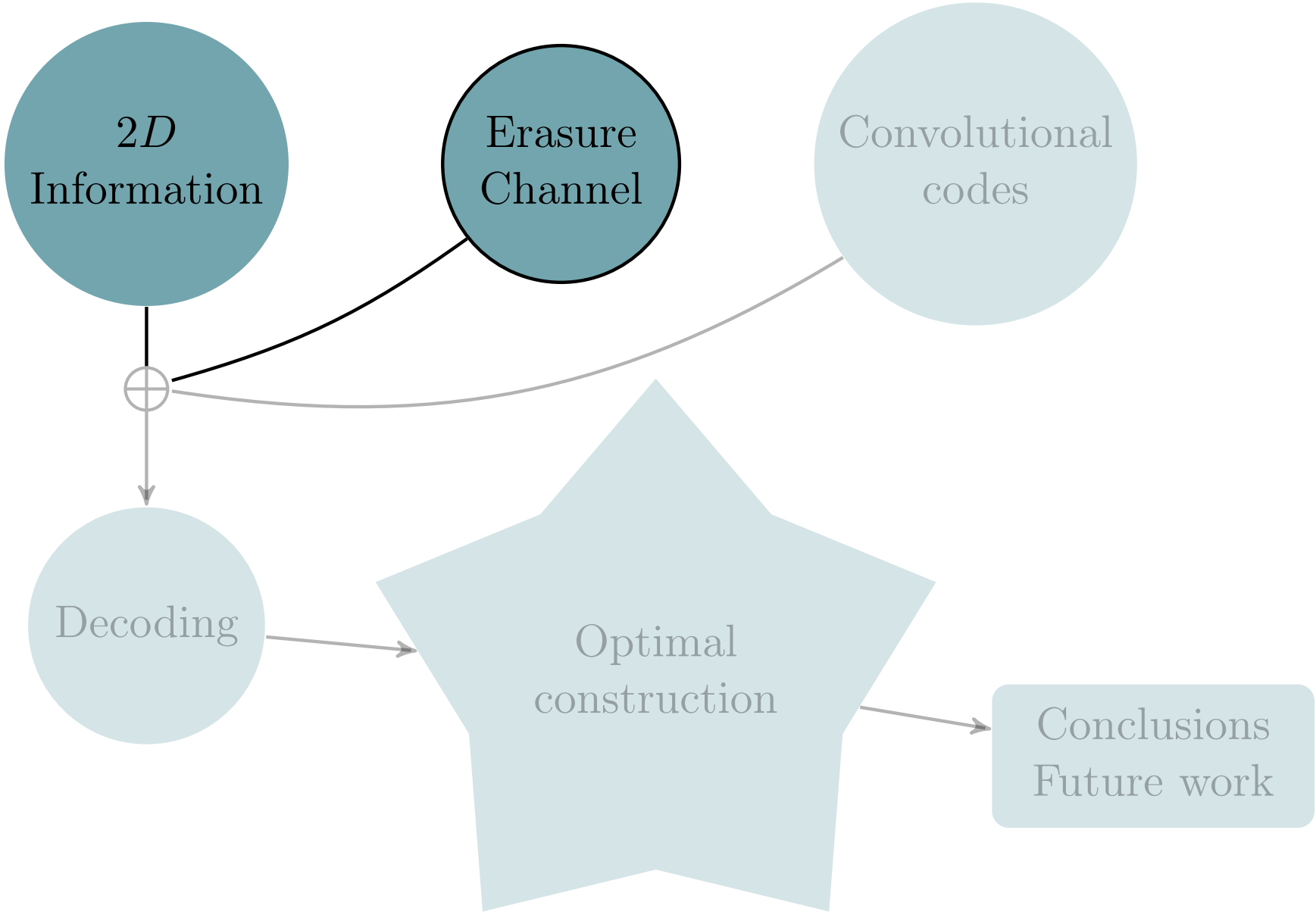


$$\{u(0,3), u(0,4), u(0,5), u(0,6), u(2,1), u(4,1), u(5,1), u(7,1),$$
$$u(1,2), u(3,2), u(6,2), u(8,2), u(0,3), u(2,3), u(7,3), u(9,3),$$
$$u(0,4), u(9,4), u(0,5), u(3,5), u(6,5), u(9,5), u(0,6), u(3,6),$$
$$u(6,6), u(9,6), u(1,7), u(3,7), u(6,7), u(8,7), u(3,9), u(4,9),$$
$$u(5,9), u(6,9)\} \in \mathbb{F}^k$$

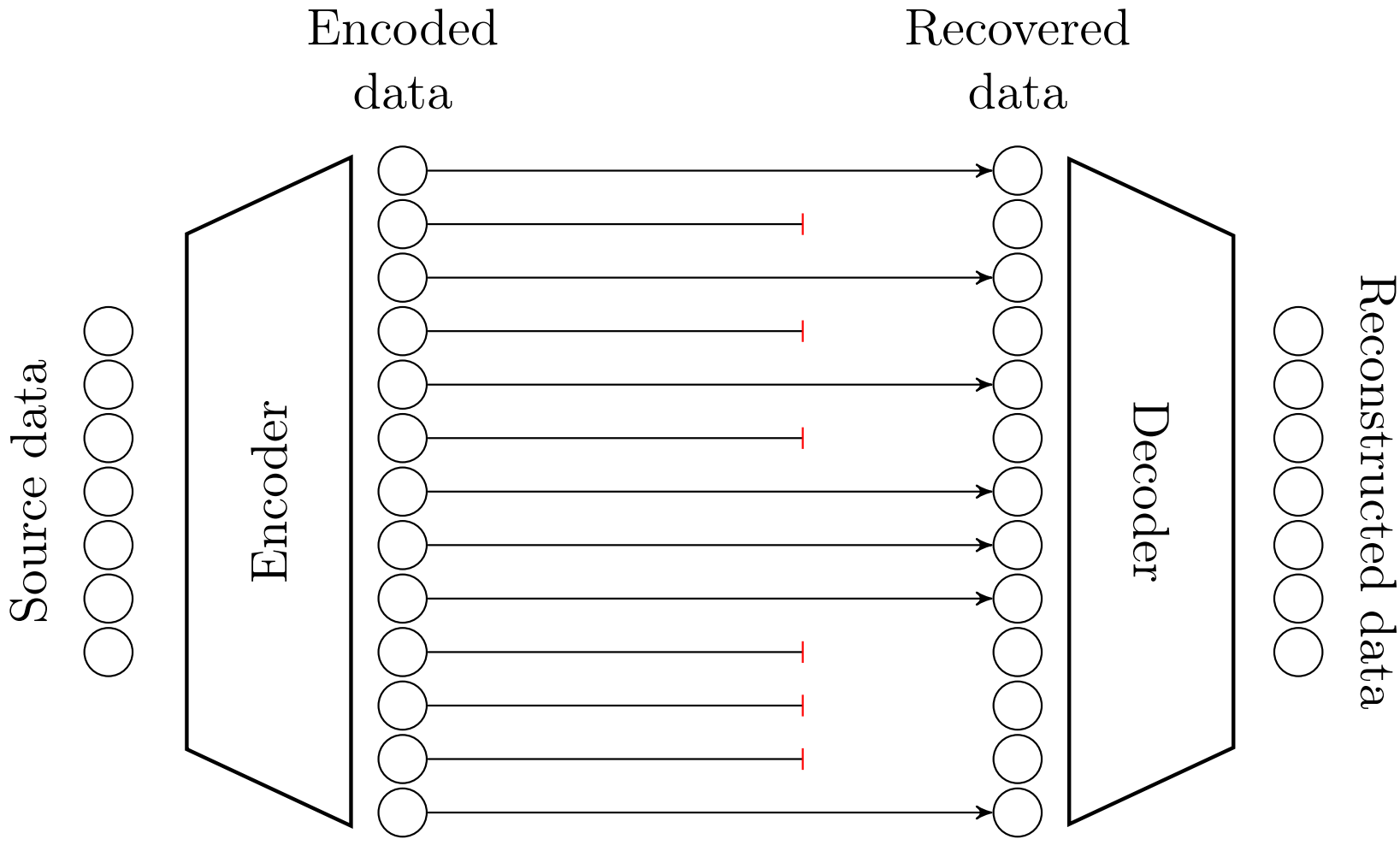
2D information

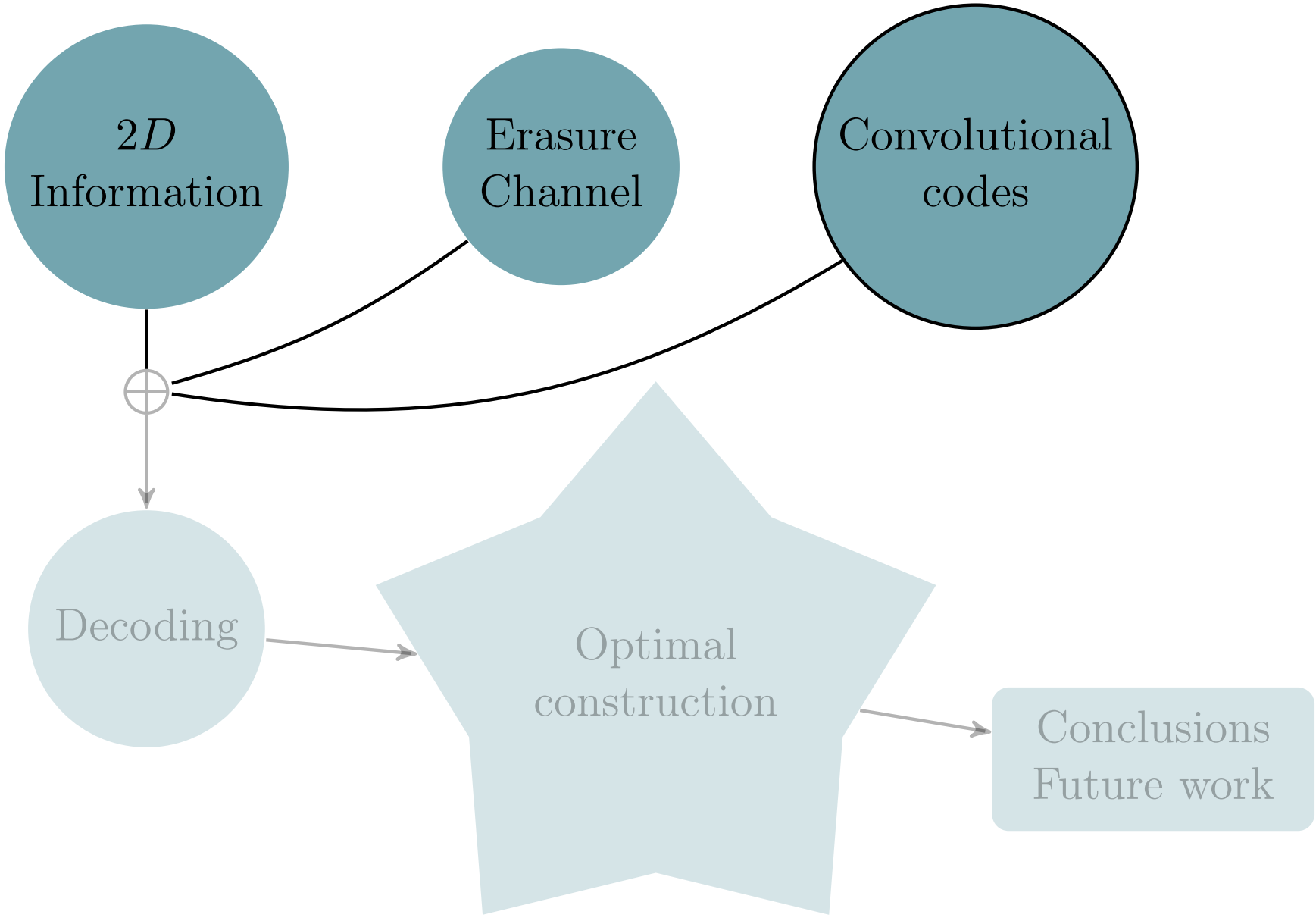


$$u(z_1, z_2) = \sum_{(i,j)\in\mathbb{N}^2} u(i,j) z_1^i z_2^j \in \mathbb{F}^k[z_1, z_2]$$


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The erasure channel




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2D Convolutional codes

- A **2D convolutional code** $\mathcal{C}$ of rate k/n is a $\mathbb{F}_q[z_1, z_2]$-submodule of $\mathbb{F}_q[z_1, z_2]^n$ of dimension k.
- Let $G(z_1, z_2) \in \mathbb{F}_q^n[z_1, z_2]$ be a polynomial matrix then

$$\mathcal{C} = \left\{ v(z_1, z_2) = u(z_1, z_2)G(z_1, z_2) : u(z_1, z_2) \in \mathbb{F}_q^k[z_1, z_2] \right\},$$

- $G(z_1, z_2)$ is called an encoder (generator matrix) of $\mathcal{C}$.
- $v(z_1, z_2)$ is a codeword.

2D Convolutional codes

- A noncatastrophic **convolutional codes** $\mathcal{C}$ of rate k/n is a $\mathbb{F}_q[z_1, z_2]$-submodule of $\mathbb{F}_q[z_1, z_2]^n$ of dimension k.
- Let $H(z_1, z_2) = \sum_{i \in \mathbb{N}} H(i, j) z_1^i z_2^j \in \mathbb{F}_q[z_1, z_2]^{(n-k) \times n}$ be a full row rank matrix, we can represent the code as the kernel of this matrix:

$$\mathcal{C} = Ker_{\mathbb{F}[z_1, z_2]} H(z_1, z_2) = \\ \{v(z_1, z_2) \in \mathbb{F}_q[z_1, z_2]^n \text{ s.t. } H(z_1, z_2) v(z_1, z_2) = 0\},$$

- $H(z_1, z_2)$ is called the parity-check matrix of $\mathcal{C}$.
- $\delta = \max\{i + j \mid H(i, j) \neq 0\}$ is the degree of $H(z_1, z_2)$
- It holds that

$$G(z_1, z_2) H(z_1, z_2)^T = 0$$

2D convolutional codes

The product can be seen as:

$$
\begin{aligned}
H(z_1,z_2)v(z_1,z_2) = & H(0,0)v(0,0) + (H(1,0)v(0,0) + H(0,0)v(1,0))z_1 \\
& + (H(0,1)v(0,0) + H(0,0)v(0,1))z_2 \\
& + (H(2,0)v(0,0) + H(1,0)v(1,0) + H(0,0)v(2,0))z_1^2 \\
& + (H(1,1)v(0,0) + H(1,0)v(0,1) + H(0,1)v(1,0) + H(0,0)v(1,1))z_1 z_2 \\
& \dots \\
& = \sum_{0 \le a+b \le \gamma} \left[\sum_{0 \le i+j \le \delta} H(i,j)v(a-i,b-j) \right] z_1^a z_2^b
\end{aligned}
$$

2D Convolutional codes

Since $H(z_1, z_2)v(z_1, z_2) = 0$ we have that:

$$\begin{cases} H(0,0)v(0,0) & = 0 \\ H(1,0)v(0,0) + H(0,0)v(1,0) & = 0 \\ H(0,1)v(0,0) + H(0,0)v(0,1) & = 0 \\ H(2,0)v(0,0) + H(1,0)v(1,0) + H(0,0)v(2,0) & = 0 \\ H(1,1)v(0,0) + H(1,0)v(0,1) + H(0,1)v(1,0) + H(0,0)v(1,1) & = 0 \\ \dots \end{cases}$$

Blockwise representation:

$$\mathbf{Hv} = \mathbf{0}$$

2D Convolutional codes

$$
\begin{bmatrix}
H(0,0) & & & & & & & & & & \\
H(1,0) & H(0,0) & & & & & & & & & \\
H(0,1) & O & H(0,0) & & & & & & & & \\
H(2,0) & H(1,0) & O & H(0,0) & & & & & & & \\
H(1,1) & H(0,1) & H(1,0) & O & H(0,0) & & & & & & \\
H(0,2) & O & H(0,1) & O & O & H(0,0) & & & & & \\
H(3,0) & H(2,0) & O & H(1,0) & O & O & H(0,0) & & & & \\
H(2,1) & H(1,1) & H(2,0) & H(0,1) & H(1,0) & O & O & H(0,0) & & & \\
H(1,2) & H(0,2) & H(1,1) & O & H(0,1) & H(1,0) & O & O & H(0,0) & & \\
H(0,3) & O & H(0,2) & O & O & H(0,1) & O & O & O & H(0,0) & \\
 & H(3,0) & O & H(2,0) & O & O & H(1,0) & O & O & O & \ddots \\
 & H(2,1) & H(3,0) & H(1,1) & H(2,0) & O & H(0,1) & H(1,0) & O & O & \ddots \\
 & H(1,2) & H(2,1) & H(0,2) & H(1,1) & H(2,0) & O & H(0,1) & H(1,0) & O & \ddots \\
 & H(0,3) & H(1,2) & O & H(0,2) & H(1,1) & O & O & H(0,1) & H(1,0) & \ddots \\
 & & H(0,3) & O & O & H(0,2) & O & O & O & H(0,1) & \ddots \\
 & & & H(3,0) & O & O & H(2,0) & O & O & O & \ddots \\
 & & & H(2,1) & H(3,0) & O & H(1,1) & H(2,0) & O & O & \ddots \\
 & & & H(1,2) & H(2,1) & H(3,0) & H(0,2) & H(1,1) & H(2,0) & O & \ddots \\
 & & & H(0,3) & H(1,2) & H(2,1) & O & H(0,2) & H(1,1) & H(2,0) & \ddots \\
 & & & & H(0,3) & H(1,2) & O & O & H(0,2) & H(1,1) & \ddots \\
 & & & & & H(0,3) & O & O & O & H(0,2) & \ddots \\
 & & & & & & H(3,0) & O & O & O & \ddots \\
 & & & & & & H(2,1) & H(3,0) & O & O & \ddots \\
 & & & & & & H(1,2) & H(2,1) & H(3,0) & O & \ddots \\
 & & & & & & H(0,3) & H(1,2) & H(2,1) & H(3,0) & \ddots \\
 & & & & & & & H(0,3) & H(1,2) & H(2,1) & \ddots \\
 & & & & & & & & H(0,3) & H(1,2) & \ddots \\
 & & & & & & & & & H(0,3) & \ddots \\
 & & & & & & & & & & \ddots
\end{bmatrix}
$$

Matrix $\mathbf{H}$ with $\delta = 3$

2D Convolutional codes

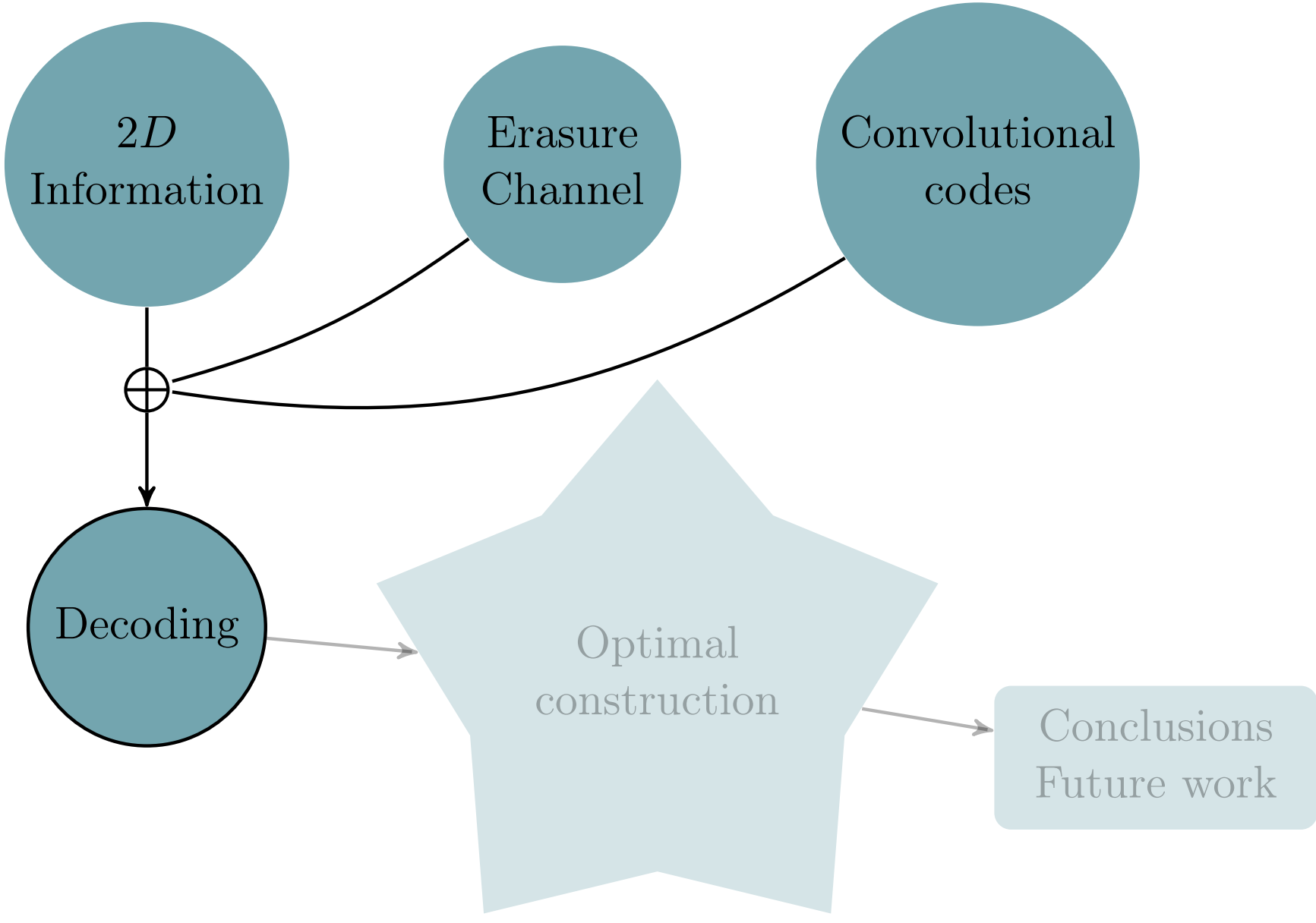
Let $v(z_1, z_2) \in \mathbb{F}^n[z_1, z_2]$ be a codeword of a 2D convolutional code, we define its **support**:

$$\mathrm{Supp}(v(z_1, z_2)) = \left\{(i, j) \in \mathbb{N}_0^2 \,|\, v(i, j) \neq 0\right\}$$

After transmitting the codeword we receive $\hat{v}$ with erasures:

$$\mathcal{E}(\hat{v}(z_1, z_2)) = \{(a, b) \in \mathrm{Supp}(v(z_1, z_2)) \,|\, v(a, b) \text{ has an erasure}\}$$

$$\bar{\mathcal{E}}(\hat{v}(z_1, z_2)) = \mathrm{Supp}(v(z_1, z_2)) \setminus \mathcal{E}(\hat{v}(z_1, z_2))$$


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Decoding 2D Convolutional codes

Now we construct the following:

- $\mathbf{H}_{\mathcal{E}}$ submatrix of $\mathbf{H}$ indexed by $\mathcal{E}$, and
- $\mathbf{H}_{\bar{\mathcal{E}}}$ submatrix of $\mathbf{H}$ indexed by $\bar{\mathcal{E}}$, and
- $\mathbf{v}_{\mathcal{E}}$ subvector of $\mathbf{v}$ indexed by $\mathcal{E}$, and
- $\mathbf{v}_{\bar{\mathcal{E}}}$ subvector of $\mathbf{v}$ indexed by $\bar{\mathcal{E}}$, and

We can construct:

$$\mathbf{H}_{\mathcal{E}}\mathbf{v}_{\mathcal{E}} + \mathbf{H}_{\bar{\mathcal{E}}}\mathbf{v}_{\bar{\mathcal{E}}} = \mathbf{0}$$

And solve the following system to recover the information:

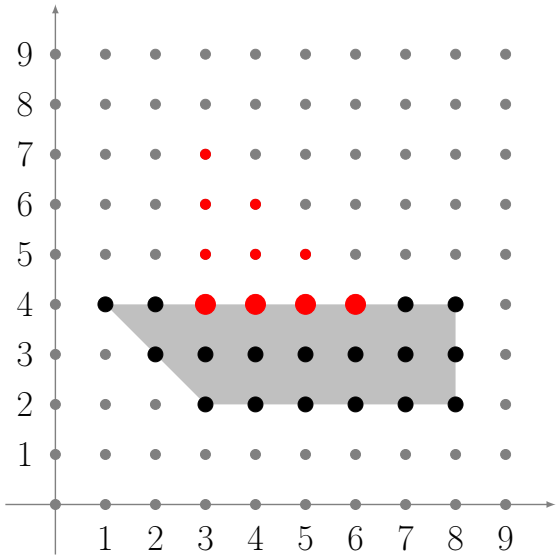
$$\mathbf{H}_{\mathcal{E}}\mathbf{v}_{\mathcal{E}} = -\mathbf{H}_{\bar{\mathcal{E}}}\mathbf{v}_{\bar{\mathcal{E}}}$$

Decoding Convolutional codes on lines

J.-J. Climent, D. Napp, R. Pinto and R. Simões, Decoding of 2D convolutional codes over an erasure channel. Advances in Mathematics of Communications, 2016, 10(1): 179-193. doi: 10.3934/amc.2016.10.179

Horizontal lines: Lets consider that in the set of erasures $\mathcal{E}(\hat{v}(z_1, z_2))$ a subset of them are in an horizontal line:

$$\mathcal{E}_h(\hat{v}(z_1, z_2)) = \{(r, s), (r+1, s), \ldots, (r+t, s)\}$$

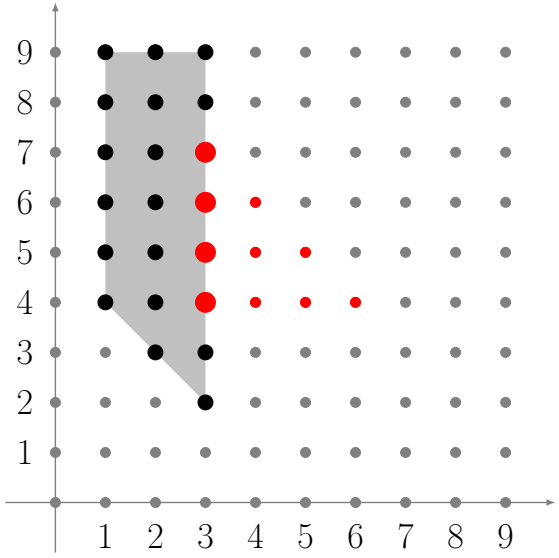


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Vertical lines: Lets consider that in the set of erasures $\mathcal{E}(\hat{v}(z_1, z_2))$ a subset of them are in a vertical line:

$$\mathcal{E}_v(\hat{v}(z_1, z_2)) = \{(r, s), (r, s+1), \ldots, (r, s+t)\}$$

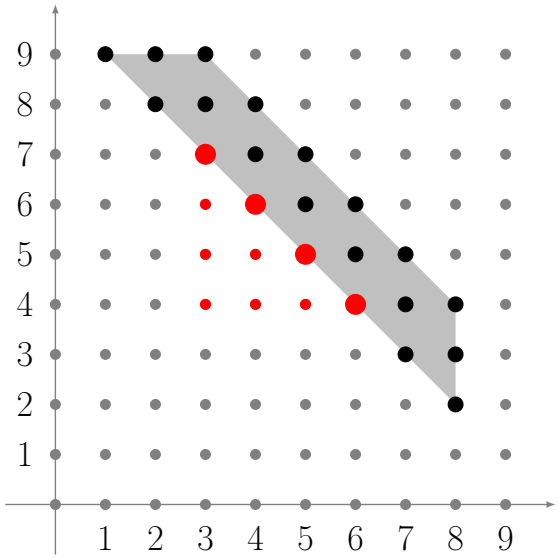


Decoding Convolutional codes on lines

J.-J. Climent, D. Napp, R. Pinto and R. Simões, Decoding of 2D convolutional codes over an erasure channel. Advances in Mathematics of Communications, 2016, 10(1): 179-193. doi: 10.3934/amc.2016.10.179

Diagonal lines: Lets consider that in the set of erasures $\mathcal{E}(\hat{v}(z_1, z_2))$ a subset of them are in a diagonal line:

$$\mathcal{E}_d(\hat{v}(z_1, z_2)) = \{(r, s), (r-1, s+1), \ldots, (r-t, s+t)\}$$



Decoding Convolutional codes on lines (horizontal)

We can construct the following system to solve the erasures in a horizontal line:

$$\mathbf{H}_{\mathcal{E}_h}\mathbf{v}_{\mathcal{E}_h} = \mathbf{a}$$

where

$$\mathbf{H}_{\mathcal{E}_h} = \begin{bmatrix}
H(0,0) & & & & & & & \\
H(1,0) & H(0,0) & & & & & & \\
H(2,0) & H(1,0) & H(0,0) & & & & & \\
\vdots & \vdots & \vdots & \ddots & & & & \\
H(\nu,0) & H(\nu-1,0) & H(\nu-2,0) & & & & & \\
 & H(\nu,0) & H(\nu-1,0) & \dots & \dots & H(0,0) & & \\
 & \ddots & & \ddots & & \ddots & & \\
 & & \ddots & & \ddots & & H(0,0) & \\
 & & & & \ddots & & \vdots & \\
 & & & & & H(\nu,0) & H(\nu-1,0) & \\
 & & & & & & H(\nu,0) &
\end{bmatrix}$$

Decoding Convolutional codes on lines (horizontal)

The matrix

$$H^h(z) = H(0,0) + H(1,0)z + H(2,0)z^2 + \cdots + H(\nu,0)$$

is the parity check matrix of a 1D convolutional code

$$\mathcal{C}^h = \mathrm{Ker}\, H^h(z).$$

Furthermore,

$$v^h(z) = v(0,r) + v(1,r)z + v(2,r)z^2 + \cdots + v(\nu,r)$$

is a codeword!

Decoding Convolutional codes on lines

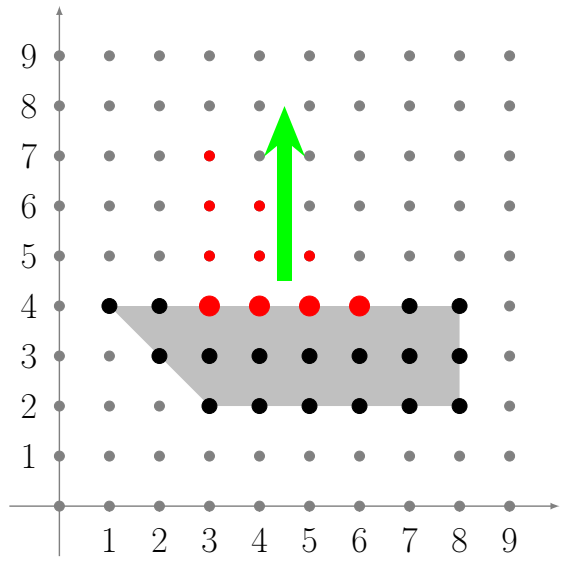
From this decoding process an optimal construction is a 2D convolutional code $\mathcal{C}$ such that:

$$\begin{aligned}\mathcal{C}^h &= \operatorname{Ker} H^h(z),\\ \mathcal{C}^v &= \operatorname{Ker} H^v(z) \text{ and}\\ \mathcal{C}^d &= \operatorname{Ker} H^d(z)\end{aligned}$$

are optimal codes (Maximum Distance Profile).

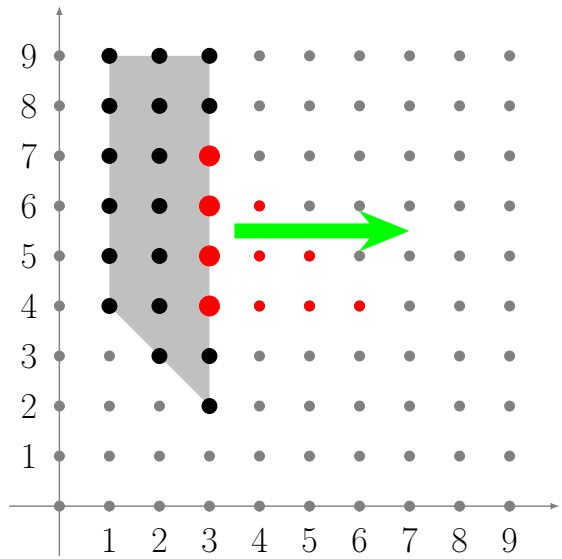
Decoding Convolutional codes on lines

1. First: **Horizontal lines**



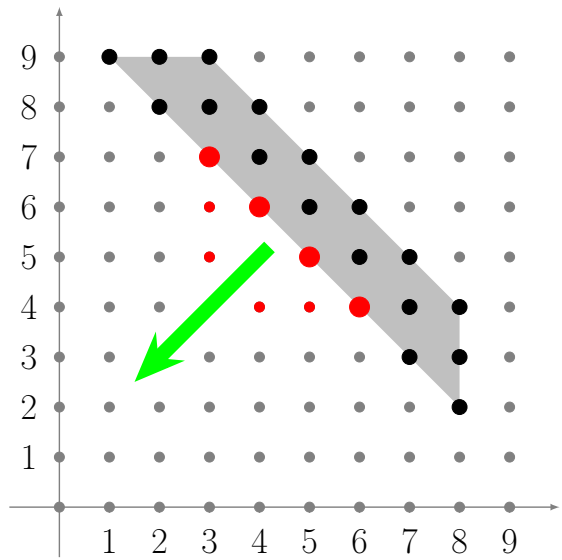
Decoding Convolutional codes on lines

1. First: **Horizontal lines**
2. Second: **Vertical lines**



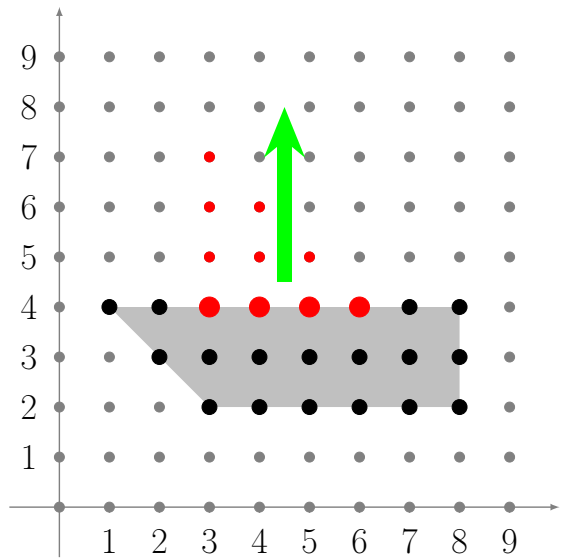
Decoding Convolutional codes on lines

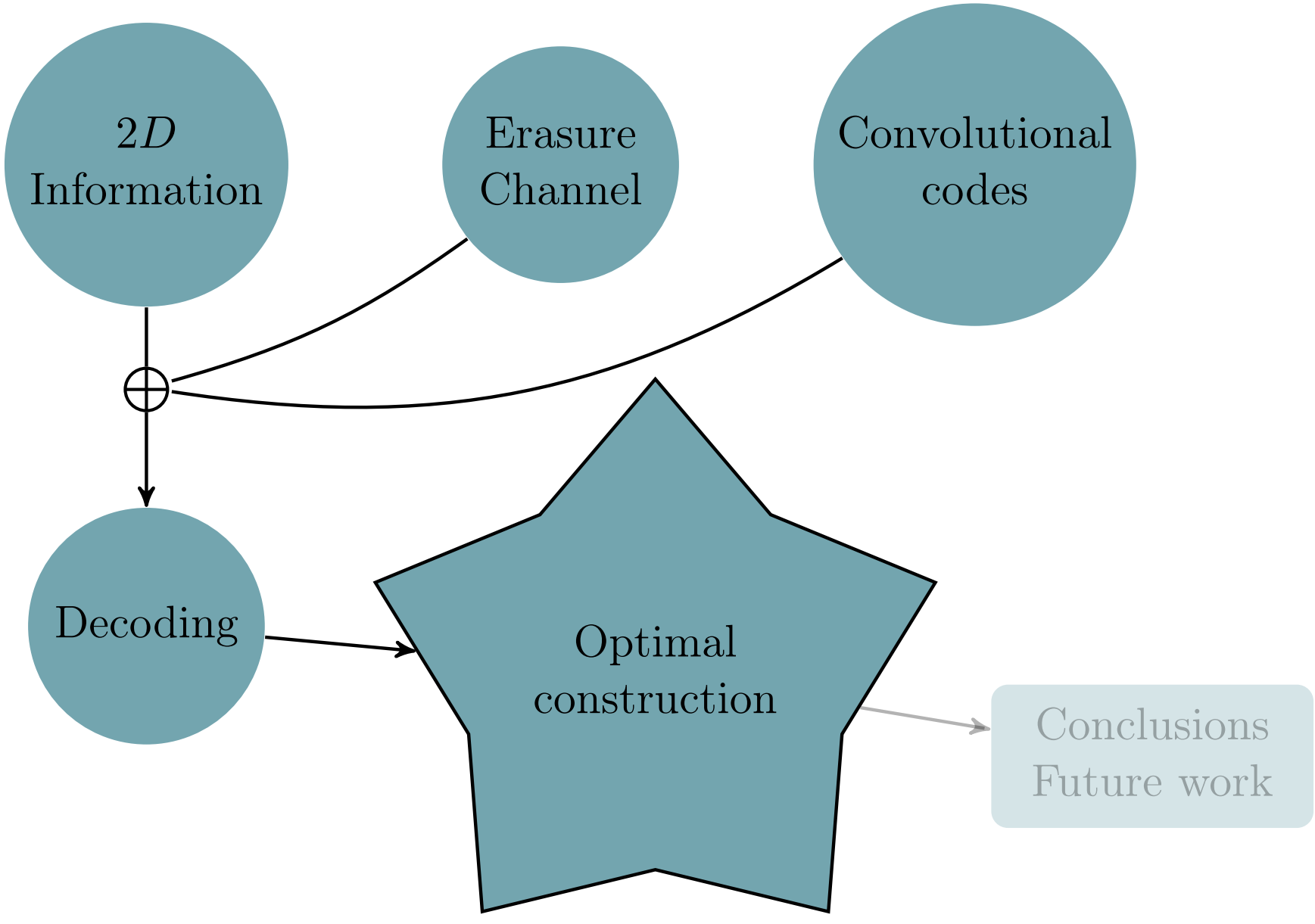
1. First: **Horizontal lines**
2. Second: **Vertical lines**
3. Third: **Diagonal lines**



Decoding Convolutional codes on lines

1. First: **Horizontal lines**
2. Second: **Vertical lines**
3. Third: **Diagonal lines**
4. and repeat...




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Construction

Let $\hat{\mathcal{C}}$ be an (n, k) non-catastrophic 1D convolutional code with parity-check matrix $\hat{H}(z) = \sum_{i=0}^{2\nu} H_i z^i$. Consider the 2D convolutional code $\mathcal{C}$ defined by the parity-check matrix

$$H(z_1, z_2) = \sum_{i=0}^{\nu} \sum_{j=i}^{2i} H_j z_1^{2i-j} z_2^{j-i}$$

$$= H_0 + H_1 z_1 + H_2 z_2 + H_2 z_1^2 + H_3 z_1 z_2 + H_4 z_2^2 + \cdots + H_\nu z_1^\nu + \cdots + H_{2\nu} z_2^\nu.$$

Construction

$$H(z) = H_0 + H_1 z + H_2 z^2 + \cdots + H_{2\nu} z^{2\nu}$$

$$\begin{array}{|cccccc|}
\hline
H_{2\nu} & H_{2\nu-1} & H_{2\nu-2} & & & \\
\vdots & \vdots & \vdots & & & \\
H_4 & H_5 & H_6 & \cdots & H_{\nu-2} & \\
H_2 & H_3 & H_4 & \cdots & H_{\nu-1} & \\
H_0 & H_1 & H_2 & \cdots & H_\nu & \\
\hline
\end{array}$$

Construction

$$H(z) = H_0 + H_1 z + H_2 z^2 + \cdots + H_{2\nu} z^{2\nu}$$

$H(0,2\nu)$	$H(1,2\nu-1)$	$H(3,2\nu-2)$		
$\vdots$	$\vdots$	$\vdots$		
$H(0,2)$	$H(1,2)$	$H(2,2)$	$\cdots$	$H(\nu-2,2)$
$H(0,1)$	$H(1,1)$	$H(2,1)$	$\cdots$	$H(\nu-1,1)$
$H(0,0)$	$H(1,0)$	$H(2,0)$	$\cdots$	$H(\nu,0)$

Construction

We will consider the 1D convolutional codes associated to $\mathcal{C}$ which its parity-check matrices are as follows:

$$
\begin{aligned}
H^h(z) &= \sum_{i=0}^{\nu} H_i z^i \\
&= H_0 + H_1 z + H_2 z^2 + \cdots + H_\nu z^\nu. \\
H^v(z) &= \sum_{i=0}^{\nu} H_{2i} z^i \\
&= H_0 + H_2 z + H_4 z^2 + \cdots + H_{2\nu} z^\nu. \\
H^d(z) &= \sum_{i=0}^{\nu} H_{\nu+i} z^i \\
&= H_\nu + H_{\nu+1} z + H_{\nu+2} z^2 + \cdots + H_{2\nu} z^\nu.
\end{aligned}
$$

Construction

Theorem

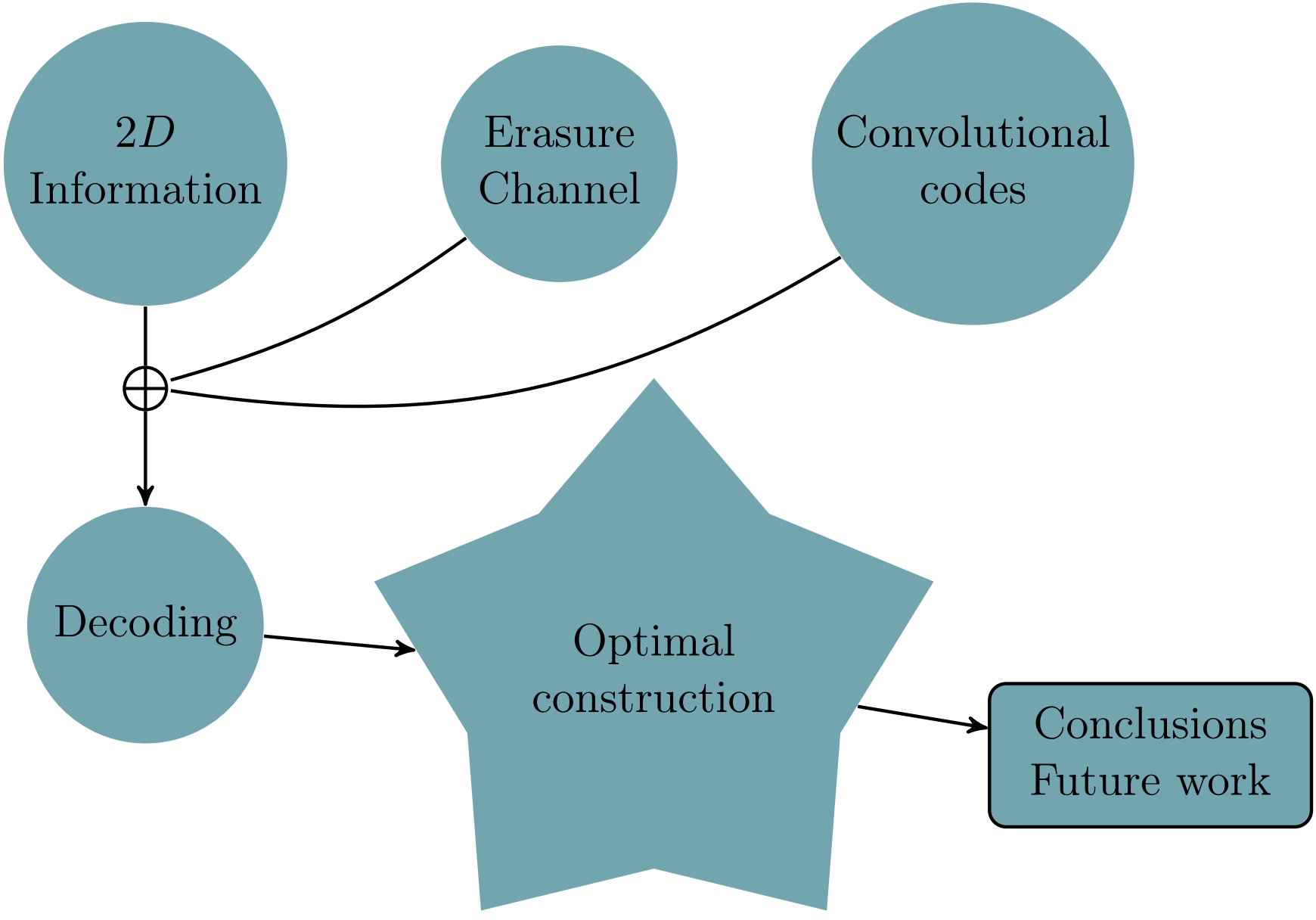
Let $n, k, \delta \in \mathbb{N}$ with $k < n$ and $(n-k)|\delta$ and let α be a primitive element of a finite field $\mathbb{F}_{p^N}$ with $N > (L+1)2^{(\nu+2)n-k-1}$. Then $H(z) = \sum_{i=0}^{\nu} H_i z^i$ with

$$H_i = \begin{bmatrix} \alpha^{2^{in}} & \cdots & \alpha^{2^{(i+1)n}-1} \\ \vdots & & \vdots \\ \alpha^{2^{(i+1)n-k-1}} & \cdots & \alpha^{2^{(i+2)n}-k-2} \end{bmatrix}$$

for $i = 0, \ldots, \nu = \frac{\delta}{n-k}$ is the parity-check matrix of an (n, k, δ) complete MDP convolutional code.



J. Lieb: Complete MDP convolutional codes, Journal of Algebra and Its Applications, Vol. 18, No. 06, 1950105 (2019) doi: 10.1142/S0219498819501056


$2D$
Information
Erasure
Channel
Convolutional
codes
Decoding
Optimal
construction
Conclusions
Future work

Conclusions

- Other decoding algorithms

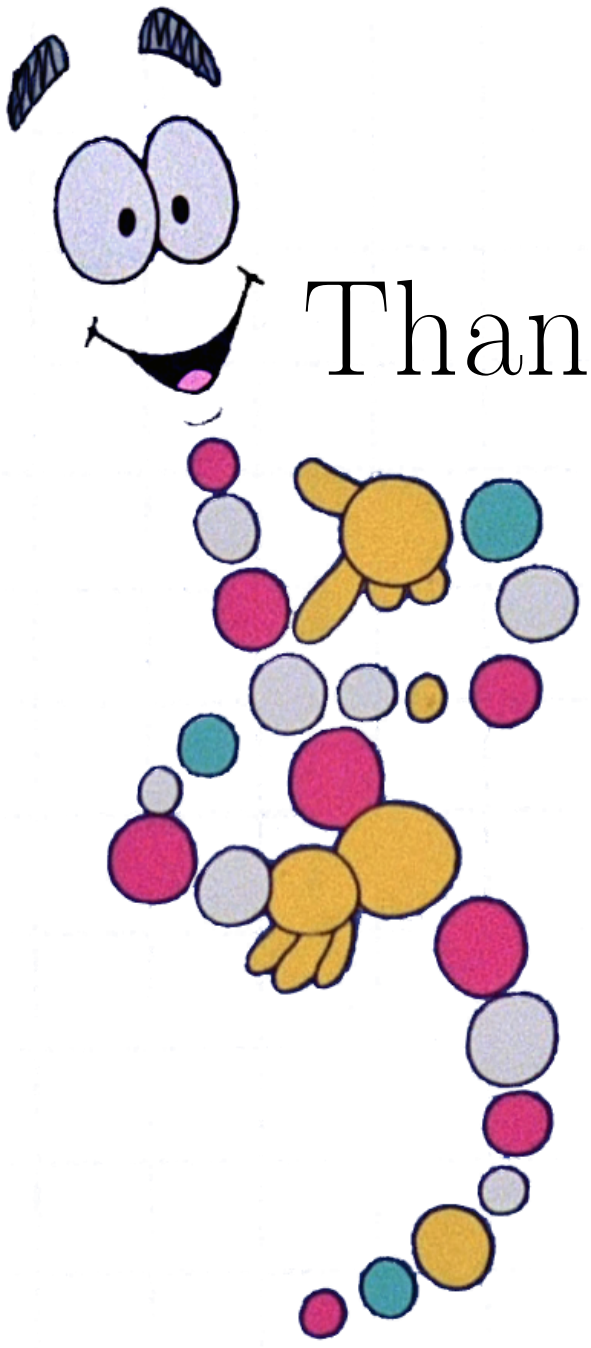
  

  Lieb, J., and Pinto, R., A decoding algorithm for 2D convolutional codes over the erasure channel. Advances in Mathematics of Communications, **2023**,17(4), 935-959.

- Other constructions with smaller field
- Decoding on lines for nD convolutional codes
- Using 2D metrics for decoding.

  Iglesias, J. I., Napp, D., Perea,C., Pinto, R. and Simoes R., Separation set distance for 2D Convolutional Codes. Available in: RIA repository



Thank you for your attention!

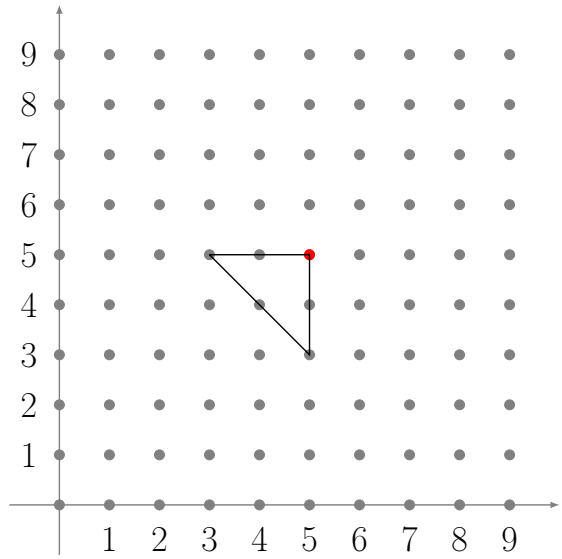
Questions?

Comments?

Suggestions?

Current work

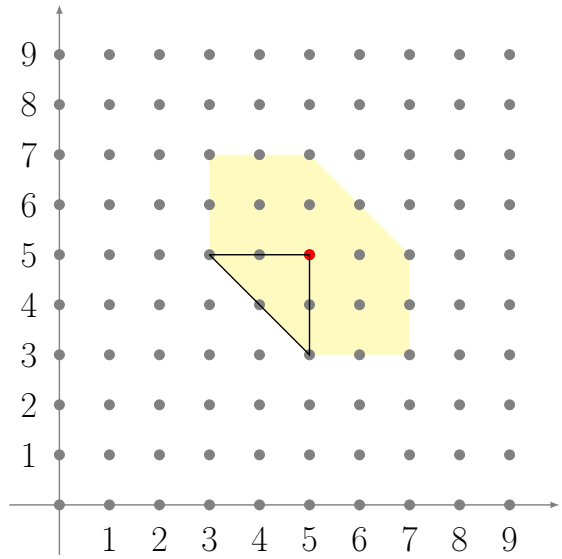
We are working on the Hexa-distance:



- Behaves very similar to the column distance.
- Allows the correction of sparse erasures

Current work

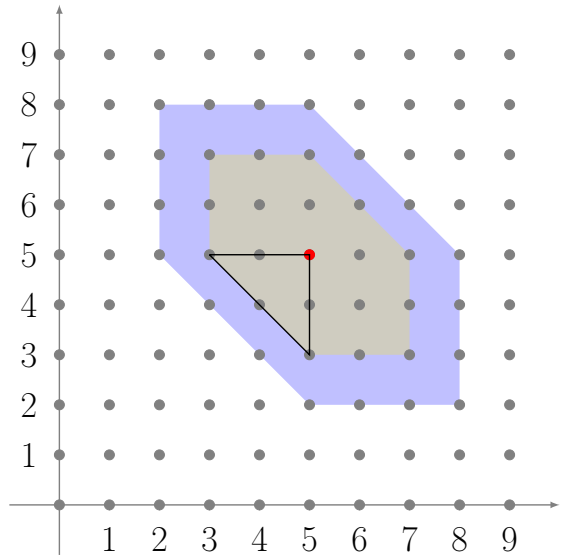
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